THE HISTORICAL CONSTRUCTION OF MATHEMATICAL KNOWLEDGE

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Is the historical reconstruction of mathematical knowledge possible ?

1. Learning Mathematics, Teaching Mathematics.

The paradigm still largely current in the teaching of mathematics, and criticised by Lakatos (1961), Dawson (1969), myself (1976) and others is of a Formalist methodology ; starting with a list *of definitions and axioms*, which are produced without much motivation or explanation and are artificial and complex. There are followed by carefully worded *lemmas* ant *theorems*, and for each theorem, a *proof.*

Students are still required to follow this path, and even though their questions may officialy be encouraged, these questions soon get lost in the intricacies and technical details of the exposition, and can often be dismissed as a lack of "mathematical maturity".

This may be regarded as a caricature of mathematics teaching, but the standard presentations often appear virtually meaningless to students who have no idea of either the teacher's reasons for such an approach, nor of the motivations of the mathematician and the context of the original problem-situation. Much of mathematics is still presented as an "end - product" and a "tool - bag" of techniques to be applied.

It is clamed that this deductive style is the essence of mathematics, and that while discovery and creation of new ideas may be mentioned in passing, these ideas are rarely put into context or explained, so that the presentation of any new ideas must be formulated in this "rigouros" manner to the student. (1)

Thirty years has passed since Lakatos'first critical attack, and still many institutions remain impervious to change. The tradition of the mathematically gifted is difficult to break, and the Formalist-deductive defences are strong. So much so, that while some changes have taken place at primary and secondary school and college level, these changes have been gained only after a great deal of effort, and they require a considerable amount of energy to maintain. (2)

The general popular conception of mathematics is still as it was.

2. Mathematical Activity

Even a cursory glance at attempts to define "Mathematics" show they are dependent on time, place, context and culture. Some classic definitions now appear entirely out of place, particularly since the rise of abstract approaches to mathematics and the broad generalisations developed since the latter part of the nineteenth century. Mathematics itself has not only developed as a subject area, but has taken on new roles and new practices.

The Intuitionists philosophy, Lakatos and later the ATM in England (3) began to describe mathematics as an aspect of *human activity*, thus avoiding some of the earlier pitfalls, but creating others. It is a truism to say, "Mathematics is what mathematicians do", but who the "mathematicians" are is not is clearly problematic.

This emphasis on the activity, has led to the interest in what we now call process ; and considerable efforts have gone into explicating this particular label for a complex of ideas like *mathematisation, generalisation and modelling*, together with many other aspects involving human communication like *language*, *visualisation*, and so on. (4) Since the late 1960, the literature in this area has been considerable.

We can detect in this literature, undercurrents of common belief such that in the process of learning mathematics, it is somehow created or re-created by the learner, and that some elements of this creative activity are claimed to be a common experience for children, pupils, students and mathematicians. A necessary consequence of this belief is that discussion and communication are vital to secure the growth of mathematical knowledge.

3. The Development of Mathematics and the relevance of History.

Before the mid 1960's there were a number of widely held assuptions about the relevance of the history of mathematics to mathematics, which we now realise had clear consequences for the methodology of teaching and learning of mathematics. Because of these assuptions, it was believed that mathematics develops quite differently from the sciences, having no recourse to heuristic or inductive reasoning, and having no relation with the contemporary social, philosophical or metaphysical contexts.

Crowe (1992) list these assuptions as :

a) Philosophers and teachers of mathematics maintained that mathematics has a deductive structure ; therefore the task of the historian of mathematics was to trace the development of these deductive chains for particular areas of mathematics. The only exception to this would be when a new set of axioms was announced, when the "deductive engine" would then be employed. Since mathematics was *purely rational*, the only criterion for judging new mathematical entities was whether they follow deductively from prior premises.

b) Mathematics is cumulative. The standard example is non-Euclidean geometry which does not contradict, and therefore does not replace Euclidean geometry. Mathematics is also improving. The poor mathematics of the past is replaced by more refined mathematics of the present.

c) Mathematics is free from metaphysics. Unlike science, the nature of the entities in mathematics is *a priori,* unquestioned and unquestionable.

d) Rigour, proof and certainty are independent of time. Once proved, a theorem remains true for ever.

e) There are no revolutions in mathematics. For example, Fourier (1822, p.7) "...this difficult science is formed slowly, but is preserves every principle it has once acquired : it grows and strengthens itself in the midst of many variations and errors of the human mind". Many other examples of this kind of certainly can be found in the statements of mathematicians almost up to the present.

Crowe's account of his research and the writing of his "History of Vector Analysis" (Crowe 1967) shows how every one of these assumptions (except perhaps the last) was shown to be unfounded. (5)

While we may admit that mathematics is a human activity, we also must recognise that the mathematics created by people has, in some sense, a life of its *own..* The problem is that mathematics as a product human activity is identified as a *body of Knowledge* and then becomes alienated from the very activity which has been producing it. There is a stong sense in which humans work on known mathematics and consolidate it (Wilder, 1968), and this tension between the individual and the social is a strong theme in Wilder's book.

We can regard mathematics as a living, growing organism which acquires a certain autonomy from the activity thas has produced it. The mathematics originally produced and in a sense owned by individuals becomes the property of the mathematical community, and as a communal activity, it then develops its own laws of growth, its own dialectic, and *defines away its past.* (6)

Defining away its past means that in the contemporary scene, mathematics is supposed to be produced according to the Formalist paradigm as described in the assumptions above. We are so busy trying to show that mathematics is about *anything*, that we obliterate the something it was originally about. (7) However, if we want to look behind the Formalist curtain, we are immediately concerned with working with and reconstructing for ourselves part ot this dialectic of ideas.

What I want to claim is that reconstructing this dialectic, finding arguments, motivations, problem-situations, conjectures, justifications, and so on, can only be done through the study of the history of the subject itself, and by attempting some kind of reconstructions of these historical situations. Following Lakatos, the meaning of rational has changed from the strict deductive sense indicated above (3a), to a meaning which includes "heuristic". This reconstruction is necessarily speculative and inexact and of course depends on the interpretation of available data. (8) Furthermore, the particular interpretations of the available data depend upon the philosophical standpoint of the person doing the interpreting, as I describe below.

4. The Reconstruction of History.

The fundamental purpose of history, *the general study of change through time*, is highly relevant not only to the mathematics we study today, but also to the communication of mathematics at all levels.

In the study of the history of mathematics we are concerned with two general aspects of equal importance. First, the past itself - the documents, records, events, and so on, which together are called the "facts" - because they form the basis of the past theories from which today's theories and techniques are derived. In reading the documents we attempt to discover the state of the concepts involved, so the importance of the records concerns not only arranging them to determine a sequence of events, but studying them to discover the empirical facts about the problems, and the theoretical facts about the solutions to those problems. In this sense, concepts form part of the basic data of the history of mathematics. (9)(10)

Secondly, from the assembled facts we then have attemps to reconstruct and interpret the past. The importance of a particular event or concept, the raising of its status from a mere fact about the past to a fact of history, depends entirely on the interpretations we might put upon that fact. These interpretation range from the conscious accounts of events and attempts at reconstruction by expert historians of mathematics, through to the unconscious interpretations made by the working mathematician or the teacher communicating mathematics.

Any programme for the rational reconstruction of the history of mathematics has to consider the effect of the many different and possible ways in which the raw data (facts, opinions and inferences) may be collected, organised and interpreted, and the variety of sources from which this data may be obtained.

These approaches can be themselves orgainsed under four broad headings : empirical reconstruction ; conceptual reorganisation ; social, economic and cultural development ; and patterns of discovery.

a) Empirical Reconstruction.

For a long time, this is what most people understood by the history of mathematics ; the process of examining the sources, gathering together the facts, arranging them in chronological order, and setting out to give an impartial account of the historical progress of mathematical ideas. The attempts to reconstruct past mathematics by the examination of the documents, and to reveal the motivations for this mathematics by discovering the relevant problems of the time, are generally underlined by a belief in the "progress paradigm" referred to above ; that is, that the mathematics of the past is somehow less well developed, less complete and less correct than it is today. This kind of approach is often called "inductivist" or "internalist". It sets out to show the development of mathematics in history, the physical and mathematical problems tackled, the new mathematics resulting from research, and the application of this new mathematics to both physical and theoretical problems as an evolutionary progression, where mathematics is getting increasingly better, and by implication, the mathematics of the past is gradually discarded as unsuitable, flawed, or incorrect.

This approach must necessarily be selective, and so the attempt to give an impartial account of all the facts is clearly impossible. There have been many textbooks written on the history of mathematics with this underlying philosophy that show the authors bias, a particular example is the classic text of Cajori (1896) which sets out not only to give an inductivist account of the history of mathematics, but also to suggest that our teaching may mirror the historical progress.(11)

b) Conceptual Reorganisation.

This concerns both current mathematics and the interpretation put upon the past. Contemporary mathematics influences value-judgements about past mathematics by describing them in terms of current concepts, and by deciding, consciously or unconsciously, whether a particular piece of mathematics has relevance or merit, or whether a particular theorem is proved rigourosly or not. Judging the past in terms of the present is a danger common to all aspects of history, not only mathematics, but it is particularly dangerous in mathematics and difficult to avoid *because of the concept of abstract mathematical structure*.

The structures of mathematics raise deep philosophical and psychological questions which have relevance for the study of the history of mathematics because we are concerned with the central concepts by which structures are described, how they came into being and to what extent they may be complete or still evolving, and the contingency or the inevitability of their rules of operation. An example to cite here would be the dominance of the concept of the abstract group, and how it is possible to "see" group structures in the mathematics of the past. (12)

This is in clear contrast to the empirical-inductivist approach outlined above which merely discards past mathematics as imperfect and incomplete.

The outcome of the unthinking application of modern standards of rigour to history is the complete writing out of history from contemporary textbooks. This, in itself, can be taken as writing of mathematical history by implication. Not only does the textbook tend to reverse the historical order, beginning with the currently accepted definition and attempting to "criticise" these by raising unmotivated and sometimes incomprehensible objections, it also removes the background of mathematics, the important "memory" of the mathematical culture, which today is available only to a few.

With every phase in mathematical history comes a reformulation, a consolidation, a new beginning. This is often considered necessary purely from the practical point of view, the proliferation of writing is such that for sheer efficiency of communication, choices have to be made. Students are thus cut adrift from the background of mathematics and it is important that both teachers and students should be aware of these dangers.

c) Social, Economic and Cultural Development.

Here we look at history from the general standpoint of forces external to the theory and the structures of mathematics. This approach examines how social changes can determine the centres of mathematical development ; how various kinds of patronage encourage the free development, the priorities and the fashions of mathematics ; the influence of individuals on research programmes ; the technical and social problems considered amenable to mathematics ; and the demands of investors and the restrictions imposed by economic conditions. All of these aspects are embedded in a particular culture at a particular time. Theses influences are important, for while they may not decide the detail of mathematical theory, they often determine its general direction and its rate of development. (13)

Societies can also act as "carriers" of mathematical knowledge ; this is evident in many situations where different kinds of specialised mathematics is used, from shopkeeping to engineering. Theories themselves can have "social existence" in at least two ways ; one where the practitioners of the theory actively contribute to its development, and one where those who do not contribute directly recognise that the theory has a valuable contribution to make to their own field, or to mathematics in general.

Apart from people, some of the obvious carriers of mathematical knowledge are text books, journals, pamphets and various other ephemera ; but today we also have other media, film, video, the computer disc, and electronic mail.

It is possible that a theory can have *mathematical* existence, but if it is not accepted and used by the mathematical community, it is not *socially* esistent, and therefore may have little or no influence on the subsequent development of mathematics. (14)

d) Patterns of Discovery.

Investigation of patterns of discovery is the domain of the philosopher as well as the historian,, and in a sense the fact that historians may try to find out, as far as is possible, exactly what a particular mathematician discovered, or what concept they were struggling with, implies the possession of an underlying philosophy, a belief in some kind of logic of discovery. The difference between historian and philosopher lies in the emphasis of interest ; the philosopher is primarily concerned with the analysis of problem situations for the understanding of theoretical systems and critical arguments, while the historian's first concern is the reconstruction of those problem-situations. The philosopher uses the historical facts as his data, while the historian establishes those facts. Obviously neither activity excludes the other.

We are concerned here with the attempts to build a philosophy of mathematics by investigating the creative intellectual processes of individuals, and exploring the contributions history can make in the formulation of a logic of discovery of mathematics, and a psychology of invention. It is through history that this heuristic might be studied. This area also concerns the greatest single problem mathematics has ; that of *communicating its relevance to the culture at large and making itself intellectually accessible at all kinds of levels.* This cannot be done without a good philosophy of mathematics, which, in turn, must draw on history for much of its data.

It is important to distinguich between the hypothetical *reconstruction* of the problem-situation, which is a conjecture about the actual problem of the mathematician at the time, and the problem of *understanding the reconstruction.* Confusion of the meta-problems and meta-theories of the historians of mathematics and the problems and theories of the mathematicians in history can lead to a great deal of argument.

Bachelard (1969) defined an epistemological profile to be an analysis of an individuals understanding of a concept, and an epistemological obstacle as the concept or method held by the individual that prevents a breakthrough into a new epistemological state (for example, Hamilton and the discovery of quaternions). Bachelard's story of science recognises the need to explain the past in terms of its own concepts and acknowledges rejected results as permanently valid achievements. (15)

5. Hermeneutics, Mathematics and its History.

Classical hermeneutics arose in the nineteenth century as a theory resulting from the repeated efforts to improve the interpretations of ancient texts. Hermeneutics is concerned with the character of understanding, particularly as it is related to written text, but here we could extend the idea of text, into the more general sense used today. (16) In this sense, it can be concerned with a wide range of activities, from textual analysis to the nature of historical understanding and the philosophy of communication.

Hermeneutics has traditionally been used to distinguish between the human sciences and the natural sciences. Because of its attention to meanings, hermeneutics distinguishes the human sciences from those where the elimination of the human elements have become a working principle. In fact, the underlying belief that the human element can be eliminated, and is in fact irrelevant, is widespread above all, in mathematics.

However, the mathematical entities and the relations between them are embedded in a complex symbol system made by and for human beings, and the meanings of this symbol system must be understood by anyone who wants to do or to use mathematics.(17)

Since history is already an interpretative activity, and since in the history of mathematics we are continually investigating the meanings of the texts available to us, it seems clear that the certainty (in the absolute sense) of what the writer of the text is communicating is continually brought into question. A good example of this is seen in the publication of the "nonmathematical" ideas of Isaac Newton (Fauvel 1988) where the mathematical establishment in England steadfastly ignored his philosophical and metaphysical beliefs which may have led him to formulate the concept of "force". From a hermeneutic point of view, "F = ma" requires considerable interpretation. (Rogers 1990)

6. Relevance for Learning.

Both creation and discovery are the concerns of pedagogy. Heuristic is used to assist the creative development of mathematical concepts by encouraging in the learner a situation where they may be able to achieve their own conceptual reorganistions. Heuristic also encourages the discovery of mathematics by providing a set of maxims for the procedural analysis of analogous situations. The extensions and elaborations of Polya's broad plan by Burton (1984) and others is an example of attempts to put this heuristic into practice.

By studying the history of mathematics we can examine aspects of the processes whereby mathematics was developed, and also become aware of the important contexts ; psychological, social, cultural, economic, etc within which mathematics exists.

It must be said that while the classroom learning of mathematics cannot in any sense mirror history, we have many lessons to learn from the different historical viewpoints outlined above.

NOTES.

(1) The formalist approach to teaching seems to originate from Christan Wolff in the 18th centrury as quoted by Erich Wittman (1992) :

"In my lectures I have mainly paid attention to three things, 1. that I did not use any word which I had not explained before in order to avoid ambiguities or logical gaps, 2. that I did not use any theorem which I had not proved before, 3. that I continuously linked the definitions and the theorems to each other, thus forming an un-interrupted logical chain.

Everybody knows that these are the rules which are followed in mathematics.

He who compares the mathematical way of teaching with the logical approach treated in my book on reasoning will find that the mathematical way of teaching is nothing but a careful application of the rules of reasoning. Therefore it does not matter if one follows the mathematical way of thinking or the rules of reasoning as long as these are correct. As I have shown *that mathematical thinking reflects natural thinking and that logical reasoning is only a distinct elaboration of natural thinking, I can well claim that my teaching followed the natural modes of thinking"* (Wolff 1726/1973, 52-54). (The italics are mine).

Of course, there are serious flaws in this belief ; perhaps this official dogma is more of a "working principle" for many mathematics teachers, and recent critical views of this approach have been repeated and reviewed in the popular book by Davis and Hersh (1981), (274-284).

(2) Here I speak of changes that have taken place in England over the last twenty years or so, and imply that in spite of these changes, much of the style of school mathematics is heavily influenced by the Formalist paradigm. See, in particular, the work on Investigations by the ATM (from 1966), which led to the work of Burton (1984) and Mason, Burton and Stacey (1985), and others at the Open University on Problem Solving and Mathematical Thinking.

An attempt to institutionalise some of this approach has recently appeared in the English National Cirriculum under Attainment Target 1 ; "Using and Applying Mathematics".

(3) Inutitionism has its roots in Kantian philosophy, and the idea of a *priori* perceptions. While I do not make any claim that the more recent approaches of the Popper/Polya/Lakatos school are in the same tradition, the similarity lies in the more recent emphasis on the human individual taking a much more central role in the creation of mathematics, and the exploration of what that central role entails.

The introduction of the social and cultural context of the creation of mathematics is an entirely recent development.

(4) For example, for language investigations see Pimm, Laborde and others and for imagery and visualisation the work of Goldin, Dreyfus, and Tall reported in the recent Proceedings of the Psychology of Mathematics Education (PME) conferences and in Nesher and Kilpatrick (1990).

(5) I note that Crowe (1975) insists on using the wording "in mathematics". This means that we can allow that "revolutionary" changes may occur in symbolism, nomenclature, metamathematics, methodology and even in historiography, but all of these changes do not appear to change the mathematics itself.

It seems to me however, that this leads us to the problem of defining what mathematics is ; a path that contemporary philosophers decline to follow (see my remarks in section 2 above). The question of defining and identifying "revolutions" in mathematics is comprehensively addressed in Gilles 1992.

(6) Wilder's concept of "consolidation" describes the situation where successive mathematicians refine particular concepts, and text book writers re-organise whole areas of mathematics according to new categories of abstraction, or particular modes of presentation. It is interesting to consider that the act of identifying mathematics as a separate autonomous body of knowledge, seems to imply thati it is possible to propose a series of "laws" of evolution for the subject.

These and many other problems concerning the relationship between mathematics, its crators and society, have been addressed by Wilder, Bloor, Crowe, D'Ambrosio, Kitcher, and others (see bibliography).

(7) Rather tongue in cheek, (pas sérieux) this is a paraphrase of Russell (1956) "...mathematics is the subject in which we do not know what we are talking about, nor whether what we are saying in true."

(8) This is a substantial claim, but much of the groundwork has already been done. Notice that the claim is not that we cannot teach mathematics without knowing its history, but that the historical background provides ideas and information upon which we can base our teaching approaches. This avoids the requirement that every teacher of mathematics should be a historian of mathematics as well!

See, for example Freudenthal (1983), and the work of the IREMS in France is producing many commentaries and epistemological papers on the processes of historical discovery.

(9) Even "facts" are controversial. See, for example Hoyrup (1991) where the consideration of the same ancient cultural practices by academics of different disciplines can lead to very different interpretations of what was actually taking place at the time.

(10) "Concepts" are dangerous things. There is little agreement as to what a concept is, and many attempts to broaden out the idea by using such terms as ; concept networks, schemata, frameworks, and so on. These is also the elusive fluidity of "concept as object" and "concept as tool" described by Douady and Perrin-Glorian (1989).

(11) Cajori's book, A History of Elementary Mathematics whit hints on Methods of Teaching, came after his more substantial work on history of mathematics (1893), and shows the influence of Herbert Spenser and the ideas of the "biogenetic law" (ontogeny recapitulates phylogeny). This theme is repeated in other books on the teaching of mathematics from an historical point of view, for example see Branford (1908).

(12) May (1972) rated this approach to history as one of the "cardinal sins". See, for example, many of the historical notes to Bourbaki which imply that past events are direct antecedents of present theories, and Zeeman (1974).

(13) There are many examples : the development of skills, techniques and notations in mathematics brought about by the organisation of society into city-states and centres of trade ; the development of writing, communication and the invention of printing ; the development of technology ; the needs of war, and so on. See, for example, Bloor (1991) and Wilder (1981).

(14) The idea of mathematical exsistence means that an individual has constructed a theory which is consistent according to some declared criteria, but which is not generally accepted and incorporated into pratice by other mathematicians. This is theoretically possible, but it is difficult to find examples because we would be unlikely to know about them. Perhaps we could consider the circle squarers and angle trisectors (cranks, to some) as examples, while some extreme forms of intuitionism or constructivism may also be considered. For social existence, the mathematics does not necessarily need to be "alive" at the moment. It is sufficient that we have evidence that mathematical practices and theories have been used by a group of people at some time or other.

(15) Many examples can be cited in science. For example Phlogison Theory is a valid achievement, but a rejected result. Similar episodes in mathematics have simply been ignored.

(16) Text, in the contemporary sense, includes all forms of written recording like music and dance notation, other media like photographs, film and video, and other graphic forms like posters or works of art.

(17) For example, Cohen (1985) suggests four criteria for identifying whether a scientific revolution has occurred :

a) the testimony of contemporary witnesses (including scientists' assessment of their own work) ;
b) the ctitical examination of the documentary history of the subject ;
c) the judgement of competent historians, particularly historians of science and of philosophy ;
d) the general opinion of working scientists today.

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