IREM PARIS 7
UNIVERSITE PARIS DIDEROT

On the historical phenomenology of probabilistic concepts - from the didactical point of view

Ewa Lakoma

Institute of Mathematics, Military University of Technology, Warsaw, Poland

Résumé

Cet article présente la phénoménologie historique des concepts probabilistes - du point de vue didactique. Une de plus importantes hypothèses didactiques est que la double nature du concept de la probabilité, qui est décrite par Ian Hacking (1975) du point de vue historique, semble être aussi principale dans le processus de l'apprentissage de la probabilité que dans la histoire de la probabilité.

1. Introduction

In this article I would like to present some results of my research studies in probability and statistics education.

Generally speaking I am especially interested in creating such approach to stochastics teaching which would respect students' natural cognitive development. Therefore I search first of all for answers to the four fundamental questions:

~ What are the natural ways of creating probabilistic concepts?

~ What are the natural ways of probabilistic reasonings, ways of probabilistic problem solving?

~ What is the natural language which pupils create and use when thinking and communicating?

~ What are symptoms of understanding probabilistic concepts?

According to the ideas of Hans Freudenthal (1983), I have reached for history of probability in order to recognize the natural ways of probabilistic thinking. History of mathematics seems to be a very important and very useful tool in eductional research. Following the old authentic or reconstructed mathematical reasonings helps to understand students' natural ways of thinking.

In the case of probability the results seem to be especially interesting. The dual character of probability concept, which was described from the historical point of view by Ian Hacking (1975), seems to play as important role today in the process of learning probability and statistics as it did in history.

In this article I will sketch very briefly the main points of the historical phenomenology of probabilistic concepts and then I will present some perspectives of using this knowledge to explore the process of probability learning and to create an approach to stochastics teaching which regards student's actual cognitive abilities.

2. The dual character of probability concept

Probability is one of the most important and also - in opinion of many mathematicians - one of the most controversial mathematical concepts (see Fine 1973, Freudenthal 1974). These controversies are mainly connected with two problems: with the problem of acceptance more than one - Kolmogorov's axiomatic - probability theories,
and with the problem of natural ways of probability learning which needs much more "practical" and "common sense" frame rather than formal, strongly abstract approach - usualy not understandable by students - who are not becoming mathematicians.

In many papers and books on history of mathematics the date of 1654 is quoted as the origin of probability calculus. In fact, that was the time of a very famous correspondence between Pascal and Fermat - concerning solutions of problems involved by haphazard. This popular opinion creates an illusion that all earlier attempts to solve some probabilistic problems were episodic and hadn't any impact on development of that theory which in the moment of its appearance had already mathematically advanced form.

There are many hypotheses which try to explain such sudden emergency of probability theory. One of them is the atmosphere of that time in Europe which was particularly good and which stimulated scientific development in many disciplines. That was the time of famous discoveries of Isaac Newton, prosperity of Port Royal with its fundamental work: "Ars Cogitandi" ("Logic"). It is very difficult to find another so fruitful period in history of mathematics and science.

In opinion of Ian Hacking (1975), this kind of explanation is not convincing enough. Many analyses of old authentic or reconstructed materials point out that probabilistic reasonings were present in various human activities already in ancient time. People tried to solve some problems connected with haphazard, with gathering statistical data and also with philosophical considerations in point of randomness, necessity etc.

In opinion of Hacking, the probability concept which was defined in mathematically acceptable form about 1660 couldn't appear so suddenly. Hacking argues that it must form successively during centuries, starting from the very beginning - from the ancient probabilistic ideas. These pre-origins and pre-conditions of Pascal's probability concept determined the very nature of this intelectual object - "probability"- which we use from the time of Pascal until now. Moreover, these pre-conditions for the emergence of probability anticipated the space of various possible theories about probability, like statistics, inductive logic, theory of inference, quantum mechanics and so on.

So, it seems that the most important period for the development of probability concept is the time before Pascal and Fermat.

During the attempts of reconstruction and understanding, how various probabilistic mental objects were formed, it is very useful to consider the specific feature which characterizes the nature of the Pascal's concept of probability. In opinion of Hacking, this is the duality of probability.

According to Ian Hacking (1975), the concept of probability has a dual nature. Two aspects of this concept can be distinguished:

~ epistemological - implied by the general state of our knowledge concerning the given phenomenon, related to the degree of our belief, conviction or confidence, which arise in connection to an argument - related to this phenomenon - and are supported by this argument;

~ aleatory - related with the physical structure of the random mechanisms under consideration and with the admission of its tendency to produce stable relative frequencies of events.

The first aspect gives the basis for the "chance calculus" and the other - for the "frequency calculus".

In opinion of Hacking, before the time of Pascal these two aspects were not consciously used. In probabilistic reasonings they were present independently.

The conscious sticking them together, the subtle contrasting and verbalising it - which was made mainly thanks to Pascal - has caused an act of illumination in the way of probabilistic thinking. From that moment the concept of probability has a dual, mathematically mature character.

3. The periods in historical development of probability

When we follow the history of probability we can distinguish five main periods:

1) prehistory - until the time of Pascal and Fermat:
   - posing and solving concrete elementary problems which have arisen from every day life;
2) the emergence of probability theory (as a science):
   - from the second half of XVII until the beginning of XVIII century (Jacob Bernoulli);
   - the time of forming the concepts of probability and expected value, the theorems of addition and product of probabilities;
   - applications: demography, social insurrance, estimation of errors in observations etc. (Pascal, Fermat, Huygens, Leibniz, Hudde, de Witt, Graunt);
3) from 1713 - "Ars Conjectandi" of Jacob Bernoulli - until the first half of XIX century (de Moivre, Laplace, Gauss):
   - the definition of Laplace, geometrical probability, statistical probability;
   - the limit theorems, laws of large numbers;
   - applications in science;
4) the second half of XIX century - the School of Petersbourgh (Tschebyschev, Markov, Lapunov):
   - generalization of the limit theorems;
   - probability as a part of physics;
   - searching for some logical fundamentals of the theory;
5) nowadays:
   - the axiomatics of Kolmogorov (1933),
   - many theories of probability (Fine 1973): personal probability of Bruno de Finetti, frequencial probability of Richard von Mises, works of Martin-Loef, Kolmogorov, Popper, Jeffreys, Keynes, Savage, Ramsey.

The historical process of forming probabilistic concepts can be presented as "the cup of probability":

The
cup of probability

fortuna — probare
casus — probus
aleatory — haphazard — probatus — epistemological
veritas — probabilis
truth — probabilitas
fair share
fair stake
expectatio
probability
verisimilis
verisimilitudo — dual
frequency
by chance
value
possible
probable
equipossible
equiprobable
event frequency
independence probability

From the didactical point of view the most interesting periods are the first two - that is the time of crystallizing the fundamental probabilistic concepts.

4. Two aspects of probability in pre-pascal reasonings
When we search for ancient probabilistic ways of thinking we recall the old solutions concerning gambling. We can easily recognize in these reasonings the aleatory aspect of probability.
Playing random games is treated as the first invention of human society.
A predecessor of the die was the most common random mechanisms: it was called the astragalus or talus. This is a 'knucklebone' or heel bone of a running animal (sheep, goat, deer). This bone is so formed that when it is thrown, it can land only on the one of four surfaces. Many such examples - well polished and engraved - are regularly found on the sites of ancient Egypt, Summeria and Assyria. It seems that mental objects connected with empirical frequencies and averages should be as old as the rolling of astragalus.
Moreover, we know some examples of taking decisions on the ground of empirical results of using random mechanisms (the Bible, talmud). It suggests that simple arithmetic connected with randomness was present in acient time.
The important question is: Wheather in ancient time people were able to recognize equally probable events? Did they use the model of even chances?

Modelling a throw of talus by means of that model was not adequate. Some experiments which were made with old authentic astragali show that frequencies of four surfaces were not equal: 0,39; 0,37; 0,12; 0,12. But you can meet some examples of regular cub dice (i.e. in Cairo Museum) which are perfectly fair - specially prepared and polished. It suggests that ancient people were able to recognize equally probable events, and they tried to compare results of experiments with a very simple and naive theory. In their arguments they used mental objects expressing the aleatory aspect of probability concept.
It is also worth to stress another specific notion which is connected with gambling: the concept of a honest game. Ancient people didn't understand this notion in the same way as it is accepted at present. For them the term "a honest game" was understood as a "according to the rules of play" even if these rules were evidently unfair. However, that kind of interpretation is well known also today - we can meet it when we listen children and young people playing games.
The epistemological aspect of probability we can recognize in ancient philosophical considerations connected with the notions of randomness and necessity.
The word "probability" which we connect with the concept emerging about 1660 - is much older. The latin adjective "probabilis" was used in medieval ages. It meant: "worth of approbation", "worth of approval" - approval of authorities.
So, the medieval author is saying that the proposition is "worth of approval" - because in his opinion it has the marks of truth or it is better supported by evidence than any other hypothesis. "Probabilis" was connected to the medieval notion of opinion. It refered to the authority of those who accepted the given opinion. From this point of view "probabilitas" suggests an approbation - with regard to the proposition accepted, and with regard to the authorities who accepted it. "Probabilitas" refers to the arguments which are presented in favour to the opinion, and from this point of view it suggests "a capacity of being proven". "Probability" has rather pejorative meaning - the proposition under consideration is merely probable because the proposition is not strictly demonstrated as are propositions which are properly scientific.
The best example of using the medieval notion of "probabilitas" we can find in works of Galileo. Galileo (1632) called the opinion of Copernicus as "improbable" because of many experiences which contradicted the annual movement of earth and because of the authorities of Ptolemeus and Aristotle. The Copernicus' hypothesis was improbable - but true. One century later Leibniz said that that hypothesis was "incomparably the most probable". For Galileo, probability has to do with an approaval. For Leibniz, it is what is determined by evidence and reason.
Galileo thought that the approval ought to correspond to the evidence, not to the weight of authorities. In his famous dialogues of Sagredo and Salviati we can find a discussion concerning the velocity of a body: Velocity of the body rolling down an inclined plane is a function of only the height of the plane.
The Galileo's explanation is as follows: "What you say seems to be very probable, but I wish to go futher and by an experiment so to increase the probability of it that it shall amount almost to absolute demonstration."

It is very clear indication of it that, in opinion of Galileo, experiments can increase probability almost to demonstration.
Connections between the old "probabilitas" and new - mature - notion of probability are rather astonishing. In opinion of Hacking, there was possible to create this concept when the notion of evidence appeared. The crystallizing of the dual concept of probability became possible when the notion of evidence has been defined in "Ars Cogitandi" (1662).
In medieval ages the proposition was probable when it was approvable by authorities. When that became possible to accept a new testimony - testimony of nature, the probability was connected with regularites and frequencies. The new proof - evidence - gave a range of probability to hypotheses and theorems - they became worth of approval. That was done by means of many observations which created the basis for predictions. In opinion of Hacking this whole process gives an explanation such a duality of probability concept.
It is also characteristic for medieval probabilistic reasonings that the word "probability" was not used to consider some problems concerning "haphazard". People used in this context such mental objects like "ability", "propensity", "proclivity", "facility", or "frequency". These objects were sticking together with "probabilitas" in time of Pascal and formed the dual probability.

5. The Problems of Points (Stakes) and the Problems of Dice - posing and solving
Various versions of "the Problems of Points" and "the Problems of Dice" belong to the most important and the most representative probabilistic tasks which were considered by many people during the whole history of probability development. Many different forms of those probabilistic problems are known. Considering the old authentic or reconstructed solutions of those problems lets us recognize different ways of probabilistic thinking and makes possible to distinguish various mental objects which were used in the process of solving problems.
I will quote the Problem of Points in a form given by Luca Pacioli (1494):
"Two football teams are playing a match. The winner team is that which scores 60 points as the first. The stake in this match is 22 ducats. For some reason the game was stopped at the point where one team had 50 points on its account and the other 30 points. What part of the whole stake each team should obtain?"
The rule which Luca Pacioli used in order to solve this problem was as follows:
To share a stake between both players according to the amount of games which every player has already won before the interruption. So, the amount of prize depends on it what would happen if the whole game were continued. This kind of reasoning is typical for many other authors of pre-pascal time.
Niccolo Tartaglia in his argument of this problem has made a "half-step" toward a mature solution - he took into account also possibilities of each player to win if it would be possible to continue the whole game. Nevertheless, the solution of Peverone (1558) has become accepted as a right one. Peverone thought that the prize should be shared according to the chances of win which both players have when the whole game is continued.
It is very important to stress that such metamorphosis in reasonings concerning the Problem of Points is characteristic not only for people who lived many

centuries ago but also is very common for probability learners nowadays - independently of their age (Lakoma 1990, 1998).
When we follow the old solutions of various Problems of Dice which are known from the XIII century we can notice that many of them based mainly on the intuitions of symmetry of a random mechanisms or on symmetry of the whole probabilistic situation under consideration. People used in their arguments various mental objects: "facility", "tendency", "proclivity", "ability" of dice to produce given outcomes and also "frequency" of results obtained on the ground of a "long observation". Galileo gave the most spectacular examples of such reasonings. Let us consider his solution of the "Problem of throw three dice". The problem to solve was as follows: "With three dice 9 and 12 can be made up in as many ways as 10 and 11. Each can be decomposed into 6 partitions. However it is known from long observation that dice players consider 10 and 11 to be more advantageous than 9 and 12 ?!"
Galileo seemed to recognize this problem as a conflict between an experiment (a long observation) and a model (of even chances). His explanation is following:
"Some numbers are more easily and more frequently made than the others; it depends on their being able to be made up by more variety of numbers. In particular, 6 partitions of 9 and 12 break down into 25 permutations, while 6 partitions of 10 and 11 decompose into 27 permutations. If permutations are equally probable, then 11 is more advantageous than 12 in the ratio 27:25."
We can recognize the hypothesis that permutations are equally probable, versus the hypothesis that partitions are equally probable. The second hypothesis is false, does not fit to the facts.
In probabilistic reasonings of the pre-pascal time we can also find another splendid ideas of modelling of probabilistic situations - in which the concept of symmetry plays a very important role. One of such examples is the reasoning of Cardano concerning a throw of a die in which he used probabilistic mental object - "an equality of series": "I am able to throw 1, 3 or 5 as 2, 4 or 6. The wagers are therefore laid in accordance with this equality if the die is honest, and if not, they are made so much the larger or smaller in proportion to the departure from the true equality."
All these mental objects, which were used in reasonings concerning dice, assert a "propensity", "tendency" or "disposition" of the dice to produce certain stable frequencies on repeated trials. On the other hand, we can notice that many authors based in their arguments mainly on the notion of symmetry and they treated probability as a part of the whole and also as a ratio of equally possible events. They also used the concept of proportion. Both aspects of probability - aleatory and epistemological - were present rather independently. However, when we analyse the ways of probabilistic thinking of Cardano and Galileo, we can find clear anticipations of the dual mature probability concept.

6. Probabilistic reasonings characteristic for the time of Pascal and Fermat
The "Problem of Dice" in a form done by Chevalier de Mere (1654) is as follows:
"How many times should one throw two dice so that the chance of throwing six on both dice will be at least equal to the chance of not obtaining this result at all?"

This version of the problem is very famous not only because it is still not clear if he tried in his reasoning to confront a simple mathematical model - "the rule of three" - with results of "long observation" or with other "mathematical theories" (Ore 1960) but mainly because it involved the greatest discussion in the history of probability development: the correspondence between Pascal and Fermat in which they both presented their solutions based on the probability concept in its mature dual form.
Let us follow the famous solution of a Problem of Points given by Pascal:
"The prize in the whole play is 64 pistoles and this player will obtain it who wins 3 games as the first. Assume that one of the player won 2 games and the other - one. They play another game and if the first wins, he obtaines the whole prize; on the other hand, if the second player wins this game then both players have 2 winnings. In that situation each of them obtains his own part of the prize: 32 pistoles. But notice that if the first player wins this game then he will obtain 64 pistoles, and if he loses - he will obtain 32 pistoles. If the players decide to share the prize before running this game, then the first should say: I surely have 32 pistoles, because when I lose I obtain it, but the remaining 32 pitoles can be obtained either by my adversary or myself - the chances are even. Let us share the sum into 2 halves, and I surely obtain 32 pistoles. Thus the first player sholud obtain 48 and the other 16 pistoles."
The solution of Fermat is following: "Suppose player A need to win 2 games more in order to win the whole play, and player B needs 3. Thus in order to finish the play it is sufficient to play a maximum 4 games. The possible results are in the table below:

1	A	A	A	A	A	B	B	A	A	B	B	A	B	B	B	B
2	A	A	A	A	B	A	A	B	B	B	A	B	A	B	B	B
3	A	A	B	B	A	A	A	A	B	A	B	B	B	A	B	B
4	A	B	A	B	A	A	B	B	A	A	A	B	B	B	A	B

In 11 cases player A is a winner and in 5 - player B. Thus player A should obtain 11/16 of the prize and player B - 5/16."
The solution of Fermat based on theoretical considerations of equally probable outcomes. Both authors relay in their arguments on the concept of symmetry. All solutions of Pascal and Fermat have the clear character of paradigmatic examples: in the concrete reasoning we can easily recognize the general method this kind of a problem. However the authors seemed to be not yet able to verbalise evident analogies in their ways of thinking - they considered them as diferent methods.
Both sorts of problems became a matter of interest also for Huygens, Leibniz, de Witt, Jacob Bernoulli and many others. Their solutions involved a great development of various probabilistic concepts.
From among such a various styles of resonings I would like to consider the Huygens's style of probabilistic thinking which seems to me particularly important from the historical and also from the didactical point of view.

7. The crystallizing of the concept of "expectation"

Christiaan Huygens in his work "De ratiociniis in aleae ludo" (Calculating in games of chance) - 1657 - presented a complete probabilistic theory which was

constructed according to the canon of Archimedes (a list of axioms, theorems) and based on two fundamental concepts: probability and expected value.
In the main part of that work various solutions of the Problems of Points and the Problems of Dice are presented.
At the beginning of that theory the notion of chance is defined: "Although in random games results are unknown the chance of player to win or to lose has a value. For instance when player bets that he will obtain 6 points on the die in the first trial it is unknown wheather he wins or loses. However it is possible to count how much his chances of winning exceed his chances of losing."
In reasonings concerning gamble Huygens used the notion of fair price of a game: "If we are invited to gamble with a given schedule of prizes depending on various outcomes, we would like to pay the faire (honest) price for taking a part in this gamble. It would be the price of selling by the player his rights to be a winner in this game." In order to count "the fair price" Huygens used the model of a fair lottery in which all tickets are symmetric and each of them can be drawn "as easily as" any other. The "fair price" is the most fundamental notion of Huygens's theory. From the historical point of view it is treated as the first explicite verbalised notion of expectation (expected value).
When we follow the history of probabilistic concepts we can pose the hypothesis that the process of forming the dual probability concept was accompanied by crystallizing the notion of expectation (expected value). These two processes were interlaced and supported each other already in the old pre-pascal time. The notion of expectation in a more sophisticated form has become present only about 1660. We can find it in reasonings which were led by these authors who also understood well the duality of probability and were able to use its nature in a proper way (i.e. Blaise Pascal, Christiaan Huygens). Both concepts: probability and expectation were usually confronted and distinguished each other. It is evident especially in definition given by Huygens:
"Expectatio - the chance of profit - is worth for somebody as much as he is able to pay as if he buys this chance in the fair and honest game." (Daston 1980)

8. Final remarks

The analysis of old authentic and reconstructed probabilistic reasonings shows that both aspects of probability - aleatory and epistemological - become inseparable and pierce each other starting from the time of Pascal (about 1660). Before that time these aspects were developed separately. So, the history points that the condition sine qua non to understand the probability concept is making conscious the dual nature of this concept.
When we observe the process of historical development of probabilistic concepts we can notice their main characteristic features. First of all it is evident that people in the past considered concrete problems which arose from needs of everyday life. In order to solve them people tried to observe the random phenomenon from which the problem arose and to discover some regularities of this phenomenon. Then some argumentations were made which were led to some conclusions related to the problem under consideration. All these activities were connected with theoretical modelling of random phenomenon.

Conclusions which were made on the basis of that model were not categorial and were usually verified by practice which confirmed or questioned a fitness of the model. Models were involved and created by concretization and schematization from the point of view of the problem to solve. There were treated as a more appropriate when they could better fit to considered "reality".
Moreover, it seems that involving in the history the concept of expectatio and careful distinguishing it from the probability made the probability calculus more understandable and clear form many people in the past and let the theory develop more intensively.
When we analyse old probabilistic reasonings we can easily notice that all of them arose thanks to great discussions and serious disputs, because of exchanging arguments and reasonable convincing of adversaries. All these activities demonstrate that probability development has a strong interactive nature.
This diachronic view on the development of probability concept allows to pose the main didactical hypothesis - in the synchronic perspective - that the dual nature of probability concept seems to play as important role today in the process of stochastics learning as it did during the process of historical development of probability. Regarding this fundamental conclusion allows to organize the process of probability and statistics learning according to the student's cognitive development at every stage of education (the Local Models' approach - see Lakoma a.o. 1996, 1997) and also to explore the natural students' ways of probabilistic thinking (Lakoma 1990, 1997).

References:

Daston L. J., 1980, Probabilistic Expectation and Rationality in Classical Probability Theory, in: Historia Mathematica 7, p. 234-260.
Fine T. L., 1973, Theories of Probability, an Examination of Foundations, N.York.
Freudenthal Hans, 1974, Mathematics as an Educational Task, D.Reidel, Dordrecht.
Freudenthal Hans, 1983, Didactical Phenomenology of Mathematical Structures, D.Reidel, Dordrecht.
Hacking Ian, 1975, The Emergence of Probability, Cambridge Univ., London.
Lakoma Ewa, 1990, The Local Models in Probability Teaching (in Polish), doctoral thesis, Warsaw University, Department of Mathematics, Warsaw.
Lakoma Ewa, 1992, Historical Development of Probability (in Polish), CODN-SNM, Warsaw.
Lakoma Ewa a.o., 1996, 1997, Mathematics 2001 - series of textbooks for students and guides for teachers for secondary school, WSiP, Warsaw.
Lakoma Ewa, 1998, On the Interactive Nature of Probability Learning, in: Proceedings of CIEAEM-49, Setubal, July 1997 (in print).
Ore O., 1960, Pascal and the Intuition of Probability Theory, American Mathematical Monthly 67, p.409-419.

Didactique et épistémologie : quelques aspects d'une recherche de terrain...

C. Hauchart, pour le
Groupe d'Enseignement Mathématique[1] *(GEM)*
Louvain-la-Neuve, Belgique

Le texte ci-après tente de mettre en évidence quelques particularités du type de recherche que nous pratiquons au GEM, ainsi que nos motivations et objectifs essentiels. Il témoigne de l'expérience de notre groupe, pas du tout pour présenter ce dernier en exemple, mais pour illustrer cet exposé à partir de ce que nous connaissons bien. Nous décrirons quelques résultats de notre groupe, en particulier d'un projet d'enseignement qui résulte d'une collaboration (projet désigné par AHA, Approche Heuristique de l'Analyse[2]).

1. Le Groupe d'Enseignement Mathématique.

1.1 Le GEM à ses débuts

Le groupe d'enseignement mathématique s'est créé vers la fin des années 70. Une de ses caractéristiques est d'être constitué de membres d'origines diverses : des étudiants de deuxième licence en mathématiques qui ont choisi de réaliser leur mémoire (travail de fin d'étude) dans le domaine de l'enseignement des mathématiques (à l'époque, ils étaient entre 5 et 10 chaque année année), des enseignants du secondaire (entre 20 et 30), et des universitaires (une petite dizaine). Cette diversité a été source de richesse.

La raison d'être visible du groupe était l'encadrement de ces travaux de fin d'étude. Mais par delà ce prétexte et cette contrainte (à la fin de l'année académique, un document sérieux devait être rédigé), dès le début, notre objectif premier, celui auquel se sont ordonnés les autres (les mémoires, la formation initiale des maîtres, les prestations de formation continuée, les recherches épistémologiques) était d'enseigner des mathématiques :

> *"Notre position radicale a alors été la suivante : nous avons pris pour objectif premier d'enseigner des mathématiques et de la faire sur le terrain le plus quotidien des classes, avec toutes les contraintes que cela suppose. En ce faisant, nous acceptions que nos autres actions (les mémoires et la formation initiale des maîtres, la formation continue des*