REVERSING THE CUSTOMARY DEDUCTIVE TEACHING OF MATHEMATICS BY USING ITS HISTORY: THE CASE OF ABSTRACT ALGEBRAIC CONCEPTS

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The customary approach to the teaching of mathematics is deductive. In this way however, the motivation for introducing new mathematical concepts, theorems or theories is hidden, so that a sufficiently deep understanding is difficult to be acquired. At the same time, the interelation between very different at a first glance, mathematical concepts or domains, is not easily revealed.

On the other hand, following the **basic steps in the historical development** of a certain mathematical subject and presenting questions and problems that have served as prototypes in this development, the subject can be taught with sufficient explanation of the motivations for introducing new concepts or axioms. Thus the interelation between different mathematical concepts or theories can be presented with clarity and the student can be given hints for further study of more advanced results that fall outside the scope of a course on the particular subject considered.

Such an approach may be called **genetic** in the sense that the presentation of a subject is neither historical nor strictly deductive but is such that one investigates a problem only after having been led to it in a **natural** way. Moreover the emphasis is less on **how to use** theories methods and concepts and more on **why** these theories methods and concepts can provide an answer to mathematical problems.

As an example of this approach, we consider the presentation of the interelation of the following concepts: (i) complex number, (ii) rotation group, (iii) isomorphism and homomorphism of abstract algebraic stuctures, as it can be taught

- at an **elementary** level in a secondary school mathematics course
- at a more **advanced** level in an undergraduate university course.

More specifically, we try to present aspects of the important developments in the 19th century algebra, that had been motivated by the geometrical interpretation of complex numbers. As already mentioned, **the presentation is not strictly historical but it is inspired by its basic steps**. Moreover, unecessary complications are avoided by using concepts and methods that emerged partly as a consequence of these developments.

From the analysis of such examples follows that the general procedure to be followed in a genetic approach is based on

(i) a general knowledge of the subject's history,

(ii) the determination of the key steps of the historical evolution,

(iii) their presentation given possibly in a modern formulation, but certainly inspired by the historical process,

(iv) the formulation of many details in sequences of exercises of increasing level of difficulty, so that each one presupposes (some of) the preceding ones. In this way any course or textbook following a genetic approach can be kept to a reasonable size.
Among the many advantages of such an approach belong

(i) the natural presentation of the subject, in which the logical gaps are kept to a minimum

(ii) problem solving becomes an essential ingredient of the presentation indispensable for a complete understanding

(iii) many correlations with other subjects are revealed. Therefore it becomes possible to understand the unity of a mathematical domain.
As far as our example is concerned, it can be convincingly argued on the basis of classroom observations, that in a deductive teaching, students have severe difficulties to grasp the meaning of **abstract** algebraic concepts like group, algebraic field, vector space, isomorphism of abstract structures etc. The basic reasons for this are that

(i) such concepts are introduced in their most general sense right from the beginning

(ii) usually students have no experience of **many mathematically relevant concrete examples**.

On the other hand, looking at the historical evolution of such concepts, it may be asserted that each one was introduced and established when
(i) concrete problems had already appeared to which this concept offered a (usually not easily substituted) solution, e.g. the concept of a group and the problem of solving by radicals the n-th degree polynomial equation
(ii) it became necessary to use sets having the same algebraic structure but the elements of which are of a totally different nature, e.g. both vectors in analytic geometry and solutions of linear ordinary differential equations can be considered as vectors of an abstract finite dimensional vector space.

Therefore, history teaches us that abstract algebraic concepts must first be introduced through mathematically relevant (i.e. non-artificial) concrete examples.

In our case study we have examined how complex numbers became a legitimate mathematical object through their geometrical interpretation in the early 19th century. This involves their trigonometric representation and the accompanying association with rotations of the Euclidean plane, based on de Moivre's theorem.

More specifically and in a modern terminology, this geometric interpretation can be described by saying that complex numbers of unit length form a group under multiplication,which is isomorphic to the group of plane rotations. Moreover, similarity transformations in the plane taken together with rotations generate an algebraic field isomorphic to the field of complex numbers.This fact played an important role in the evolution of algebra in the 19th century, naturally leading to the question:
"Is it possible to generalize complex numbers in such a way so that a similar relation to space rotations and similarities exist?" In terms of an analysis of the relation of complex numbers to plane rotations this question can be formulated as follows:" Is it possible to find a field, which extends the complex number field, so that its multiplication group is isomorphic to the group generated by rotations and similarities in space?".

This question lies behind Hamilton's main motivations in his research that led to the discovery of quaternions. In a genetic approach we show how one can avoid Hamilton's complicated and zig-zag procedure, by using the (historically antecedent) concept of a matrix and arrive at quaternions in a simple and lucid way.

Finally this relation of quaternions to space rotations can be formulated in a compact and elegant form, as was first done by Cayley, a fact puzzling Cayley very much, thus motivating him to look for a deeper interpretation. Using the concept of a matrix, we can show that Cayley's exploitation of an idea of F.Klein can be presented in an elementary way, leading to the homomorphism of the group of space rotations to the group of the 2X2 unitary matrices (or "complex plane rotations"!). At the same time this reveals the intimate topological relation of the two groups, which seen from a modern viewpoint, is the basis of the puzzling relation of quaternions to space rotations, refered to above.

In our approach many details have been given as interesting and illuminating exercises which may motivate the student (i) to study the subject further (ii) to study other more advanced results, for which hints have been given through the genetic presentation of our original subject.

To summarize what has been said, we may say that our aim is the description in detail of a concrete example of how the customary presentation of mathematics can be reversed by using its history as an essential ingredient.