Using Ancient Astronomy to Teach Trigonometry: A Case Study

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Introduction

Trigonometry, a subject of immense utility and beauty, is nevertheless difficult to convey to students in a form that allows them to be practitioners; that is, masters rather than servants. The definition of the sine and other trigonometric functions is easily followed, but once the student returns from the picture to the universe of mathematical symbols, perception vanishes and is replaced by a vague collection of algorithms for symbolic manipulation inherited from past experience in mathematics courses. A number of my beginning college-level calculus students, for instance, are proud to have solved the equation $\sin\theta + \cos\theta = 1.3$ for θ as follows:

$$\theta(\sin + \cos) = 1.3$$

$$\theta = \frac{1.3}{\sin + \cos}$$

Often these are students who perfectly well grasped the geometric content of the trigonometric functions. Their leap into symbolic mathematics has been accompanied by a jettisoning of conceptual awareness. Hence their ability to use trigonometry in new situations is severely hampered or completely curtailed.

As a beginning teacher, my initial reaction was to join my colleagues and blame the secondary mathematics program, which in my locality gears its students to algorithmic performance. The syllabi are so large that many teachers are forced to spend their time training students in drill problems designed for performance on similarly-structured exams. It seems to me after several years' experience that this criticism applies equally well to many North American, and possibly European introductory college-level mathematics programs. While our students may perform well on technical examinations soon after the class ends, they are not learning mathematics. Particularly, they lack a grasp of:

- **Why** the subject exists. Mathematics in history often arises (directly or indirectly) from an enquiry in some other discipline. Many students regard mathematics as pure algorithmic training, internally justified, and thus find little motivation. (I intentionally do not refer to "applications" of mathematics, since that implies that the theoretical edifice was created independently of the application.)

- **What** the mathematics means. An algorithmic approach enforces the students' belief that life in a mathematics class consists of mimicking a mechanical symbol-manipulating device. Surprisingly little grasp of the geometric or quantitative meanings of the symbols is retained in the long term, and I have found that students strongly resist having to change their conceptual base to break the barrier between thinking in math class and thinking in real life. They have, after all, survived so far with this distinction.

- **How** to ask mathematical questions, and how to pursue an answer. We are often frustrated that students display little originality and few exploratory instincts. Of course, it is unrealistic to expect them to have this ability if they have not been encouraged to "go beyond the paper" in their understanding.

It is a common refrain that history can provide a fertile ground to address these problems, and with some reason. Presenting the early development of a subject within its cultural and scientific context can give rise to the original motivations in a natural way. Through this, the role of the subject and its importance can become clear. If the historical goal of a subject can be made to be the students' goal in an historical project, they take ownership of the task and can participate in a clearly defined, and extra-mathematically important, achievement.

The presentation of history, however, must be planned carefully to achieve these objectives. If implemented without care, historically-based teaching may not solve, and could even lead to, many of the unfortunate effects above. My historical materials for the classroom are developed with several points in mind. Firstly, the linguistic style is quite casual. This helps to overcome a conception that mathematics is performed only by those predisposed to algorithmic thought, and begins to break down the barrier between mathematical and other ways of thinking. Secondly, the enthusiasm and excitement for the subject comes through clearly (partially aided by the casual tone). Enthusiasm is an infectious disease that we should not fear spreading to our students! Thirdly, a clearly defined goal determined by historical need provides relevance, interest, and a desire to pursue the project to its conclusion. Finally, a constant movement back and forth between from the geometric to the symbolic to the geometric blurs the false dichotomy between these two ways of understanding. Once this conceptual barrier has been overcome, the initiative to pose questions and to find one's own solutions comes naturally. After all, many mathematical disciplines arise from asking the "obvious" questions!

Trigonometry is one of my students' most feared and least understood topics. It arises in introductory calculus to an inevitable shared and terrified hush. Although the students have seen it in more than one previous course and can wield identities with some ability, a little discussion reveals that they know little of the content of an identity beyond that it has "something to do with angles". More than half at some point make the fundamental mistake of treating the functions and arguments interchangeably, producing for example the curious solution to the trigonometric equation at the beginning of this paper. Clearly the connection between the geometry and the algebra does not exist for them in mathematical practice.

The early history of trigonometry presents a nice case study for use in the classroom. It arose directly from problems in astronomy, and in fact remained a subdiscipline of astronomy for about 1000 years. The required background knowledge in astronomy is minimal, and comes naturally to a student trying to make sense of the motions s/he sees in the sky. The results can be rewarding: with only a little additional help, at the end of the study the student will be able to predict the Sun's location on any day, and with a set of lunar positions can determine whether an eclipse will occur at a certain time. This was one of the historical motivations for trigonometry, and provides the climax for the case study. Additionally, the historical abandonment of the chord in favour of the sine becomes apparent in a very practical way. The mathematical theory that emerges from the students' explorations corresponds to the historical discovery of many of the basic trigonometric identities and functions. It is presented here in the spirit of Ptolemy's *Almagest.*

My materials are still a work in progress, and may be handled quite roughly. They may be reorganized to a certain extent, sections may be omitted without much difficulty, and modern functions replaced with the ancient ones and vice versa. I have decided to use the ancient chord function instead of the modern sine, because it is the obvious function to choose given the problems Hipparchus and Ptolemy faced, and the theory behind the chord table is more natural with it, but I have made a concession to simplicity by using a base circle of 1 unit rather than the ancient 60, since it is an unnecessary and unilluminating complication in the classroom. For a similar reason I use decimal arithmetic rather than the astronomers' base 60.

The original intended audience for this presentation was a mathematics class for 17–18 year olds working in small groups. Occasionally throughout I ask one or more questions in a shaded box, with a difficulty level assigned to it. I suggest the students to explore these on their own, or better in groups of two to four. I have asked most of the questions marked Easy or Medium to my classes with success; the harder questions are for those more accustomed to geometrical or other theorems and proofs. Some may be omitted entirely or dealt with in the classroom.

An excerpt from the handout is included below. The entire paper is about 20 pages long and too extensive to present here in its entirety. I will be delighted to provide copies of the complete work to those interested; please write me at the listed address.

Excerpt of the Presentation

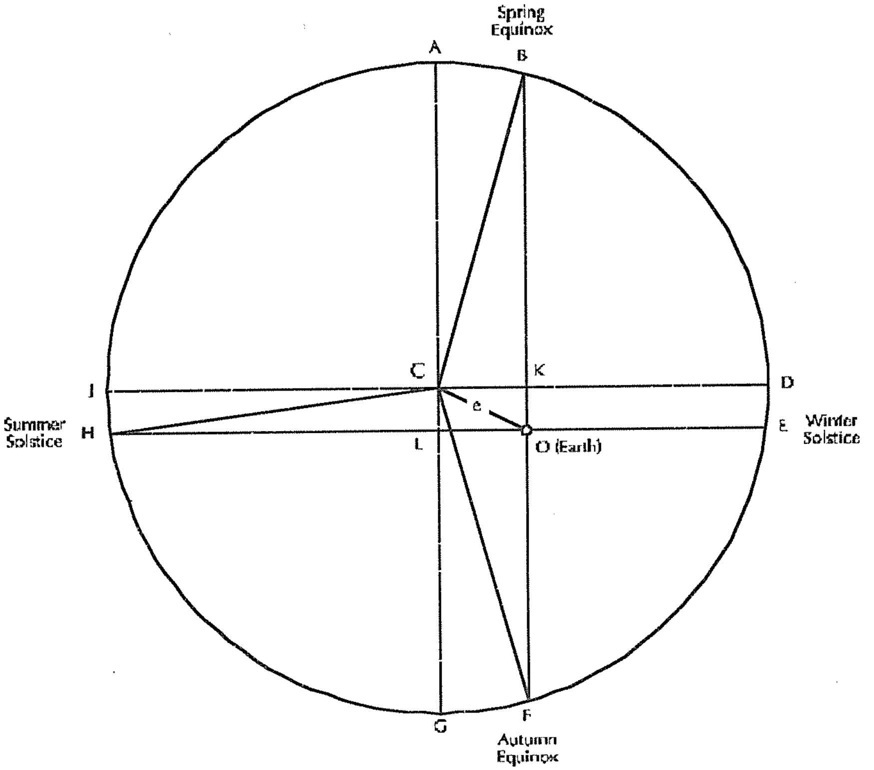
Sundance: Greek Astronomy and the Birth of Trigonometry

The handout, which can be given to a group of students as independent study or presented in class, begins with a description of the obvious patterns of motion observable in the sky. From this, the need to understand the paths of the Sun and Moon for agricultural, astrological and meteorological purposes arises. This leads to Hipparchus' desire to model the motions of the Sun and Moon, which could then also be used to predict eclipses by determination of the times when they occupy the same, or diametrically opposite, positions. The basic solar model, a circular orbit around the Earth at its centre, is slightly altered to account for the Sun's apparent variable speed. This leads naturally to a need to measure lengths in a circle where only angles are given. This excerpt picks up after a discussion of how Hipparchus might have constructed a trigonometric table to find these lengths.

The Final Solar Model and Eclipses

With a table of chords, Hipparchus could go ahead and find the precise values of the eccentricity and the angle at which to place the centre of the Sun's orbit, which in turn would allow him to tell where the Sun would be at any time. We will follow Hipparchus' journey, assuming now that we have a chord table and can find the chord of any angle. The diagram below is the same as our earlier model of the Sun's motion except that I've added some letters to indicate points and some extra lines we'll need. You'll remember we worked out earlier that the arc of the circle for the summer is 91.172°; from the picture this is –HCF. We can use the same reasoning to find the angle for the spring:

$$\angle BCH = 0.98564° / day \times 94\tfrac{1}{2}\, days = 93.143°.$$



Question 6 Hard	*From these facts, can you work out the angles <JCH and <ACB?*

Question 6 (alternative) Medium	*Which angles in the picture correspond to the amount that the spring arc is greater than 90º? How many degrees is this sum? Repeat this question for the summer, replacing the word "sum" by "difference". Now, use the two pieces of information you have to find the values of each of the angles that makes up the sum and difference.*

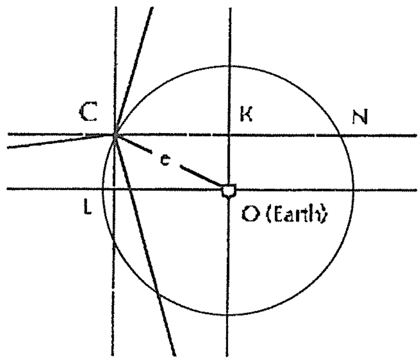
So far we haven't needed any chords, because all we were handling were arcs, not lengths. To find the eccentricity $e = OC$ we're going to need some! Hipparchus' strategy relied on using one of the right-angled triangles near the middle of the diagram, either OCL or OCK. For either triangle, the hypotenuse is the length we're after. Hipparchus had the good fortune to live well after Pythagoras (although the Pythagorean Theorem was also known long before even Pythagoras!), so he was able to use the Pythagorean Theorem, which for the triangle OCL says that $OL^2 + CL^2 = OC^2$. Thus, our problem is now to find both *OL* and *CL*.

Question 7 Medium	*Remember that the angle <JCH is 0.985º. From this, and assuming you can work out any chord from your chord table, how could you find out the length CL?*

You should find that *CL*=0.0172 units. To get *OL* you can use the same process, but with the angle <ACB instead, and you'll find that *OL*=0.0377 units. Then we turn to Pythagoras and we find:

$$e = \sqrt{CL^2 + OL^2} = 0.0414 \text{ units.}$$

We're almost there! Now we need to know the angle < KOC, which will tell us the direction in which the centre of the circle will be. This is a little trickier, so I'll draw part of the picture of the solar model again with an extra circle added, as a clue.



Question 8 **Medium**	*Find the size of <KOC. You'll need the chord table, but you'll need to use it backward: you'll know the length of a chord and you'll want to use the table to find the angle that has that value for the chord length.*

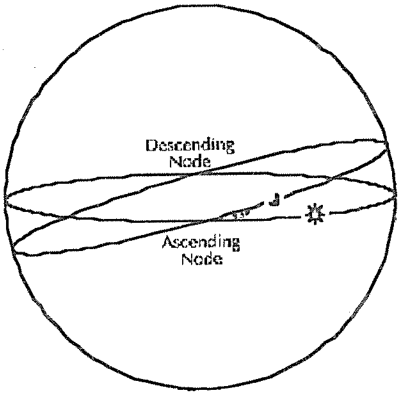
The solution to Question 8, using the small circle drawn around the right-angled triangle, is similar to the common practice in ancient trigonometry. There is a much simpler solution using the modern sine function. If you know the sine, see if you can find it. In fact, many of the methods you see here have simpler solutions using the sine instead of the chord. That's why the sine is used today, and the chord has all but disappeared.

So, now we know that the direction of the centre of the Sun's circle is 65.5º west of north, so to speak, on our picture. That means that the Sun is furthest away from the Earth (its **apogee**) when it is in this direction from the Earth as well, which is in late spring. We want to be able to find out where the Sun is on particular days and particular times, so let's find out what day of the year the Sun is at its apogee as an example for calculations. We know that the Sun is at the point *B* on March 21, the spring equinox, and that it moves 0.98564º/day.

Question 9 Easy	*Find out what day the Sun arrives at its apogee, using the information above.*

We're now able, with a little chord work, to find out where the Sun is on its orbit circle at any day of the year, just by remembering the crucial 0.98564º/day. That takes care of the most important of the seven wandering stars. Hipparchus moved next to the Moon, and either he, Ptolemy, or someone between them dealt with the planets as well. We don't precisely know how Hipparchus dealt with these other objects, but we do know what Ptolemy did, and it gets very complicated, although it's very accurate. Ptolemy used Hipparchus' model for the Sun, though: its accuracy was good enough for even Ptolemy's standards, and was certainly good enough to be quite reliable for predicting eclipses.

We will not go through Ptolemy's model for the Moon, but let's look at enough details to understand the way to predict eclipses. The motion is a complicated combination of circles, but it all happens on a circle that is tilted 5º from the ecliptic (see below). The circle itself moves around the celestial sphere, so that the places where the Moon's circle and the Sun's circle intersect (the **ascending and descending nodes**) move slowly along the ecliptic.



Question 10 Medium	*Why would it be important for the purpose of eclipse prediction to know how close the Moon is to the ascending and descending nodes?*

Let's suppose know that we know where the Moon will be on certain days of the year (I'll give the information to you). From this, we know how to find where the Sun will be, and if the conditions are right, we'll have a solar eclipse!

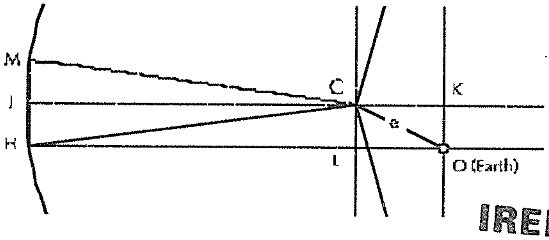
Solutions to Questions

6. Since the spring angle is <BCH=93.143º and <ACJ=90º, we know that the sum of the two smaller angles is $90° + \angle ACB + \angle JCH = 93.143°$. The summer angle, similarly, is $90° + \angle FCG - \angle JCH = 91.172°$. Adding these two equations together, we find

$$180° + 2(\angle ACB) = 184.315°,$$

which gives <ACB=2.158º. From this number we use the spring angle sum equation above to get <JCB=0.985º.

7. Draw a line straight up from H, as below, until you reach the circle on the other side of J. That point is M on the diagram below. Then the segment HM is the chord of *twice* <JCH, so we can find the length of HM by using the chord table to evaluate $\text{Crd}(2 \times 0.985°) = \text{Crd}(1.97°)$, which is 0.0344 units. But HM is twice CL, so that CL=0.0172 units. (Alternatively, using modern trigonometry, $CL = \sin \angle CHL = \sin \angle JCH$.)



8. The line *CK* is half of the chord (*CN*) of the angle <*CON*. But <*CON* is twice the angle we want, <*KOC*. So, since *CK*=*OL*=0.0377 units, we know that *CN*=0.0744 units. The next step would seem to be to look up the angle corresponding to this in a chord table, but remember that the table gives chords assuming that the radius of the circle is 1 unit, and our circle's radius is *e*=*OC*=0.0414 units, which is much smaller. The trick is to set up a new measuring system temporarily, where the radius *e* is 1 Unit long. In these new Units, we have

$$CN = 0.0744\,units \cdot \left(\frac{1\,Unit}{0.0414\,units}\right) = 1.797\,Units.$$

Now our chord table can tell us that this number of Units corresponds to an angle of 130.9º. The angle we want is half of this, so now <*KOC*=65.5º.

9. The angle from *A* to the apogee is 65.5º, as we just worked out. From our previous work, we know that <*ACB*=2.158º, so the Sun needs to travel about 67.7º after March 21 to make it to the apogee. At 0.98564º/day, this is 67.7/0.98564≈69 days. That makes it May 29.

10. The Moon and the Sun are small enough so that if the Moon is far enough away from the nodes, the Sun cannot possibly be close enough for the Moon to overlap it. And that, after all, is precisely what a solar eclipse is: the overlapping of the Moon in front of the Sun.

The remainder of the paper carries the project to its goal of eclipse prediction using Hipparchus' model of solar motion and basic knowledge of the celestial sphere. The Ptolemaic model of lunar motion is a technically difficult additional factor, and data for lunar motion are provided to bypass this complication. My pedagogical aims would not be better fulfilled by including a complete theory here.