CONTINUITY AND VARIATION: THE TRANSFER FROM A GRAPHICAL TO AN ANALYTIC REPRESENTATION

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INTRODUCTION.

In this paper we want to show some of the relevant circumstances that made possible the transfer from elementary to higher mathematics. We will call "higher mathematics" the period that begins with the mathematical study of movement. That includes as a central part the origins and development of calculus. We hasten to say that we shall not follow a detailed historical path but, rather, give a perspective and, if possible, to offer an explanation in epistemological terms of some important "conceptual moments" that made possible that transfer.

Two circumstances were instrumental to this development: the identification of number to a geometrical continuum and the mathematical construction of variation.

In regards to the concept of number, it is important to consider the drastic conceptual change we can observe when comparing the Euclidean conception and the new concept of number introduced by Stevin.

It has been reported elsewhere (see [1], [2]) that the Euclidean number concept comes from the works of Pythagoras and Aristotle. There, are two facts that from our present viewpoint are astonishing: (i) *One* is not a number, and, (ii) *number* can only be applied to the study of discrete collections; in other words, there is no notion of continuity associated to the concept of number. It is because of these features, that we can consider, as a relevant problem *to try to find when number and (continuous) magnitude became integrated into the same concept.* Let us recall that Aristotle dismissed the Arrow Paradox by saying that time cannot be made of moments and lines cannot be made of points, --which is a way of saying that the category of Quantity is formed by means of two disjoint components: the discrete (number) and the continuous (magnitude).

They are reflected in mathematics as the study of magnitudes and numbers, ie: as the study of Geometry and Arithmetic. For Aristotle, continuity can be characterized as "never ending" divisibility, from which it is possible to conclude that the line cannot be composed from points. Lines and points belong to different mathematical realities.

The geometric continuum appears as an abstraction of physical continuum. Because of the characterization of continuity, as never ending divisibility, It is possible to conclude that the continuum is not made of indivisibles. On the other hand, number is the prototype of discreteness; number is a collection of units (and unity is not a number).

This scenario changed in a radical way with the work of Simon STEVIN. The Greek concept of number had come to being, as a result of an abstraction process applied to the material world. Stevin challenged the Greek viewpoint, in face of practical necessities. Nevertheless, Euclidean conceptions were so deeply rooted, that Stevin found it necessary to argue against this tradition both, from a practical viewpoint and what we consider as an epistemological viewpoint. It was not just a matter of saying that "one is a number" but of producing a substantiated conception of number to justify that assertion.

Representations and Mathematical Notions

One can observe that during the evolution of a discipline, conceptual nuclei are formed around which mathematical activity is carried out. For instance, analytical representations permitted the widening of the universe of curves to which tangents could be drawn. The associated operational field allowed the exploration of the "tangent object". This cognitive structure is what we call a conception of tangent. The associated concept lives in the formalized mathematical discourse. It is subjected mainly, to the appropriate syntactical rules.

The process by which a conception is refined, can be described in terms of Piagetian assimilations and accommodations (See [5] and [6] for a further discussion).

Human minds try to understand the world of their experiences through the construction of models. To have a model is a path to understanding. The interplay between representation and understanding has been a driving force in the historical development of mathematics. The Greek geometry is a good example: an adequate form of visual representation enables us to handle abstractions we have made from experience.

Representations carry with them a semantic dimension. When Analytic Geometry was created, the possibility of going from a symbolic representation (the algebraic equation) to a visual one (the graph) was established. The interplay between these two forms of representation leads to the construction of deeper meanings for the conceptions involved. Connections control meaning; so, the new space for doing geometry--the Cartesian plane--by allowing the encounter between the symbolic/algebraic and the visual/geometric conceptions, made possible the (new) geometric study of movement. This was instrumental for the emergence of Infinitesimal Calculus.

A new conception of number

In 1585, Simon STEVIN published a book, *L'Arithmetique,* that was called to produce an epistemic cut in mathematical knowledge: a book about the theoretical and practical aspect of arithmetic ([3]). In chapter X Stevin presents his (new) concept of number. For him, "number is that through which the quantitative aspects of each thing are revealed", (our translation). The category of Quantity had been separated into disjoint classes: discrete and continuous. Now, Stevin made no difference between the two. Number was there to represent the mathematical phenomenon of quantity. The decimal notation, comes to solve the problems posed by the tension between form and content. In fact, as there is no distinction inside the category of quantity, then there is no distinction between the object of study of arithmetic and that of geometry. The representational tool needed to cope with this new situation had to be flexible enough to deal with the problems of discrete quantity and simultaneously, with the problems of divisibility. To talk about parts of the unity, decimal notation was instrumental. The new representation was deeply linked to the new concept of number: This is, perhaps, one of the best examples of how an adequate symbolic representation becomes an instrument to explore with.

While Euclidean mathematics produced a concept of number by means of an empirical abstraction---in the sense of genetic epistemology---, Stevin's concept was the result of reflective abstraction. It is not possible now, to separate the concept and the symbolic representation. This makes clear the abstract nature of the new concept of number. Perhaps it could be said that, now, number "lives" in the mathematical discourse. In fact, it is interesting to compare, once more, the Greek conception of Number as a principle (the unity is the principle of number) and Stevin's conception of number. In Stevin's work the concept of number is justified not only because it could accomodate all the needed computation but also, because the symbolic character of his work was in line with the development of symbolic algebra. Stevin talks about "arithmetical numbers" and "geometrical numbers". According to him, if you don't know the numerical value of a geometrical number, it enters algebraic computations as an *indeterminate quantity.* This is not yet a genuine algebraic variable because of the homogeneous character implied by the geometric language used. And also, because the indeterminate quantity *is a number*, only that we don't know it. Perhaps we could say that this is exactly the role of the "unknown" in the period of the Italian efforts to solve third and fourth degree equations. In the latter case, however, we can see a transitional moment that gives rise to the departing point of the new algebraic language.

The algebraic concepts underwent a very subtle development from early sevententh century. It is plausible to say that the first conception of algebra was as a kind of generalized arithmetic. Another viewpoint is presented in Vieta's *Artem Analytecem Isagoge,* 1591. In this book, one of the cornerstones of the new algebraic thinking, we can read this: "The supreme

and everlasting law of equations or *proportions* (our italics), which is called the law of homogeneity because it is conceived with respect to homogeneous magnitudes , is this:
I. Only homogeneous magnitudes are to be compared with one another.
For it is impossible to know how heterogeneous magnitudes may be conjoined. And so, if a magnitude is added to a magnitude, it is heterogeneous with it. If a magnitude is multiplied by a magnitude, the product is heterogeneous in relation to both".

In Vieta's work, the term "magnitude" is used in a general sense; not only as a geometric magnitude. The magnitude one is looking for, when solving an equation, for instance, can be an arithmetical one ie: a number. In this respect, we want to cite J. Klein (see [2]): "What is characteristic of this 'general magnitude' is its indeterminateness, of which, as such, a concept can be formed only within the realm of symbolic procedure. [the Euclidean presentation] does not do two things which constitute the heart of symbolic procedure: It does *not* identify the object represented with the means of its representations, and it does *not* replace the real determinateness of an object with a *possibility* of making it determinate, such as would be expressed by a sign, which instead of *illustrating* a determinate object, would *signify* possible determinacy".

Perhaps the kernel of the meaning of this citation is the identification of the signifier with the signified. In this case, we can accept that the Euclidean presentation is not symbolic. This is a large step towards the constitution of a symbolic mathematics; nevertheless Vieta is still "linked" to the law of homogeneity. Anyway, we can assert that the symbolic character of Vieta's work is the result of a process of reflective abstraction. In a sense, we could say, following Klein, that the concepts of the new science are obtained by a "reflection on the total context of that concept" or, in other words, by a process that can be termed "symbol-generating abstraction". Vieta's work can be seen as, perhaps, the first work of this new discipline.

In his book *Rules for the Direction of the Mind,* Descartes considers the problem of multipling the product ab of the magnitudes a, and b, by a third magnitude c. He says that, for this to be possible, *the product ab has to be conceived as a line (our italics).* This detachment of quantities from the geometrical constraints appears as Rule XVIII; this is possible because of the resultant abstract symbolism: the quantities involved in the operational activity are abstractions of the geometrical figures, not the figures themselves. The identification of a number with the symbol used to represent it, leads to a conceptualization of number as a mental entity, not anymore as the Greek Arithmos, used to count material things. In this way, Vieta's symbolic algebra, which still is arithmetical and geometric, becomes fully symbolic---through the loss of the dimensionality of the symbols used---in Descartes' hands. Before Vieta, the main activity was the search of a formula (a set of directions rules) to compute the roots of an equations with numerical coefficients. In a sense this activity can be seen as a generalized arithmetics: all the operations involved are done on the numerical coefficients, and, eventually, also the unknown has to be "operated". With Vieta's work, this changed radically. Now, the *operation* is the new object of study. In terms of genetic epistemology, we can say that we have entered an *inter-objectal* stage of development. As we have already remarked---but we find necessary to recall---the passage to this new stage is possible because a reflective abstraction process is present.

variable and variation

The symbolic language of algebra enabled the construction of *models at a higher level of representation.* With its own operational field, as symbols could be manipulated like given quantities, they could also be related through symbolic expressions. We believe this was instrumental for the conscious study of functions as models designed to study *states. The table of values* is a good example of this kind of model. Perhaps what is more important in this activity, was the use of literal symbols as something with the capacity to take numerical values and thus, to vary on a numerical substratum.

In this kind of historical cross-section we are trying to describe, we cannot ignore Oresme's work. Up to now we have roughly described the process that led to the construction of the algebraic language, and how this was instrumental to the study of numerical variation. Nevertheless, the study of variation in a geometrical context also played a very important role. The idea of a *flowing quantity* can be found in the already mentioned Oresme's work ([4]).

He introduced geometric figures to represent the behavior of a quality. According to Oresme, the study of a body can be realized from two viewpoints: from an extensional one, for instance, the weight of the body; and from an intensional viewpoint, for instance, the body's temperature. In this case, the measurement is punctual. Reading through the chapter "On the continuity of intensities" ([4]) we can find the Euclidean conception of number as Oresme says that any measurable thing *except number can be represented by a magnitude*. This also explains why, when talking about intensities he says that "the points of a line" are a necessary fiction, used to represent a place on the studied body. Each intensional measurement one makes on a body, is represented by means of a segment. The body itself is also represented by a segment. Oresme considered the whole set of intensities as the gate towards the study of variation. This set is called a *surface latitude,* and it contains all the information about tha variation of the intensity. But here appears a shift in his viewpoint as the surface latitude is used to study the variation of *form, rendering his work a qualitative one*. Let us consider his study of velocity. This concept is conceived of as a quality adquired by a moving body. Then the terms "uniform" and "Uniformly difformed" are introduced, to name a quality that does not change with time, in the first case, and, a variable latitude with a constant rate of change, in the second case. These are the main examples considered later, by Galileo, though from a different conceptual framework. It must be mentioned that Oresme considered another possibilities. For example, he considered "difformly difform" surface latitudes. There is a total autonomy of these studies from physical constraints; perhaps we could say all this work is done *to classify by means of the representing tool.*

Oresme's way of considering variation endows his representational model with the possibility of studying motion geometrically. This is the feature we want to stand out, as it is this feature through which we want to approach Galileo's work on motion. However, in the latter case, the conceptual framework, as we have already said, is very different. Let us consider an example taken from Galileo's "Two New Sciences" (See Struik [7], pp 208-209) :

Theorem I, Proposition I

The time in which any space is traversed by a body starting from rest and uniformly acelerated is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed and the speed just before aceleration began.

There is a corresponding version of this theorem that can be found in Oresme's work, --- see Struik [7] ,pp 137. It appears as "conclusion 5", and the wording is the following:

"...It can be proved that a uniformly difform quality is equal to the medium degree, i.e., just as great as a uniform quality would be at the degree of the middle point, and this can be proved in the same way as for a surface".

The proof, consists in transforming the area of a triangle into that of a rectangle. Let us make a remark on this proposition: it is important that Galileo can represent the distance traversed by a body, *as an area. There is a correspondence between this fact and Descartes linearization of all magnitudes.*

The correspondence between physical facts and those predicted by the model, helps to configure, within the model, a certain set of normative criteria. This is in agreement with the Piagetian assertion about the common origin of an instrument of knowledge and its validation criteria.

We can summarize the foregoing discussion through the following sketch:

Phenomenon (motion)

↓

Representation (of phenomena)

↓

Extraction of operational rules from a particular physical context

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Generalization of operational rules

(loss of particular meanings, contextual dependent)

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