INTUITION AND VISUALIZATION ("ANSCHAUUNG") - A FUNDAMENTAL PRINCIPLE OF LEARNING THEORY AND ARITHMETIC INSTRUCTION IN PRUSSIAN ELEMENTARY SCHOOLS IN THE 19 TH CENTURY INTENTIONS AND REALITY

Siegbert Schmidt
(Universität zu Köln, Germany)

1.Theoretical Categories Used as a Basis of Analysis

In a talk in the section "Social History of Mathematics Education" at the ICME-6 in 1988 in Budapest (Hungary) G. Schubring (1989) described theoretical categories for investigations in the social history of mathematics education. The following categories were declared to be basic:

- Neutral knowledge does not exist: Knowledge always is influenced by intentional and social factors.

-The epistemological nature of mathematical knowledge has to be considered in comparison to sciences as well as to the humanities.

- Knowledge develops, knowledge is not static, it is not always the same.

In this paper concrete examples will be discussed with regard to the following aspects which themselves are bound by the categories just mentioned:

- Status of elementary arithmetic within a concept of general education for the "just plain folk".

- Elementary arithmetic and the teaching-learning process in arithmetic instruction in the elementary school:

Steps in arithmetic instruction and the "nature of human mind".

-Some social and political factors of the Prussian system of public education in the 19th century.

Because a piece of history shall be presented that can be sufficiently empirically founded some decisions had to be made, thus getting some further contextual constraints:

- We will talk about arithmetic instruction in the elementary school in Prussia between the beginning of the 19th century and 1872 - thus considering the time between the decline of Prussia because of the victory of Napoleon, but also the beginning of reforms in Prussia, on the one side, and the unification of Germany as an empire under the leadership of Prussia in 1871 and some essential changes in the elementary school system and the pre-service training of elementary school teachers in 1872, on the other side.

- We want to make concrete the essential features of our analysis by considering some details of the thoughts and proposals of two formerly very famous arithmetic educators - namely A. Diesterweg (1790-1866) and E. Hentschel (1804-1875).

- By crystallizing the analysis around intuiton and visualization ("Anschauung") - in this short version we must omit discussing these notions in detail - we have chosen one of the two main categories of the contemporary pedagogical discussion on education and instruction in the elementary school - namely 'intuition' (Anschauung) and 'self-directed activity' (Selbsttätigkeit).

2. Prussia Between 1806/07 and 1872: Its Elementary School System within the Frame of Political and Socio-economic Development and a New Thinking in Pedagogy

After the decline of the "Holy Roman Empire of German Nation" in 1806 and the crushing defeat of Prussia inflicted by Napoleon in 1807 a remarkable period of reforms began in Prussia: Reforms concerning the socio-economic situation, reforms of the organization of the government, reforms of the army, and reforms of the educational system were initiated between 1807 and 1814. The time after the Congress of Vienna (1814/15), however, was dominated by a conservative or even reactionary relapse. Simultaneously, we can observe an economic development in Germany -[in particular in Prussia and clearly becoming visible between about 1845 and 1873]- that has to be considered as the 'German version' of the "Industrial Revolution". This contrast between the development concerning the distribution of political and military power, on the one side, and the economic development, on the other side, marks two simultaneous but different lines of development of the order of the society. Considering our subject matter within the perspective of a social history of mathematics education we have to take into account as an essential part of the underlying frame the following features concerning the relationship between the state and public education in Prussia between 1810 and 1872:

- The system of public education as a factor of modernization and the maintainance of the pre-revolutionary[1] order of society and politics:

On the one side the authorities of the state favoured the modernization of the educational system: Economy as well as administration needed better educated people. On the other side the same authorities strictly tried to avoid the consequences of this modernization for the political power as well as for the order of the society.

- During the 19th century the states in Germany become, in fact, the rulers of the school system:

Increasingly it is the state that fixes the regulations for the different levels of the system of public education. - For the elementary school the authorities of the state in Prussia still accepted - up the 1870s - that this control should be performed in accordance and in coordination with the church.

Considering the different levels of the system of public education in Prussia during the 19th century we have to take into account that the sub-system of the elementary school was strictly separated from the sub-system of higher or advanced education. While the teachers for the Gymnasium had to study at a university, and in 19th century in Prussia these teachers were regarded as scholars (cf. G. Schubring 1991) the elementary school teachers got their training at teacher training colleges which did not not belong to the domain of higher scientific oriented education.

During the transition from the 18th to the 19th century peadgogy in Germany underwent an essential change; the following principles were parts of the anthropological basis of this new orientation of pedagogy - including education and instruction in the elementary school:

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[1] In this context the notion "pre-revolutionary" refers to the French Revolution of 1789 - thus marking those key ideas of the "right order" of society and government of the era of absolutism.

- Principle of self-determination:

In contrast to the pre-revolutionary view a human being should get his/her status in society himself/herself by means of education - and no longer by birth and membership of a certain class and its rank in the "right order" of society.
"It is education that is the main vehicle by the means of which a human being becomes a human being" (F.H.C. Schwarz, 1805).

- Principle of self-education:

All educational efforts can only be considered as stimuli, as challenges for real education, i.e., for education by oneself.
"Al education is only making someone to educate oneself" (K. Weiller,1802).

It is evident that such principles were in rather sharp contrast to those who strictly tried to maintain the pre-revolutionary order of society and political power.

Considering, especially, education in the elementary school one educator is of outstanding importance - the Swiss educator J.H. Pestalozzi (1746-1827):

- According to Pestalozzi also each member of the "just plain folk" has the right of getting educated as a human being, and only such an education as a human being could be the basis for all other forms of education and training concerning the everyday life as a citizen and the life within a professional domain.

- Within this basic orientation Pestalozzi tried to find the "absolute basis of knowledge" - thus, as he was convinced, gaining the "essence of instruction":
"I have fixed the highest, uppermost principle of instruction in the recognition of intuition (Anschauung) as the absolute foundation of all knowledge ... thus trying to find the archetype by means of which the education of our race has to be determined by nature itself" (J.H. Pestalozzi, 1801).

These educators were reflecting on education convinced that they were searching for - and finding - absolute foundations of human cognition by revealing the 'nature', the 'essence' of the human mind - insofar being quite opposite to a modellistic epistemological perspective of nowadays.

3. MAIN EDUCATIONAL OBJECTIVES FOR ARITHMETIC INSTRUCTION According to the Didactical Conceptions of A. Diesterweg (1829) and E. Hentschel (1842)

The main educational objective for arithmetic instruction in the elementary school in the sense of A. Diesterweg is this:

" ... [it is] formal education [allgemeine Bildung], that arithmetic instruction shall aim at everywhere. This latter (main) purpose only can be achieved if the student everywhere in mental arithmetic as well as in written arithmetic (...) is compelled to exert his thinking power.

It is much more challenging, better for education and therefore more important that a problem is treated and solved in six different manners than to give six different problems" (A. Diesterweg 1829).

And as consequences for instruction he considers:
- "The student shall recognize the particular ideas, concepts, laws, rules by intuition and he shall find them intuitively, and by intuition he shall achieve general and abstract ideas" (A.Diesterweg 1829).

- "Everywhere one shall stimulate the student to be self-acting, ..." (A. Diesterweg 1829).

- Instruction has to be well-grounded: To give arguments already on an intuitive level has to be an essential part within the arithmetic learning of the students - also in the elementary school.

The main educational objective for arithmetic instruction in the elementary school in the sense of E. Hentschel is included in the follwing quotation:

"The student shall learn *to do arithmetic by thinking*,
and he/she shall learn *to think by doing arithmetic* - that is one;
besides insight (comprehension) he/she shall acquire that skill, too, that life is demanding - that is the other" (E. Hentschel, 1842; emphasises in the original).

Consequences for instruction:

"The first demands *completeness, intuition, and a wide varietey* of instruction; skill, however, is only to be acquired by frequent, continuous practice (exercises)" (E. Hentschel, 1842; emphasises in the original).

4. INTUITION AND VISUALIZATION AND THE VERY BEGINNING OF ARITHMETIC INSTRUCTION ACCORDING TO THE DIDACTICAL CONCEPTIONS OF A. DIESTERWEG (1829) AND E. HENTSCHEL (1842)

◆ A. Diesterweg (1829): Extracted of the first exercise:

The teacher draws a stroke on the blackboard and speaks: |
"That is *one* stroke."
The students speak: "That is one stroke."

The teacher draws another stroke and speaks: | |
"That is another stroke; one stroke and still another stroke are *two* strokes."
The students speak: "One stroke and one stroke are two strokes."

This has to be continued up to ten. The following survey shall also be fixed on the blackboard:

|
| |
...
| | | | | | | | | |

A. Diesterweg emphasises: "The purpose of this exercise is that the students learn to represent, to think and to name the numbers from one to ten intuitively/in a visualized manner [anschaulich]".

◆ E. Hentschel (1842): Extracted of the first exercise:

The teacher makes developing the basic numbers openly to the students by iteration of the one:

First he draws a stroke on the blackboard |
and makes the students speak: "This is one stroke."

He draws a stroke under the first one and then another one beside it:

|
| | ;

the students speak: "One stroke and one stroke are two strokes."

This has to be continued until:

|

| |

...

| | | | | | | | | | ;

The students end: "Nine strokes and one stroke are ten strokes."

The visualization also should be realized by - e.g. - points, circles, crosses, blocks, beans, balls etc. E. Hentschel emphasizes: "Do not miss a variety - it is important."

After having made acquaintance with the basic numbers - in the cardinal aspect as well as in the ordinal aspect - and after having got to know the digits the continuation of instruction follows this pattern - E. Hentschel is a bit more explicit and consequent than A. Diesterweg:

> visualized mode - mental mode - written mode - applications: using numbers and measures in word problems - conncections with all the other numbers and operations considered up to now.

Addition and subtraction (according to E. Hentschel, 1842): The number 2
(E. Hentschel recommends to use blocks (cubes) or balls.)

▹ *Visualized mode*:

Teacher: How many blocks are there?
Students: Two.

Teacher (moves the blocks apart):

What is this?
Students: Two is one and one.

Teacher (pushes the blocks together):

Students: One and one is two.

Teacher (takes one block away):
What is this?

Students: Two take away one is one.

Teacher (takes the left block away, too):
What is this?
Students: Two take away two is nothing.

▹ *Mental mode*:

Questions of the teacher: How much is one plus one?
How much is two take away one?
How much is two take away two?
How much ist one take away one?

▷ *Application* (here presented in the mental mode)
Charles got a birthday present, and Santa Claus brought another present.
How many presents has he got now?

▷ *Written Mode*
1 + 1 = , 2 - 1 = , 1 - 1 = , 2 - 2 = .

It is a feature of the conception of E. Hentschel, that addition and subtraction as well as multiplication and division are considered simultaneously whereas A. Diesterweg treats one after the other and combines them afterwards. But there he recommends to make the students formulate arguments for the solutions of problems in an amnner preforming logical inferences - e.g., 6+4+-3=7:
Since 6+4=10 and 10-3=7, therfore 6+4-3=7.

In accordance with the Swiss educator J.H. Pestalozzi there was a broad consensus of opinion in arithmetic education that intuition had to be the real foundation of knowledge, and, hence, every teaching-learning process in arithmetic instruction in the elementary school had to start on an intuitive level. Insofar the relevance of intuition in arithmetic instruction in the elementary school was beyond doubt.

J.H. Pestalozzi was convinced that the "equilateral quadrilateral" (square) and the "straight line" - as he called it - were basic objects, and as such basic objects they represented "pure intuition"; they were considered as being free from any other concreteness - thus representing the unit as such within the formation of the number concept. Insofar the "equilateral quadrilateral" (square) and the "straight line" had to be the privileged forms for visualized representations in arithmetic instruction in the elementary school. According to J.H. Pestalozzi working with his "table of units" activated the "power of the intuition of the pure relations of numbers".

This kind of representing whole numbers was introduced by J.H. Pestalozzi in 1803 in the first volume of his "Anschauungslehre der Zahlenverhältnisse" ["Teaching the relations of numbers by intuition"]. Using lists of strokes as a basic visual representations for the first whole numbers by both arithmetic educators - namely, A. Diesterweg and E. Hentschel - is due to J.H. Pestalozzi.

But analyzing the meaning of intution at E. Hentschel (1842) and A. Diesterweg (1829) in more detail reveals that there are indicators for differences; and these differences can be interpreted to have consequences for the teaching-learning process.
The role of intuition / visualization in the didactical conception of E. Hentschel (1842) when learning the first whole numbers can be summarized like this:

◆ The teacher is drawing and manipulating
- graphic representations - like strokes, points, circles, crosses

or
- concrete homogenous materials - like blocks, beans, rods, balls;

and he is speaking
- giving informations - e.g.: There are five strokes -

or
- asking a questions - e.g.: What is this?

◆ The students are perceiving
- the objects - as a result of the teacher's actions (the graphic representations or concrete materials)

and

- the actions of the teacher - e.g.: | | | | | → | | . | | | ;

and they are speaking
  - repeating what the teacher has said - e.g.: There are five strokes;

or
  - answering a teacher's question - e.g.: Five is two and three.

Referring to the Piaget's difference between the figurative aspect and the operative aspect there are indicators so that we can state for E. Hentschel (1842):

- The emphasis is on the figurative aspect of thinking.
- There is no evidence that can be interpreted as an implicit aknowledgement of the importance of the operative aspect of thinking.

For example, E. Hentschel (1842) explictly states:

"The visual representations do not belong to the slates of the students contrary to the problems with digits."

Therefore the lists of strokes as visual representations of whole numbers have to be drawn only on the blackboard.

The situation is different when considering the description of the learning process of A. Diesterweg (1829). At first glance this difference seems to be marginal: In his hints for the teacher for realizing the first exercise he recommends that the students should also use their slates in order to draw lists of strokes. But combining this with a "short psychology" in a handbook for elementary school teachers (1835) this opens the door for quite another perspective of the cognitive functioning being basic here.

The role of intuition / visualization in the didactical conception of A. Diesterweg (1829) when learning the first whole numbers can be summarized like this:

◆ The teacher is drawing and manipulating
  - graphic representations - e.g.: | | | | |
    [concrete homogenous materials are allowed, too]

  and he is speaking
  - giving informations - e.g.: There are five strokes -

  or
  - asking a questions - e.g.: What is this? / How many ...?

◆ The students are perceiving
  - the objects - as a result of the teacher's actions (the graphic representations or concrete materials) - e.g., | | | | | | | -

  and
  - the actions of the teacher - e.g.: | | | | | → | | | | | | | ;

  and they are speaking
  - repeating what the teacher has said - e.g.: There are five strokes;

  or
  - answering a teacher's question - e.g.: There seven strokes now.

But then the students have to act according to oral utterances of the teacher - using their own slates and also putting questions and problems to the others; insofar there is a second phase of visual representation in which the students have to activate their intuition.

A. Diesterweg interprets the human cognitive mecanisms - in particular concept attainment and concept representation - in the light of the epistemology of the famous German philosopher I. Kant: In a form mediated by another German philosopher J. Fries he used the epistemology of I. Kant as it was published in the "Kritik der reinen Vernunft" (Critic of pure reason) as a psychological frame in order to describe and to analyse cognitive processes, as a theoretical background for his view of teaching-learning processes. According to this, a successful concept attainment for - let us say - the whole number five had to comprise these components:

- Numeration step by step of a manifold of homogenous objects -
  here: one (|), and still another one (|), and another one (|),
  and another one (|), and still another one (|);
- synthesis of this manifold -
  here: union of the strokes as a list - | | | | | ;
- idea of this synthesis of a manifold as a new unit -
  here: Thinking of the list of strokes as a new object -

here: Thinking of the list of strokes as a new object -  ;

  the latter comprises the faculty to have at disposal a method to produce a list of strokes or points or signs whatsoever as a picture, as a technical exterior representation of the number five.

This idea of a general procedure to produce a picture for a concept is called the "schema of the concept" within the epistemology of I. Kant; and such a schema is always considered as a product of the spontaneity of the human mind - a schema is rooted in the human mind but not in the objects. Sensory stimuli are necessary because the human imaginative faculty / imagination is only activated on the occasion of sensory perceptions. It is essential for a successful concept attainment that the arithmetic learning human subject

- is becoming conscious of this enumeration of the units, of the synthesis of the manifold of these units, and of the idea of this manifold as a new object

and

- is able to activate the schema to produce a new list of strokes as an exterior representation of a number whenever this is appropriate.

Therefore, using the own slates by the learners gets an essential value!

Referring again to the Piaget's difference between the figurative aspect and the operative aspect we can state for A. Diesterweg (1829):

- The emphasis is on the operative aspect of thinking.
- A. Diesterweg does not know this difference explicitly and so we have to assign him an implicit aknowledgement of the importance of the operative aspect of thinking.
  In a more principal perspective we can compare the epistemological positions of E. Hentschel (1842) and A. Diesterweg (1829) in this way:
- E. Hentschel (1842) is in line with the epistemological tradition founded by Aristotle:
  - Concepts are achieved by abstracting from sensory perceptions.
  - The essence of a concept is already contained in the sensory perceptions - but only latently contained; and it is the turn of the activity of the human mind to remove what is only accidental.
  - Constructive activity of the learner consists of the activity to abstract the concept from the perceived entities - here: The concept of whole number. Getting conceptual knowledge is due to active mapping on the side of the learner.

A. Diesterweg (1829) is in line with the epistemology of I. Kant:

- Concepts are achieved by constructing of the human mind.

  The basic conceptual components are within the human mind as a cognitive structure a priori, and concepts are revealed if the mind becomes conscious of what it is doing on the occasion of sensory perceptions - or remembrances of them. Sensory perceptions are shaped by the apriori forms of space and time; and, in particular, the apriori form of time is basic for the concept of whole number.

  - Constructive activity of the learner consists of the activity of his/her mind by activating cognitive schemas that mediate between the aprirori forms and categories and the sensory perceptions. Getting conceptual knowledge, therefore, is due to the cognitive construction of the learner - the sensory material is not considered as the real cause for concepts but only as the occasion.

Both epistemological positions have in common that imagery is in a privileged position: Every knowledge must have passed in its genesis the imagery and its forms of internal representation. Modern trials of theorizing the role and the functioning of imagery within the scientific paradigm of cognitive psychology do not make the assumption that all thought processes involve imagery or that imagery is in a privileged position as a form of internal representation.

5. Can We Learn From such an Epistemologic-Historical Analysis ?

Despite the epistemological differences there are common structural features of the basis of learning theory of A. Diesterweg and E. Hentschel:

- Intuition as an anthropological feature

A. Diesterweg and E. Hentschel - as other progressive contemporaries - were convinced that

- intuition was a property of the human mind because intuition was a part of the "nature", of the "essence" of man;
- intuition was a well-defined property of the human mind.

- Privileged position of intuition in cognition

  Intuition was considered as being principally involved in every knowledge that claimed to be true knowledge.

This universal and anthropological anchoring of intuition had - historically speaking - two progressive functions:

- This anchoring served as the basis for two statements:

  - Education - as general or formal education - has to be available even for all members of the 'just plain folk'.

  - Education - as general or formal education - is - at least for the first steps - also attainable for all members of the 'just plain folk'.

- In this perspective instruction in the elementary school got a certain 'pedagogic dignity':

  Instruction in the elementary school on the basis of intuition - and self-directed activity - could be considered as an essential contribution to education and not only as a training of some useful skills to cope with everyday life and professional situations. (But we should not have an too idealistic idea of the reality in the classroom: In 1835 in the Rhineland, e.g., one teacher hat to teach 91 students of all age groups! And we should keep in mind that the elementary school teachers were not well paid.)

On the other side this universal and anthropological anchoring contains the danger of a dogmatic stabilization in mathematics instruction, namley concerning the relation between embodiments and graphic representations on the one side and mathematical notions and structures on the other side:

- According to the 'mapping perspective' of the Aristotelian tradition one can be inclined to consider embodiments and graphic representations as carriers of unequivocal meanings -thus establishing a one-to-one-correspondance between the embodiments and praphic representations on the one side and the mathematical notions and structures on the other side.

- According to the 'constructing perspective' of the Kantian tradition one can be inclined to assign unequivoval meanings, too, to the embodiments and graphic representations - but here referring to the arguemnt that the "nature of human cognition" constructs the same ideas and notions on the occasion of certain exterior representations.

Meanwhile, however, we have sufficient empirical evidence that we cannot suppose that students - in particular in the primary school - get the same ideas/images in their minds when looking at or acting with the same embodiments or graphic representations (cf. J.H. Lorenz 1991). Also findings in the field of neurophysiology can be summarized that even the processes of visual perception are essentially constructive (cf. G. Roth 1992).

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Prof. Dr. Siegbert Schmidt
Seminar für Mathematik und ihre Didaktik
Universität zu Köln
Gronewaldstraße 2
D - 50 931 K ö l n
Germany