CONSTRUCTIVISM, EDUCATION AND THE PHILOSOPHY OF MATHEMATICS

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Dans les dernieres années le paradigme constructiviste a acquis une vaste diffusion dans l'éducation mathématique. D'autre part, les philosophies failliblistes des mathématiques ont été décrites et analysées dans nombre de livres et d'articles. Le but principal de ce travail est de discuter les relations entre la philosophie des mathématiques et les paradigmes de l'éducation mathématique (en particulier le constructivisme). Je cherche a démontrer que le paradigme constructiviste n'a pas besoin d'une philosophie faillibliste des mathématiques ni, peut-être, d'aucune philosophie, et que les attitudes des enseignants vers les mathématiques ne decoulent pas entièrement de la philosophie des mathématiques (au moins si celle-ci concerne la fondation de la certitude mathématique).

1. INTRODUCTION

In the last years the attention towards the philosophy of mathematics has considerably increased. Even in mathematics education the role of general paradigms has grown more definite. These paradigms are often related to major trends in epistemology, even though they take on ideas and motivations from other settings as well. A consequence has been the need for going deeper into the exploration of the links between the philosophy of mathematics and the theoretical frames of the research on mathematics education. In particular this has concerned constructivism, whose epistemological and philosophical roots are manifest. Some parallel has been drawn in the past between constructivism in education and constructivism as a foundational school in mathematics.[1] More recently, some studies (for example, Lerman 1989) have pointed out the differences between the two theoretical frames and the links between constructivism as a theory of learning and the fallibilist philosophy of mathematics. This issue has been widely developed in the book of Ernest (1991), in which he presents the fallibilist philosophy as a philosophical frame adequate to constructivism.

The main object of this paper is to discuss Ernest's conclusions following two main strands: from one hand, I point out some failures of the fallibilist philosophy that in my opinion make it inadequate as a philosophy of mathematics; on the other hand I try to argue that the acceptance of the fallibilist philosophy is not necessary for a constructivist researcher or educator and that there are no close links between the philosophy of mathematics and educational research and practice.

2. THE CONSTRUCTIVIST PARADIGM IN MATHEMATICS EDUCATION

The following have been recognized as the main assumptions of constructivism (Kilpatrick 1987, Vergnaud 1987, Lerman 1989):

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[1] By *constructivism* in mathematics it is generally meant a wide area of foundational trends and studies including Brouwer and Heyting's intuitionism, recursive analysis, E.Bishop's constructivism (also known as explicit mathematics) and many others. In the remainder of the paper by 'constructivism' it is meant the theory of learning.

(1) Knowledge is actively constructed by the cognizing subject, not passively received from the environment.

(2) Coming to know is an adaptive process that organizes one's experiential world.

(3) Coming to know does not discover an independent, pre-existing world outside the mind of the knower.

The acceptance of (1) only is sometimes called 'trivial constructivism', whereas the acceptance of (1), (2) and (3) is called 'radical constructivism'. There is people, for example Vergnaud, accepting (1) and (2) but not (3). The constructivist paradigm, which is very popular among researchers in mathematics education, fits very well the need for a shift in perspective on mathematical knowledge: aspects as creativity and ability at planning are more and more regarded as inherent to mathematical competence; moreover, it seems a theoretical frame adequate to the purpose of extending mathematical education to students of different cultures and styles of learning.

One of the most striking features of constructivism is its flexibility, that is the opportunity of being combined with other theoretical frames, even though conflicting as regards philosophical assumptions[1]. Various research groups in different countries adopt research frames borrowing ideas from both constructivism and Vygotskij's theories, or other theories assuming social interaction as a fundamental step in the process of construction of mathematical knowledge[2]. This means that constructivism is more used as a wide frame which can include and strengthen different ideas and practices than as a reference theory from which carefully draw all possible implications on teaching, learning and the related research. As a matter of fact, it has been argued that "as a theory of knowledge acquisition, constructivism is not a theory of teaching", that "there is no necessary connection between how one views knowledge as being acquired and what instructional procedures one sees as optimal for getting that acquisition to occur"[3] and that "epistemologies are descriptive, whereas theories of teaching must necessarily be theories of practice"[4]. Nonetheless, von Glasersfeld has identified five consequences for educational practice that follow from a radical constructivist position[5]:

(a) teaching becomes sharply distinguished from training;
(b) processes inferred as internal become more interesting than overt behavior;
(c) linguistic communication becomes a process for guiding a student's learning, not a process for transferring knowledge;
(d) students' deviations from the teacher's expectations becomes means for understanding their efforts to understand;
(e) teaching interviews become attempts not only to infer cognitive structures but also to modify them.

Kilpatrick's opinion is that "all five consequences fit the constructivist stance, but they appear to fit other philosophical positions as well". He argues that constructivist positions, in itself, do not lead to a way of teaching radically different from the other ones; the only difference is one of focus. In his opinion, the contrast between the two different perspectives on the acquisition of knowledge (construction versus transport) is only a contrast between metaphors. In particular, in opposition to the constructivist argument asserting that students' misconceptions are better understood when seen as arising from alternative constructions of meaning than as failures in communication, he remarks that both the models 'alternative construction' or 'failure

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1 A comparison of activity theory and radical constructivism can be found in Bauersfeld, 1990.

2 For example see Lerman, 1992. A survey of research in such direction can be found in Bartolini Bussi, 1991.

3 Kilpatrick, 1987, p.11.

4 Kerr, 1981, reported in Kilpatrick, 1987, p.11.

5 von Glasersfeld 1983, reported in Kilpatrick, 1987, p.12.

in communication' can lead to attempts to find out what the student is thinking.

In my opinion it is quite correct to avoid strict links between learning theories and the educational practice; an educational product is affected by the results of research in different fields, such as the epistemology of the subject matter, psychology, pedagogy and so on. Moreover, the dimension of the so called 'didactical phenomena' has to be taken into account. The mechanical transposition of theories or pieces of theories to educational practice has generally yielded poor results, such as, in Italy, modern mathematics or even misunderstood portions of Piaget's theories.

As regards the role of metaphors, any construction process is affected by the context (including linguistic communications that take place), and sometimes the influence of the context may be so strong to make the metaphor of transmission convincing. There are situations, even in mathematics, where external clues seem a necessary condition for the construction of knowledge.

But in some aspects the constructivist paradigm seems more fruitful. The transition from the metaphor of transmission to the metaphor of construction is not only a shift of emphasis but sometimes allows teachers and researchers to better understand the processes involved. This applies to misconceptions too, because often they are by no means removed by the careful iteration of transmissions. In students' misconceptions there is often much more than what has been actually transmitted, as far as already existing mental models or schemes are involved.

3. THE FALLIBILIST PHILOSOPHY OF MATHEMATICS

3.1. An outline of fallibilist views

The fallibilist philosophy is a wide area of attitudes, opinions and studies mainly related to the papers of Lakatos, Hersh and Davis, Putnam, Tymoczko and others. The main fallibilist assumptions have been summarized in the book of Ernest; in brief, their claim is that mathematical knowledge is not absolute but fallible and corrigible and formalization does not fulfils its role of warrant but rather hinders the development of knowledge. Moreover, the development of mathematics is regarded as parallel to natural sciences; in mathematics as well as in natural sciences the emphasis is not on the transmission of truth from true premises to conclusions, but on the re-transmission of falsity from falsified conclusions (falsifiers) to hypothetical premises. Apart from formal contradictions such as $p \wedge \neg p$, the potential falsifiers of a theory are the informal theorems of the (assumed) pre-existing informal theory. In fallibilist views informal mathematics is of crucial importance, because as a product it is the source of all formal mathematics. Theories are not equally likely to be falsified: elementary group theory is harder to falsify because the original informal theory has been radically replaced by the axiomatic theory. Set theory is a different question as it is "the dominant, unifying theory in which all available mathematical facts have to be explained."[1] From the focus on informal mathematics follows that the history of mathematics is of paramount importance as well.

Ernest proposes a new philosophy of mathematics called social constructivism, "which is largely an elaboration and synthesis of pre-existing views of mathematics, notably those of conventionalism and quasi empiricism."[2] I try to give a brief account of Ernest's views, especially of those I intend to criticize. I apologize for the unavoidable oversimplification; the interested reader is referred to Ernest's book for a more detailed and proper account. Social constructivism, which develops most of fallibilist views, is claimed to be a descriptive (and not prescriptive) philosophy of mathematics. Mathematics is regarded as a human construction because "the basis of mathematical knowledge is linguistic knowledge, conventions and rules,

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[1]Lakatos, 1967, p.39-40

[2] Ernest, 1991, p.42.

and language is a social construction"[1]; furthermore "interpersonal social processes are required to turn an individual's subjective mathematical knowledge, after publication, into accepted objective mathematical knowledge"; finally "objectivity itself will be understood to be social"[2]. Any logical system of knowledge depends on a set of primitive propositions or terms and "for objective mathematical knowledge these propositions and terms are to be found in the objective knowledge of natural language". So one "cannot question the fact that 'A and B' entails 'A' or that 1+1=2 without withdrawing some of the possibility of communication." Ernest claims that "our use of the key logical terms 'not', 'and', 'or', 'implies', 'if, and only if', 'entails', 'there exists', 'for all', 'is a', and so on, strictly follows linguistic rules....These rules fix as true basic statements such as 'if A, then A or B', or rules of inference such as 'A' and 'A implies B' together entail 'B'. ... Thus reasoning ... rests on the shared rules of language. The rather more abstract and powerful forms of logic used in mathematics also rest on the logic embedded in natural language use."

3.2. Some drawbacks in fallibilist arguments

In this section some fallibilist positions are criticized. In particular I focus on three issues: Lakatos' parallel between mathematics and natural sciences, the role of proofs in mathematics and the connection between mathematical logic and natural language exposed in the last paragraph of the previous section. All these points concern mathematics education and the related research, but the third one seems to me the more directly connected to practice as it involves not only the role of mathematical logic in the curriculum of primary and middle school, but also a particular perspective on language and so on learning processes.

MATHEMATICS AS A NATURAL SCIENCE

The assimilation of mathematics to natural sciences is not convincing. At this regard, Cobb's remark about the lack in mathematics of something corresponding to dismissed theories (as falsified by experimental facts)[3] seems quite correct. Mathematics cannot be regarded only as a process of discovery of mathematical truths. Mathematics is made up of theories and models and studies their properties and relationships. Discovering mathematical facts not compatible with some previous theory means formulating new theories. In mathematics, theories are to be regarded as a whole, it does not make much sense to take into account statements outside the theoretical context they are included. For example, it is not very sensible, from a mathematical point of view, to ask how many are the parallels to a given line through a given point, and which answer is a necessary mathematical truth. In the same way, it is not so interesting to ask whether the Boolean law 1+1=1 and the arithmetic fact 1+1=2 are contradictory or not.[4] The similarity between the two formulas is only apparent because they belong to different theories and then symbols are forced to assume different meanings. Even the theory of potential falsifiers is questionable. In some theories, such as group theory, the hypotetico-deductive structure is evident: xy=yx is not a necessary (or unnecessary) truth, but is introduced to separate two different classes of structures. Any substantial change in the axioms of group theory would not lead to an *improved group theory* but to *another* theory. It would not change

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1 ibidem

2 ibidem

3 Cobb, 1989, p.37.

4 Ernest, 1991, p.52. In effect neither formula is valid, for they are both atomic formulas. '1+1=1' is a theorem (usually an axiom) of the elementary theory of boolean algebras. In '1+1=2' two non primitive (1 and 2) symbols occur. If we assume standard arithmetic definitions, i.e. 1=succ(0) and 2=succ(succ(0)), then '1+1=2' becomes a theorem of first order arithmetic. I do not know any reasonable mathematical theory including both formulas as theorems. Observe that 1+1=0 is a theorem of the theory of fields of characteristic 2, 1+1=2 is a theorem of the theory of the category of sets and ¬(1+1=2) is a theorem of the theory of the category of all categories. This variety of interpretations is an enrichment for mathematical knowledge, not a limitation.

the idea of group, but would construct a different idea. The only discussion could be on which of the two theories deserves the appellation *group theory,* but this is not a very important question. The old theory should not be dismissed, maybe it could be disregarded, if the new is more interesting and promising. Most likely, the analysis of the relationships between the two theories would become a new research field. As regards sets, the question is not very different. The notion of set is largely determined by the mathematical theory. I cannot understand what informal set theory could be like. Naive set theory is contradictory, as Lakatos himself concedes. What will happen in the informal theory to those set theoretical problems posed as a consequence of the efforts to provide a suitable axiomatization? In other words, in informal set theory is replacement schema true, or one has to bound oneself to separation? And what about the axiom of choice? And what happens to the notion of set deriving from principles of choice different from AC? Which cardinals do exist for the informal theory? Why the informal theory should not be constructive? I cannot imagine how some set theory, say ZF, could be falsified by an informal theorem, but I cannot help thinking that this should produce nothing more than another set theory. We have already different set theories that provide different ideas of set. Moreover, the notion of set is not so dominant in mathematics that all available mathematical facts should be explained in it. Sets may be replaced, at least in principle, by other concepts, such as categories, topoi, or different kinds of sets could be used, such as recursively enumerable ones. The mathematical idea of set does not seem more natural than the idea of group.

THE ROLE OF PROOF IN MATHEMATICS

In my opinion the main object of proofs is not to warrant the certainty of mathematical knowledge, but to determine the domain where mathematical results hold, for any interpretation of constants, functions and predicates, and to make explicit the links among the different properties. The goal of the search for constructive proofs, for example, is not so much to increase the certainty of a theorem, but to analyse the relationship among different mathematical systems, or even to extract an algorithm. A proof may be interesting due to its simplicity and elegance, or because it is easily translated into an algorithm, or works in some weak theory or within a weak logical frame. In other words a proof belongs to mathematics, is not a warrant for it. A thorough discussion on these and other related questions may be found in Lolli (1987).

Another question is related to the meaning of a statement such as "mathematics is fallible and corrigible". If it is regarded from an anthropological perspective[1], it roughly means that mathematical theories can change or turn out inadequate or even be disregarded, which is widely acceptable. A different question is the claim that results within a specific theory are fallible: for example, if I state that in a specific theory, within a specified logical frame, some theorem holds, one thing is to claim that such theorem makes no sense, or embodies no bright idea, or is not useful, or the proof is ugly, another thing to claim that it is not a theorem in that theoretical and logical context. The first claim seems to belong to the anthropological context, the second to the experiential one[2]. In other words, for a given a theorem, I cannot be sure that it is interesting for mathematical knowledge on my own, but I can be sure myself that it is actually a theorem, at least as much as I am sure of my address or my phone number.

MATHEMATICAL LOGIC AS BASED ON NATURAL LANGUAGE

According to Ernest, it is natural language that guarantees the basis of objectivity in mathematics as well as in mathematical logic. I do not understand why we cannot question facts such as 'A and B entails A' or '1+1=2' without consequences for the possibility of communication. These formulas hold in some 'language game' and do not in some other. I doubt that they remain in force in our meta-language. Why should we be compelled to carry out our reasoning within the frame of classical logic? Why not another logic, say a non-monotonic

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[1] Cobb, 1989.

[2] ibidem.

one? And why standard arithmetic should apply to such a large domain? I completely disagree with the opinion that axioms and rules of mathematical logic should reflect the use and the meaning of the terms involved. The everyday use of words such as 'not', 'and', 'if... then...' and so on is by no means the same as in classical logic. In classical propositional calculus, for example, the truth of a complex statement is a function of the truth of the atomic statements involved. This does not happen in the natural language, which has a different semantics, also based on factors including context, purposes and shared culture. So statements of the form 'if A then B', when A is false and B true, are true according to the truth-functional semantics of mathematical logic, but they are perceived as false, and also a bit foolish, in most of 'language games'. Further examples in this vein can be found in Freudenthal, 1989.

On the other hand there is evidence supporting the idea that human reasoning is content dependent and not based on rules.[1] The view of reasoning as resting on the (assumed) shared rules of (verbal) language contrasts with the growing interest and success of research on visual reasoning.[2] On the other hand, the idea that logical competence is the basis for mathematical knowledge has been refuted by both epistemological analysis and experimental findings. There is no need of recalling here the failure of most of the attempts to teach (mathematical) logic to children as it were a tool for reasoning or solving problems. By the way, social constructivism seem more prescriptive than descriptive at this regard, as it rejects (or overlooks) forms of reasoning widely used and studied, even by constructivist teachers and researchers. In spite of the social roots of mathematical knowledge, the model of reasoning proposed is absolutist and detached from everyday experience.

4. MATHEMATICS EDUCATION AND THE PHILOSOPHY OF MATHEMATICS

4.1. The role of teachers' perspective

According to Ernest (1992), problem-solving (which is a fundamental component of mathematics education) "... is assimilated to the teacher's mathematical perspective. In other words, what a teacher understands by problem solving is largely a function of that teacher's personal philosophy of mathematics."[3] Absolutism, Progressive Absolutism and Fallibilism are recognized as the three main philosophies of mathematics held by school and college teachers of mathematics. Absolutist philosophies, which are the most widespread ones, regard mathematics as a body of fixed and certain, objective knowledge. In Ernest's view, logicism, formalism and, to a large extent, platonism are absolutist. Teachers with an absolutist view of mathematics will regard problem solving "... as the carrying out of non-routine teacher imposed tasks with determinate right answers. Problem solving is thus an activity which follows on from the transmission of mathematical content, and provides the means to apply previously learned knowledge and skills."[4]

The progressive absolutist position also views mathematics as made up of certain, objective knowledge, but in addition it accepts that new knowledge is continually being created by human activity. Popper's epistemology and the intuitionist philosophy of mathematics are included in this class. Progressive absolutist teachers "... will view problem solving as the means to develop and employ the strategies and the processes of mathematics, and to uncover the truths and structures of mathematics."[5]

The fallibilist position has been already presented in 3.1. A fallibilist teacher "... will view

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1 See Jonson-Laird, 1983.

2 A survey may be found in Dreyfus, 1991.

3 Ernest, 1992, p.294 (thesis 2).

4 ibidem, pp.294-295 (thesis 2.1).

5 ibidem, p.295 (thesis 2.2).

problem solving as the appropriate pedagogy to employ in the classroom. In particular, it is seen as a socially mediated process of problem posing and solution construction, requiring discussion for the negotiation of meanings, strategies and proofs."[1]

In the book of Ernest, teachers' perspectives are related to philosophy of mathematics, which, in turn, is linked with the problem of mathematical certainty and then with the studies on foundations (which, besides attitudes and perspectives, produce theorems as well). I cannot see such a close link among teachers' perspectives, their opinions on mathematical certainty and the results of foundational research. In other words I reject the assimilation of teachers' perspectives to their philosophy of mathematics, which Ernest[2] seems to identify with their acceptance of one out of three main classes of opinions about mathematical certainty. Opinions on mathematics and its certainty no doubt affect teachers' attitudes, but this is not necessarily related to their even implicit awareness of the foundational debate and of the positions of the different schools. In Italy there exist teachers with well defined attitudes (dogmatic or relativistic) who hardly have heard something definite about the philosophy of mathematics or the foundational research. My feeling, exaggerating a little, is that most teachers' attitudes would hardly change had Gödel proved, in place of one of his well known theorems, that ZF+ "any cardinal one can think does exist" is consistent, showing some recursive model. Joking apart, what I mean is not that teachers have no general opinions about mathematics, but that these opinions do not mainly concern the problem of mathematical certainty nor the results of foundational studies and research paradigms.

In my opinion, the factors affecting teachers' attitudes are mostly other, more complex ones[3]. In Italy, for example, the custom of regarding mathematics in some curricula as a model of abstract knowledge, detached from applications, is less an outcome of opinions about the foundations of mathematics than of fascism's cultural policy in the 20s, largely influenced by dominant Gentile's neo-idealist philosophy and by the need of maintaining society under control. Another aspect, which is a consequence of the previous one, is the role of mathematics in the curriculum: in Italy, for a long period, mathematics has contributed, more than other subject matters, to students' selection, excluding part of them from the continuation of studies. Related to these aspects, there are either absolutist or relativist teachers, but these attitudes are only partially determined by their acceptance of some philosophy of mathematics. Their role in the instructional system and in society is an influential variable as well.

4.2. Philosophy of mathematics, foundations and mathematical certainty

The fact is that the relationship between mathematics and society is very complex, and does not involve only pure mathematics, let alone foundational issues. Applied mathematics is more and more widely used in everyday life as well as technologies, such as computers, which are generally regarded as closely connected with mathematics; this most likely affects people's attitudes more than the traditional quarrels on foundations or on mathematical certainty. It is strange that a philosophy of mathematics asserting the social construction of mathematical knowledge should assume such a narrow view of the relationship between mathematics and society.

On the other hand, the empirical evidence provided to support such claims[4] is not conclusive, as the author himself concedes, as far as the classification of teachers' perspectives seems to be

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1 ibidem, p.295 (thesis 2.3).

2 Ernest, 1991.

3 About the complexity of the relationships between teachers' conceptions and educational practice see Thompson, 1984.

4 Ernest, 1991, p.297-298.

based on their behavior, and the link with the three main philosophies of mathematics is taken for granted and not proved at all.

In conclusion, I do not argue that social constructivism is a completely false philosophy of mathematics education. It is a very promising frame which can contribute to better explain a number of questions, such as the role of history and social interaction in learning processes. I think it presents an oversimplified view of mathematical knowledge and its relationships with teaching and learning. The analysis of Cobb (1989) pointing out the need for multiple perspectives seems quite correct and useful at this regard. In his view, "In experiential terms, mathematical objects are experienced as being practically real. ... if we view Platonism and mathematical truth as experiential aspects of consensually constrained mathematical activity, then, as a constructivist mathematics educator, I want students to experience intuitions of a mind-independent mathematical reality ... If students do not act as Platonists when they do mathematics they are left with nothing but empty formalisms."[1] Something analogous applies to mathematical certainty as well: many mathematical properties are experienced as being practically certain. This happens when students (groups or singles) check a computation, proof a theorem, build a model or a counterexample in their own way, following their own paths to knowledge. Non-absolutist mathematics educators may want to give students the opportunity of directly handling mathematical ideas and properties as objective, socially accepted knowledge, and they are aware that processes of this kind are an important component of mathematical acculturation.

REFERENCES

Bartolini Bussi, Maria: 1991, 'Social interaction and mathematical knowledge', *Psychology of Mathematics Education, Proceedings of the XV Conference,* vol.I, 1-16.

Bauersfeld, Heinrich: 1990, *Activity Theory and Radical Constructivism - what do they have in common and how do they differ,* occasional paper 121, Universität Bielefeld - IDM.

Bishop, Errett: 1967, *Foundations of Constructive Analysis,* Mc Graw Hill

Cobb, Paul: 1989, 'Experiential, Cognitive and Anthropological Perspectives in Mathematics Education', *For the Learning of Mathematics,* 9/2, 32-42.

Dreyfus, Tommy: 1991, 'On the status of visual reasoning in mathematics and mathematics education', *Psychology of Mathematics Education, Proceedings of the XV Conference,* vol.1, 33-48.

Ernest, Paul: 1991, *The Philosophy of Mathematics Education,* The Falmer Press.

Ernest, Paul: 1992, 'Problem Solving: Its Assimilation to the Teacher's Perspective' in J.P.Ponte *et al.* (eds.), *Mathematical Problem Solving and New Information Technologies,* Springer-Verlag, 287-300.

Freudenthal, Hans: 1988, 'Logic as a subject or as an attitude', *Atti degli Incontri di Logica Matematica,* vol.5, 27-38.

von Glasersfeld, Ernst: 1991, 'Outline of Radical Constructivism', summary of a lecture held in Trento at the school on *Psychology of Learning and Mathematics Learning.*

Johnson-Laird, Philip N.: 1983, *Mental Models. Towards a Cognitive Science of Language, Inference and Consciousness,* Cambridge University Press.

Kilpatrick, Jeremy: 1987, 'What Constructivism Might be in Mathematics Education', *Psychology of Mathematics Education, Proceedings of the XI Conference,* vol.I, 3-27.

Lakatos, Imre: 1967, 'A Renassaince of Empiricism in the Recent Philosophy of Mathematics' reprinted in Tymoczko, T. (ed.) *New Directions in the Philosophy of Mathematics,* Birkhäuser, 1986, 29-48.

Lerman, Stephen: 1989, 'Constructivism, Mathematics and Mathematics Education', *Educational Studies in Mathematics,* 20, 211-223.

Lerman, Stephen: 1992, 'The function of language in radical constructivism: a Vygotskian perspective', *Psychology of Mathematics Education, Proceedings of the XVI Conference,* vol.II, 40-47.

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[1] Cobb, 1989, p.

Lolli, Gabriele: 1987, *La macchina e le dimostrazioni,* Il Mulino.
Thompson, Alba Gonzalez: 1984, 'The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice', *Educational Studies in Mathematics,* 15, 105-127.
Tymoczko, Thomas: 1986, *New Directions in the Philosophy of Mathematics,* Birkhäuser.
Vergnaud, Gerard: 1987, 'About constructivism' *Psychology of Mathematics Education, Proceedings of the XI Conference,* vol.I, 42-54.