Volume Calculations in the Manner of 17th Century

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1. Archimedes

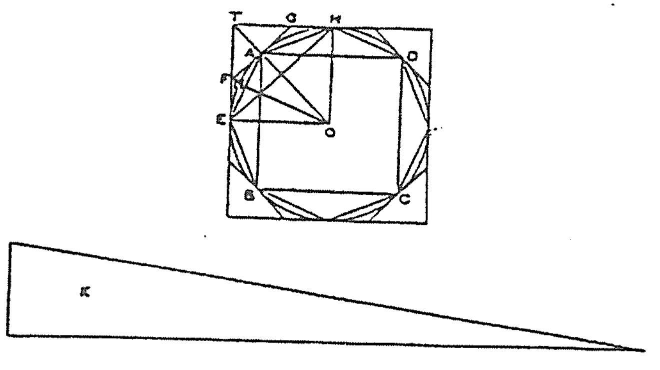
Searching for the areas of simple curvilinear figures ancient Greeks used the method of exhaustion invented by Eudoxus. The gist of the method is the so-called Eudoxus principle (Euclid X.1 ; Edwards, 1979) :

Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half,and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Eudoxus determined the area of a curvilinear figure in terms of a sequence of polygonal figures that exhaust the evaluated area, and the principle helped him to cope with this sequence where the concept of infinity intervenes.

Archimedes extended the method of exhaustion to that of compression. Unlike Eudoxus, having dealt only with inscribed polygons, Archimedes employs both inscribed and circumscribed polygons. Then the unknown area is compressed between the areas of inscribed and circumscribed polygonal figures. We illustrate Archimedes' approach by his proof of Proposition 1 from the Measurement of a Circle :

The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference of the circle.



Proof : Let ABCD be the given circle. K the triangle described. Then, if the circle is not equal to K, it must be either greater or less.

I. If possible, let the circle be greater than K.

Inscribe a square ABCD, bisect the arcs AB, BC, CD, DA, then bisect (if necessary) the halves, and so on, until the sides of the inscribed polygon whose angular points are the points of division subtend segments whose sum is less than the excess of the area of the circle ovec K.

Thus the area of the polygon is greater than K.

Let AE be any side of it, and ON the perpendicular on AE from the centre O.

Then ON is less than the radius of the circle and therefore less than one of the sides about the right angle in K. Also the perimeter of the polygon is less than circumference of the circle, i.e. less than the other side about the right angle in K.

Therefore the area of the polygon is less than K ; which is inconsistent with the hypothesis.

Thus the area of the circle is not greater than K.

Then Archimedes proved in a similar way that the area of the circle is not less than K. From both parts of the proof he concluded that, the area of a circle is equal to the area of the triangle K.

This shows that Archimedes' considerations meet the usual criteria of mathematical proof today. (If we neglect for example things like not consequent distinguishing between the area of a figure and the figure itself). Archimedes applied the method of exhaustion to a member of rather sophisticated cases and he did so with great accuracy typical of him, which may have been the reason, why the further progress of calculus was held up. As we assume, that further development could only proceed either by surpassing Archimedes, or, new ideas would have to arise. And since nobody surpassed Archimedes, the waiting for a new idea took place.

2.Kepler

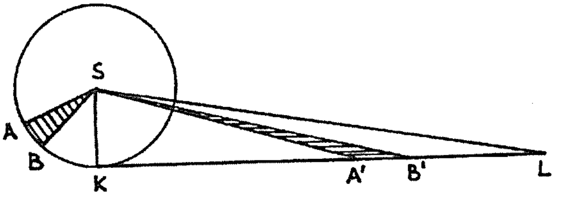
Johannes Kepler was born on 27th December 1571 in the small German town of Weil der Stadt. He lived at a time when astronomers were making attempts to work out a model of the solar system. And he was one of those who dedicated their whole lives to this mystery.

The essential part of Keplers discoveries was made at the time of his stay in Prague (1600-1612), where he was invited by Tycho Brahe, the famous royal astronomer of emperor Rudolf II. This Danish astronomer had a passion for sophisticated instruments and used them to make measurements of the sky which were very good and very accurate by the standards of the period. And it was precisely the results of the observation Brahe had collected that were invaluable to Kepler, for he himself had never been a good observer.

On the account of Tycho Brahe's twenty years of observations of the sky Kepler calculated the orbit of Mars. To his astonishment he found out that it is not a perfect, divine movement along a circle, but that the course is an ellipse with the sun in one of its foci. He also found that this rule was valid for other planets as well - and today we speak about the first of Kepler's laws of planetary motion. Then this exceedingly diligent arithmetician arrived upon the second law, according to which the area of the figure created during period t by the segment sun - planet (the sun is understood as a fixed point) is ct, where c is constant. Both rules were published in the year 1609.

Kepler arrived at these discoveries through an enormous amount of astronomic calculations. And just these, for us unimaginable calculations gave Kepler not only a great arithmetic routine, but trained his feeling for estimations, for tables of figures which approximate the continuous motions and for interpolation and extrapolation. He learned to see continuous movement through the sequences of numerical values.

Kepler excellently used these abilities in a non astronomical field, in geometry. Let us present his way of evaluation of the area of a circle (Kepler, 1987).



Kepler wrote : "*I divide the circumference into as many parts as there are points on it, which is an infinite mumber. We consider each of them to be the base of an isosceles triangle with altitude r*". I will present the further Kepler's considerations slightly condensed : If we unfold the circumference into the line segment KL then the (infinitely small) arc AB will be unfolded into the line segment A'B' and the segment ABS will be transformed into the triangle A'B'C' of the same area. All of these triangles will form the right-angled triangle KLS with the area KL.KS/2 equal to the area of the circle. It is $S = \frac{1}{2}.2\pi r.r = \pi r^2$ in our symbols.

3.Remarks to both approaches

Archimedes gave the rigorous proof for the area of a circle in the manner which he applied to a wide range of problems. The proof is a typical reductio ad absurdum argument. The assumptions that the area of the circle is greater, or less than the area of the given triangle, entail a contradiction and therefore the areas must equal. This approach can be applied generally, but requires one thing : One has to know the result to be proved. In the case of a circle it was not a problem, because the formula has been known before Archimedes, but for other evaluations Archimedes used the technique which he called "mechanical method". To this, rather sophisticated, method (Edwards, 1979, Bero, 1989) Archimedes attributes the discovery on heuristic grounds of many of his results, prior to providing them with rigorous proofs. But Archimedes did never mention his method when providing the proofs and his treatise entitled The Method was only discovered in 1906. Why so ? Archimedes shared the Greek "horror of the infinite" and the mechanical method required for example to consider plane figures as consisting of a set of line segments, which he could not "afford" at all. The motivation of the origin of the method of exhaustion stemmed, in our opinion, from the effort to avoid the concept of infinity. Here, Eudoxus replaced an infinite process, by a finite process which could be repeated infinitely.

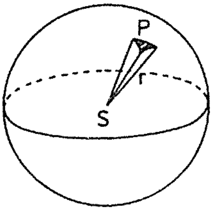
On the contrary, the awe of the infinity hab been long forgotten in Kepler's time. It was to the great merit of scholastic philosophers who had finally overcome the Greek "horror of the infinite" and had prepared a fruitful foundation for Kepler.

Kepler's evaluation, as opposed to that of Archimedes, is hardly acceptable as a mathematical proof. (On the other hand, Kepler's approach provides us with a very good insight into the crux of the formula for the area of a circle, which cannot be said about Archimedes's proof at all). Nevertheless, this is not the reason for its underestimation it. Not in the least. Kepler was completely aware of this and simply said : it's as simple as that, and if you don't like it, you can do it in the way Archimedes had done. Kepler was familiar with Archimedes' ideas and he even described his evaluation of the area of a circle as the idea which lay behind Archimedes's proof.

Whatever the origin, Kepler developed the idea into a special method of his own. This was published in 1615 when he issued his book Nova stereometria doliorum vinariorum (New Stereometry of Wine Barrels), and this only mathematical book of his represents an important milestone in the development of calculus. He presented here almost 90 evaluations of the volumes of the solids of revolution.

4. Kepler's method

What do we refer to as Kepler's method ? It might make the point clearer if we illustrate this in terms of a particular task. For example, the volume of sphere. Kepler imagined the sphere to be decomposed into large number of pyramids, each having its vertex at the centre and its base on the surface of the sphere, with the height equal to the radius of the sphere. Slightly changing the shape of these pyramids and arranging them together, he got a new, large pyramid with the base and the height equal to the surface area and to the radius of the sphere respectively. This gives $V = 1/3.S.r = 4\ \pi r^3/3$ in modern-days terms.



Generally, Kepler dissected a given geometrical figure into an infinite number of very small particles, which after "suitable reshaping" he put together in an "advantageous fashion" to get another geometrical figure, the volume or area of which was known to him. As the fundamental priciple he assumed that the volume and area of the former must equal the volume and area of the latter.

Importunately arising question for us is the question of the nature of these "very small particles". We can only hypothesize Kepler's point of veiw here, because he did not account for the issue.

Edwards (1979) thinks that Kepler regarded these "small particles" to be infinitesimal figures. We would argue that Kepler did not think so. For example, when he derived the formula for the volume of a torus by dissecting the torus into infinitely many thin slices, he distinguished the thickness of slices at the opposite sides of them (Kepler). This would not have been necessary if he had taken them as infinitesimals.

Kepler's method seems to be, and is in fact, very universal, but there are at least two stumbling-blocks. Both the suitable reshaping of particles and putting the particles together in the appropriate fashion require a first-rate, refined intuition. And this is, what he had reaped when doing his vast astronomical calculations we mentioned above. And this is what we, the owners of tables, calculators and computers, lack (Bero, 1993).

5. Cavalieri

Another step in the history was taken by Bonaventura Cavalieri. His method was based on the principle that is known as Cavalieri's theorem (Edwards, 1979) :

If two solids have equal altitudes, and if sections made by planes parallel to the bases and at equal distances from them are always in a given ratio, then the volumes of the solids are also in this ratio.

While Kepler had aimed at getting results, afforded himself an almost "free play" with infinity, leaving a logical precision to those who cannot forget of Archimedes, Cavalieri attempted an infinitesimal technique with a better developed logical base. He considered geometrical figures created by the use of indivisibles whose nature, however, still remains unclear. Sometimes he considered geometrical figures as formed by the movement of an indivisible, and sometimes just as if consisting of an indefinitely large number of indivisibles.

Cavalieri's indivisibles were always of lower dimension than the geometrical figure formed by them. Here he differs significantly from Kepler, whose indivisibles are of the same dimension as the figure created by them.

6. Teaching

The methods described above are of great importance for the historical development of calculus. But can they also be of any significance for the teaching process ? We can hardly think of Archimedes's method of compression in these terms, but to us Kepler's method seems to be simple enough for the pupils at secondary level. We believe that considerations of this kind can serve as very useful propedeutics for the calculus. Experimental teaching in this fashion, as described in Bero, 1993, provides children with a feeling for limit processes and builds their intuition with respect to infinity in general, which are the abilities that are so indispensable for understanding the calculus. It was rather interesting, from theoretical point of view, how easily pupils adopted Kepler's way of thinking, and how naturally they turned to that of Cavalieri whenever it was advantageous (without having any knowledge his method). The results showed that for children both Kepler's and Cavalieri's methods are very challenging and inspiring.

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