HISTORY OF SCIENCE A CRITICAL AND CONSTRUTIVE TOOL FOR THE MATHEMATICS CURRICULUM (1)

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RESUME

History of mathematics offers outstanding examples of simple and at the same time powerful ideas which organize their surroundings, ideas connected with each other in a transparent network. Rather PASCAL'S "esprit de finesse" rules the process of thinking and to some extent by analogy learning, too.

Some impressive examples show that even less able students can take profit by such an organization of mathematical knowledge. There is more hope that a new quality of mathematics teaching might result from the epistemological and historical point of view rather than from the currently flourishing empirical research, categorizing and doctoring merely the symptoms.

Reacting to some psychological research of the international PME-group (FISCHBEIN, VINNER) I show an example of using history of science to overcome certain shortcomings of the common curriculum with respect to the ideas of tangency centered around a memoir of DIDEROT.

Another example is developed from an idea of Jacob BERNOULLI for the teaching of elementary algebra found in his Ars Conjectandi. How does this translate profitably to the classroom of junior highschool, where algebra is normally used to find out how old grandma is ? Read the Appendix!

ABOUT ORGANIZING IDEAS

There is a lot of lip-service and well-intended general advice for the use of history in math teaching, but very few worked out examples of the kind I would like to talk about.

(1) dedicated to Christian Houzel, my talk at the Le Mans Conference profited greatly from numerous discussions with him, when I was visiting professor at the University of Paris (XIII) (March-June, 1984), teaching on tropics from the history of mathematics, and also from discussions with Jean DHOMBRES (Nantes).

In the history of mathematics I was looking out for ideas

- influential in the development of mathematics
- simple and useful, even powerful

which at the same time could act as

- "centers of gravity" within the curriculum
- knots in cognitive networks.

In that sense I call them organizing ideas.

In the course of history new central ideas developed by reorganization of the old stoks of knoledge allowing to draw a better general map from those "higher points of view".

"La vue symoptique" brings to light associations hitherto hidden.

This does not work by "longues chaînes de raison" (DESCARTES). Not Euclid's "l'esprit de géométrie" rules the process of cognition and by analogy the process of learning, but rather a mode of thinking related to Pascal's "esprit de finesse" paving short ways from a few central points to many stations.*)

PASCAL himself provides a splendid example. When sixteen years old he reorganized the knowledge about conics handed over by DESARGUES unveiling the "mysterium hexagrammicum", ever since called Pascal's theorem : the high point surrounded by a lot of close corollaries.

Having cut organizing ideas out of the historical context one is left with the hard task to process them into more or less comprehensive teaching modules (or even schoolbook chapters). The products can be problem fields in which very few organizing ideas instead of dozens of theorems operate as a means for problemsolving.

Concentrating on few simple and the same time powerful ideas which organize their surroundings and which are connected one another within a simple network, offers help for the less able student too. His inability derives to a large extent from the fact that he is unable to organize his thought with respect to a complex field in which the connections are presented in the usual plain logical systematical way

*) "...il faut tout d'un coup voir la chose d'un seul regard, et non par un progrès du raisonnement, au moins jusqu'à un certain degré..." (PASCAL, Pensées, ed. Lafuma 512 : the difference between l'esprit de géométrie and l'esprit de finesse)

and where teaching is used to administer only spoonfuls of the subject matter, disconnected and without depth.

EXAMPLES GIVEN AT LE MANS

Ransacking the history, pasting together ideas, interesting problems of very different provenience, even some routine exercices etc. might not always please the feelings of the "real" historians.

But junior high school teachers should be pleased to find the "real bricks" for interesting problemoriented teaching modules of the kind drafted or worked out at the Technische Universität Berlin. At the Le Mans conference mainly two examples were shown :

- An idea of Jacob BERNOULLI for the teaching of algebra one version for the normal classroom, another for the talented pupil. (15)
- The powerful idea of tangency - problems solved by means well below the $\frac{dy}{dx}$ introduced by a memoir of Denis DIDEROT.

What BERNOULLI'S idea looks like you can find out by leafing through the paper at the appendix.

With respect to the other theme I will give some key words because at the moment there is available only an outline in German. (18)

(1) A good start: DIDEROT'S Mémoires sur différents sujets de mathématiques, Paris 1748, ch. XIV. The involute of a circle as a tool for geometrical constructions ; on unwinding curves from HUYGENS' Horologium oscillatorium (especially on the cycloid).

(2) On the construction of algebraic equations (DESCARTES, géométrie ; NEWTON, Arithmetica universalis). Confrontation of geometrical, algebraical and analytical methods.

(3) Level lines as a simple geometrical method to solve optical and dynamical problems of e.g. ALHAZEN, REGIOMONTUS, GALILEO. Comparison with the anlitical versio (LAGRANGE'S multipliers).

(4) Several tangent constructions by dynamic methods from ARCHIMEDES to NEWTON and the "opposition" (like HUYGENS in his Horologium).

(5) Tangency and algebra of (even "long") polynomials. Translation x = x'+a together with "cutting off tails" (linear approximation) offer powerful applications of the basic algebraical techniques well below the $\frac{dy}{dx}$ for the junior highschool. (Freeing the 13-16 year olds from a lot of exercises of the type, how old is grandma, if...)

MORE EXAMPLES FROM TU BERLIN - AN ANNOTED BIBLIOGRAPHY

Copies will be sent on reguest free of charge, except (14).

(1) G. Booker, Mohry B., Stowasser R.J.K., Geometry reborn a unifying idea for teaching geometry drawn from the history of mathematics. Preprint 1/1984, 22 pages. Tu Berlin, Math. Dep.

A proposal of G.W. LEIBNIZ to found geometry on distance (and symmetry) from a letter to HUYGENS (1679.9.8) prepared for pupils (age 11-13). An attractice "opening problem" with a nice family. An interesting use of symmetry to solve problems that have relied on applications of "Pythagoras".

(2) B. Mohry, Otte M., R.J.K. Stowasser, Trennlinien. Problemsequenzen zu einer integrativen unterrichtsidee. Teil 1, Mathematiklehrer (Mathematics Teacher : a German Journal) 1/1981, pp. 21-27. tei 2, 2/1981, pp. 25-27.
(2) is the more comprehensive version of (1).

(3) G.W. Leibniz Trennlinien und Trennflächen zur Grundlegung der Geometrie. Mathematiklehrer 1/1981, pp. 28-29.
A part of the letter from LEIBNIZ to HUYGENS (sept.8,1679) mentioned in (1).

(4) R.J.K. Stowasser, A textbook chapter from an idea of

PASCAL. For the learning of mathematics, vol. 3, num.2, 1982, pp. 25-30. (French and German versions are available). PASCAL'S idea to develop quick algorithms for remainders discovered by watching the clock. The idea grasped from the paradigm, not abstracted from a lot of examples. (ARISTOTELES' difference between teaching and research). Nombabylonic faces of clocks and divisibility rules.

(5)R.J.K. Stowasser, STEVIN and BOLZANO - practice and theory of real numbers. Preprint 4/1983, 15 pages, TU Berlin, Math. Dep.
LAGRANGE in his "leçons élémentaires sur les mathématiques données à l'école normale en 1795" teaching about 1300 prospective teachers from all over France, foreseen to form the staff of local teacher colleges, presents a very simple but universal method to solve geometrical construction problems. The infinite halving procedure, similar to the metric system, used already by STEVIN to solve engineering problems, became later with BOLZAMO a very simple proof method. One and the same proof scheme works and can be successfully applied by even high school students to all the fundamental theorems about real sequences and continuous or differentiable functions (theorems about cluster points,..., intermediate, mean, extremal values,...). For the general opinion these facts lies in the deep sea. But actually they float on the surface, held together and ruled by the simple idea of repeated halving. The insight, that measuring and proving are like head and tail of a coin, should soon have important consequences for highschool teaching.

(6)G.W. Leibniz, Im Umkreis einer fundamentalen Idee : Ähnlichkeit. Mathematiklehrer 1/1981, pp. 11-12.
An undated manuscript (1679 ?) on analysis situs (analysis of position), in which LEIBNIZ polemizes against Euclid's cumbersome handling of the simple idea of similarity and presents his own powerful approach. He looks at invariants when changing units by linear transformations. A very modern idea, which could unify several chapters of school mathematics. Some lessons in the spirit of

LEIBNIZ' idea : (8), (9).

(7)R.J.K. Stowasser, Eine Anwendung von LEIBNIZens Methode der Maßstabsänderung : Das Pendelgesetz. Mathematiklehrer 1/1981, pp. 12-13.
An application of LEIBNIZ' method of changing units by linear transformation : the law of the pendulum as a result. (With reference to (6)).

(8)J. von den Steinen, Problemsequenzen zum Thema y = c.x in der Geometrie (1. Teil). Mathematiklehrer 1/1981, pp. 13-17.
Problemsequences concerning y = c.x used in elementary geometry as a reaction to LEIBNIZ' idea of similarity (With reference to (6)).

(9)dito, (2. Teil)
Mathematiklehrer 2/1981, pp. 28-31
Continuation of (4).

(10)Alte und junge Anwendungen der Kongruenzrechnung in der Schule : PASCAL, GAUSS, RIVEST.
Preprint 1/1985, 19 pages, TU Berlin, Math. Dep.
Old and young applications of the idea of congruence (the number-theoretic concept).

(11)R.J.K. Stowasser, Mohry B., Die Idee der Rekursion und der Isomorphie im Umkreis von STEINERs Raumteilung und EULERs vertauschten Briefen : für talentierte Schüler verfaßt. Mathematiklehrer 2/1983, pp. 2-10.
The idea of recursion and of isomorphy : problem sequences prepared for talented students (age > 14). A dissertation of Jacob STEINER "Einige Gesetze über die Teilung der Ebene und des Raumes" (1826) ("Some theorems about the division of the plane and the space") taken as a basis.

(12)R.J.K. Stowasser, A collection of problems from the history attractive for teaching recurrence methods.
Preprint 2/1985, 36 pages, TU Berlin, Math Dep.
Addressed to teachers to create interesting teaching units from the collected basic material. More about the content

in the textbook for students = (14).

(13) R.J.K. Stowasser, Ransacking history for teaching. Zentralblatt für Didaktik der Mathematik 2/1978, pp. 78-80. Aselect from (12).

(14) R. Stowasser, Mohry B., Rekursive Verfahren. Ein problem-orientierter Eingangskurs.
Schroedel Verlag Hannover (West Germany) 1978, 104 pages.
The booklet "Recursive Prodedures - a problem oreinted preparatory course" concentrates upon the fundamental idea of recursion as a tool for modelling, especially combinatorial situations, rather than for proving.
Difference equations treated before differential equations, linear methods as heuristic means to find say BINET'S formula (1843) for the FIBONACCI sequence (1202), polynomial interpolation etc. offer a good chance for general practice and preparation of the harder material to follow in the Analysis, Linear Algebra, Stochastic Courses.
The booklet explains in a manner suitable for 16 to 17 year olds a philosophy of math teaching with combines POLYA'S problem solving approach and WAGENSCHEIN'S (WITTENBERG'S) paradigmatic teaching.
Although the materials appear as a rule in a somewhat historical setting, we do not follow TOEPLITZ' unsuccessful genetic approach. Irreverently "Ransacking history for teaching purposes " may offer better chances to reach the classroom.
The result : a collection of 14 problem areas : Simple patterns ; LUCAS' problem ; Coloring, handshaking ; Bachet de MEZIRIAC'S weighing ; Jacob BERNOULLI'S figurate numbers ; Jacob STEINER'S intersection problems ; From Leonardo of PISA to BINET ; PASCAL numbers : MONTMORT-BERNOULLI letters ; Population problems.
More about the autors' philosophy and the solutions of the problems can be found in a booklet : Stowasser/Mohry Didaktik und lösungen zum kursbuch Rekursive Verfahren Schroedel ; Hannover 1980, 49 pages.

(15) R.J.K. Stowasser, Breiteig T., An idea from Jacob BERNOULLI

for the teaching of algebra : a challenge for the interested pupil.
For the learning of mathematics, vol 4, num. 3, 1984, pp. 30-38. Jacob BERNOULLI'S Ars conjectandi (1713) offers a readable discussion of the binomial numbers and derived from these, formulas for evaluating the power sums $(1^K + 2^K + 3^K + \ldots n^K)$ - the BERNOULLI polynomials. "The wonderful properties of the figurate numbers" traced by Bernoulli where "eminent secrets from the whole of mathematics are hidden..." (Bernoulli) can be obtained by interested pupils. But there is enough stuff for the normal classroom too : A family of problems for the general repetion of basic algebraic techniques.

(16) Ch. Keitel-Kreidt, Papamastorakis E., Mathematiklernen aus der Mathematikgeschichte. Auswahl aus DESCARTES "Geometrie". Beiträge zum Mathematikunterricht 1984, pp. 192-195. To learn mathematics from the history of mathematics. A selection from DESCARTES' "Geometry". A prospectus of a booklet in the making centered around geometrical construction problems analysed by algebraic means.

(17) R.J.K. Stowasser, An excerpt of DURER'S treatise on mensuration with the compasses and ruler en lines, planes, and whole bodies to be used for teaching.
Preprint 3/1983, 14 pages, TU Berlin, Math. Dep.
DURER'S constructions of regular polygons of n sides for all $n < 17$ as a starting point to reflect on algebraic equations and to get the idea of continuous fractions.

(18) F. van der Blij, kießwetter K., Stowasser R.J.K., "Forshungsaufgaben" für Lehrer und Schüler.
Reprint 2/1984, 16 pages, TU Berlin, Math. Dep.
"Research problems" for teachers and students. A collection of 6 pieces under that heading in the journal "Mathematiklehrer" 1981-1983. Intention : to stimulate the teachers themselves to learn mathematics by doing. Being led to certain interesting points in the history there is some hope that they might explore the offered problem areas and finally be tempted to create an attractive <u>centered problem family</u> for the classroom.

Topics

- Recursively generated number arrays which LEIBNIZ learnt from HUYGENS and PELL as starting points
- The problem field "geometric constructions" (with various) tools) opened by the research program of Jacob STEINER
- The problem field "plane linkages" introduced by David HILBERT
- Tangential problems solved by means below $\frac{dy}{dx}$, motivated by a memoir of Denis DIDEROT
- Fragments from EULER about the BERNOULLI approximation for the roots of algebraic equations.

(19) R.J.K. Stowasser, Geometry and problem solving. Proceedings of the International Colloquium on Geometry Teaching organized by the Belgian subcommission of ICMI 1982 at Mons (ed. G. Noël), pp. 207-226.
Message : The renewal of geometry teaching means above all the emancipation of more elaborate knowledge and of its logical systematical Euclidean organization. Geometry should not be taught en bloc and self-reliant but in a network of overlapping problem families around a few simple fundamental ideas linking together geometry, algebra, number theory and analysis (!). This structure can hinder passive learning, support problem solving activities, train intelligent behaviour in new situations. The paper shows by examples in what way math history is helpful.

REMARKS

(1) An overview of historical aspects for mathematics education offers the miscellaneous collection : Historische Aspekte für Mathematikdidaktik und Unterricht. Ed. R.J.K. Stowasser. Zentralblatt für Didaktik der Mathematik 4/1977, pp. 185-213 and 2/1978, pp. 57-80. Authors : J. M. Bos, H. G. Flegg, H. Freudenthal, P. S. Jones, M. Otte, L. Rogers, I. Schneider, G. Schubring, R. J. K. Stowasser.

(2) I cannot but mention one book not related to the Berlin stable at all, which was very much useful for me when preparing the chapter "problèmes des tangentes" for my seminar at Paris XIII : J. P. CLERO, E. LE REST, La naissance du calcul infinitésimal au XVIIème siècle. Paris 1980.

Jakob BERNOULLI (1655-1705)

A provisional picture of his work

A very clever young man, yet frustrated and undecisive due to an extremely severe education, find true satisfaction only in the dream world of mathematics, in spite of all knowledge about the world around him. Starting out from the obvious questions of pratical mathematics, he never abandoned his special liking for problems of applied mathematics. His numerous investigations interspersed with mechanical ideas speak for themselves. Admittedly, he almost exclusively concentrates on the theoretical side of the problem, not that much on the experimental side which at his time was beyond the available aids.

JAKOB, who was above all an autodidact, took hold of DESCARTES' conception of mathematics and came to infinitesimal mathematics though BARROW, WALLIS and LEIBNIZ. Initially he did not know anything about other contemporary work in this direction later on he only marginally heard about different methods - working them over critically rather than creatively. To be able to continue his work, he pines for praise and success, which fits his fundamentally shy nature. If both are denied him, he is not able to let things rest, but instead tries to force his point of view. Occasionally he adopts the wrong methods and goes too far, but still, on the whole, we can only speak of this man with the deepest respect.

In his efforts to protect his rights as an inventor, JAKOB is rather intolerant, particulary of his brother, with whom he is nevertheless very close related in character and whom he knows and understands better than anybody else : he is deeply hurt when the younger man denies him his due recognition. Certainly, JOHANN was the more lively and quicker thinker and more elegant in his presentation. And yet, he admittedly needed just as much competing conversation with his brother and when JAKOB died, JOHANN was hardly able to develop really creative and new ideas. Feelings of inferiority characterize JAKOB'S position to LEIBNIZ and MENCKE : quite unfoundedly he feels put at a disadvantage by them. Only after being admitted to the Academy of Berlin, JAKOB recognizes that LEIBNIZ had nothing against him whatsoever, as he had hitherto believed. Their correspondance, being revived at an unhappy moment,does not yield anything significant, for LEIBNIZ is getting old and no longer able to follow with clear insight JAKOB's strange ideas (law of large numbers, isoperimetric problem).

JAKOB has hardly any elegant ideas. Those produced by him - almost a little titanically - are primarily founded on the logical use of DESCARTES' attempts in connection with calculus, the formal elaboration of which was so ardently promoted by JAKOB. Generally he is not very found of geometrical considerations where they appear in his infinitesimal attempts, they are not always clearly enough established. In this field JOHANN is very much superior with his short and elegant conclusions. On the other hand, the fact that JAKOB firmly stuck to cartesian coordinates, which he not often abandons for modified polar coordinates and only rarely for other coordinates, was of great importance to the formal development of differential coordinate geometry. Despite of it's strive for generality, JAKOB'S scientific work was more strongly fixed on ascertainable details which stand in loose connection next to each other without growing together in a unified theory. This kind of attitude suits far better the light scientific essay in periodicals and the playing with interesting simgle problems, than the condensed presentation in a book.

The only major scientific work JAKOB has written and left unfinished - ars conjectandi - is in the first place a collection of single problems with only occasional contributions to a general theory. Primarily JAKOB was a professional mathematician and as such a powerful research personality, not as genial and full of ideas as LEIBNIZ, but a profound thinker, tenacious and logical - one of the great pioneers in the field of modern mathematics.

(A piece of a memoir of J. E. Hofmann quoted in the German journal MATHEMATIKLEHRER 1/1981, pp. 22-23, translated by Marie Christine Cuyvers)

An Idea from Jakob Bernoulli for the Teaching of Algebra: A Challenge for the Interested Pupil*

ROLAND STOWASSER, TRYGVE BREITEIG

* Translated and slightly condensed from the German in *Mathematik-lehrer*, 1/1983

1. The effective method of Jakob Bernoulli for power sums: Motivation and pattern

> *What in former epochs occupied the mature minds of men is in later times made accessible to boys.*
> *Hegel*

In Jacob Bernoulli's famous book on probability—in his *Ars conjectandi* (the art of conjecting) from 1713—we find a readable discussion of the binomial numbers. Derived from this we find formulas for evaluating the power sums—the Bernoulli polynomials. Jakob is obviously not a little proud of this achievement:

> *"Using this table I have, within half a quarter of an hour, found that the 10th powers of the first thousand numbers sum to*
>
> *91 409 924 241 424 243 424 241 924 242 500*
>
> *From this we see how hopeless is the effort which Ismaël Bullialdus has shown in writing his very extensive 'Arithmetica Infinitorum'. He has not achieved much, as he has calculated only the power sums for c = 1 to c = 6, with the greatest effort. This is just a part of what we have reached in one page."*

Even less sensible than the calculation of the insulted Ismaël Bullialdus—acquainted with Desargues, Mersenne, Roberval—is the blind summation:

$$S(1000^{10}) = 1^{10} + 2^{10} + 3^{10} + \ldots + 1000^{10}$$

$$S(n) = \sum_{j=1}^{n} j = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$S(n^2) = \sum_{j=1}^{n} j^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$S(n^3) = \sum_{j=1}^{n} j^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

$$S(n^4) = \sum_{j=1}^{n} j^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

$$S(n^5) = \sum_{j=1}^{n} j^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

$$S(n^6) = \sum_{j=1}^{n} j^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$

$$S(n^7) = \sum_{j=1}^{n} j^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2$$

$$S(n^8) = \sum_{j=1}^{n} j^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$$

$$S(n^9) = \sum_{j=1}^{n} j^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{1}{12}n^2$$

$$S(n^{10}) = \sum_{j=1}^{n} j^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - 1\,n^7 + 1\,n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$$

Bernoulli polynomials

The binomial theorem would scarcely reduce the amount of calculation. But the Bernoulli polynomial dramatically reduces the work, and the calculating is fun!

$$
\begin{aligned}
\frac{1}{11}\cdot 1000^{11} &= 90\ 909\ 090\ 909\ 090\ 909\ 090\ 909\ 090\ 909\ 090\ \tfrac{10}{11}\\
\frac{1}{2}\cdot 1000^{10} &= 500\ 000\ 000\ 000\ 000\ 000\ 000\ 000\ 000\ 000\\
\frac{5}{6}\cdot 1000^{9} &= 833\ 333\ 333\ 333\ 333\ 333\ 333\ 333\ 333\ \tfrac{2}{6}\\
-\ 1\cdot 1000^{7} &= -\ 1\ 000\ 000\ 000\ 000\ 000\ 000\ 000\\
+\ 1\cdot 1000^{5} &= 1\ 000\ 000\ 000\ 000\ 000\\
-\frac{1}{2}\cdot 1000^{3} &= -\ 500\ 000\ 000\\
+\frac{5}{66}\cdot 1000^{1} &= 75\ \tfrac{50}{66}\\
\hline
S\,(1000^{10}) &= 91\ 409\ 924\ 241\ 424\ 243\ 424\ 241\ 924\ 242\ 500
\end{aligned}
$$

Also, in the era of the computer — as in Bernoulli's time — the search for such effective rules (algorithms) for solving problems already posed is a main occupation of mathematicians.

In school one can find many occasions to study an algorithm with respect to its effectiveness. We have tried to find certain recipes to make concious step-by-step improvements. One example is the teaching of the so-called conjugate theorem. With a sufficiently extensive table of squares (like that of Crelle), you can solve multiplication tasks easily by subtraction:

$$(n+m)(n-m) = n^2 - m^2$$

which can be written more directly

$$a \cdot b = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

2. Remarks on the teaching proposals for grade 10 (or 9)

The attempt to teach by examples claims much preparation and care. But it preserves the pupils and the teacher from the worst that can happen: the deadly slumber of the eternal repetition of a canalized curriculum.

Wagenschein

Jakob Bernoulli was interested in sums obtained from the *Pascal triangle*. We start by taking certain sums out of the simple *multiplication table* as an opportunity to ask for fast calculating methods. One of our results will be for instance, that the product

$$\left[\frac{n(n+1)}{2}\right]^2$$

is fairly good for the value of $\sum_{i,j=1}^{n} i \cdot j$

On the way to this, pupils knowing the usual algebraic techniques should be completely armed, so they may also use the Bernoulli polynomials (in the beginning without proofs). And they will use this algebraic means so frequently that this teaching proposal could be titled, "A family of problems for the general repetition of basic algebraic techniques".

This teaching proposal (besides leading to other facilities) could also serve as a preparation for "harder matters" (calculus, linear algebra, probability) in the higher grades. For pupils coming from different classes, put together for the purpose of equalizing individual differences in knowledge and skill, and with the goal of deepening and widening the basis for the "harder matters" to come, it could serve as an "exemplary" approach, in the Wagenschein meaning.

As advantages of this teaching proposal we can count

— the unity and strength of the Bernoulli method of solution;
— the unity of the content (the multiplication table carries a coherent system of problems so to speak within itself);
— the importance of the results for the later integral calculus;
— the possible adaption to very different levels.

The proposal, sketched in sections 3-7, is made for usual class teaching. Therefore we avoid there the sum (Σ) signs, so as not to give the quick reader an impression of the elegance of the formulas, thinking that this is "higher mathematics" which has no place in secondary school. For a working community of interested pupils, however, we would not hold back such elegance. In a free working group, released from pressure, the problem field can be so widely exposed, that the connection between the higher arithmetical sequences and the Bernoulli polynomials and the Pascal numbers appears. "The wonderful properties of the figurate numbers", traced by Jakob Bernoulli, where "eminent secrets from the whole of mathematics are hidden..." may thus be obtained by interested pupils.

In sections 9 and 10 we sketch a teaching procedure for such a group. The "normal" teaching procedure has a weakness which we can nevertheless appreciate. It does not lift away from the average pupil the burden of indulging in intensive mathematical thinking. Though much effort is made to facilitate the creative exploration of the material through appropriate exposition and illumination, the burden is still there. But our suggestions are not for specially gifted pupils only.

Many schools have programmable calculators or computers at their disposal. This proposal should actually offer material for clever uses in this connection. Here we leave the necessary didactical considerations to the teacher, fearing that otherwise our notice would take the baroque form of a Bruckner symphony, which can only open itself after repeated listening when we have learnt to withstand its heavenly length.

3. The pattern example LEDIA as a starting point

Each problem that I solved became a rule which served afterwards to solve other problems.

Descartes

The multiplication table offers many kinds of summation problems. We start by calculating the sums along the *le*ft-running *dia*gonals.

·	1	2	3	4	5	6	. . .
1	1	2	3	4	5	6	. . .
2	2	4	6	8	10	12	. . .
3	3	6	9	12	15	18	. . .
4	4	8	12	16	20	24	. . .
5	5	10	15	20	25	30	. . .
.	.	.	.	.	.	.	
.	.	.	.	.	.	.	

$$\text{LEDIA } 5 = 1\cdot 5 + 2\cdot 4 + 3\cdot 3 + 4\cdot 2 + 5\cdot 1$$

We would probably not calculate

$$\text{LEDIA } 10000 = 1\cdot 10000 + 2\cdot 9999 + 3\cdot 9998 + \ldots + 10000\cdot 1$$

with paper and pencil, but with a programmable calculator or computer. In calculating LEDIA 10^6, will the calculator boil over?

What do we have to tell it when handling such big numbers? For the program we must express the "structure" in a much better way: with variables.

With a little intuition and with routines from elementary algebra, we can arrange this in a much simpler way.

$$(1)\ \text{LEDIA } n = 1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + \ldots + (n-1)\cdot 2 + n\cdot 1$$
$$= (1+2+3+\ldots+n)\cdot n - (1\cdot 2 + 2\cdot 3 + \ldots + (n-1)\cdot n)$$

At both ends of the expression the reformulating was blocked up for a while. With the summation sign we would not have observed this.

The first part will be replaced with a simple formula by using the Bernoulli polynomial. In the second part, it comes out that the summands are nearly square numbers

$$k(k+1) = k^2 + k$$

Then we have

$$\text{LEDIA } n = \frac{1}{2}n^3 + \frac{1}{2}n^2 - (1^1+2^2+\ldots+(n-1)^2) - (1+2+\ldots+(n-1))$$
$$= \frac{1}{2}n^3 + \frac{3}{2}n^2 + n - (1^1+2^2+\ldots+n^2) - (1+2+\ldots+n)$$
$$= \frac{1}{2}n^3 + \frac{3}{2}n^2 + n - \frac{1}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n - \frac{1}{2}n^2 - \frac{1}{2}n$$

$$(2)\ \text{LEDIA } n = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$$

Now the calculating of LEDIA 10^6 has become child's play, and we have, like Jakob Bernoulli, fun by sensible calculating.

$$\text{LEDIA } 10^6 = \frac{1}{6}(10^{18} + 3\cdot 10^{12} + 2\cdot 10^6)$$
$$= 1\,000\,003\,000\,002\,000\,000 : 6$$
$$= 166\,667\,166\,667\,000\,000$$

When n is not a power of 10, we would prefer the factorized expression

$$(2a)\ \text{LEDIA } n = \frac{1}{6}n(n+1)(n+2)$$

This is easily obtained by a quadratic completion:

$$n^2 + 3n + \frac{9}{4} - \frac{9}{4} + 2 = (n + \frac{3}{2} + \frac{1}{2})(n + \frac{3}{2} - \frac{1}{2})$$

We should like to state an intermediate result here:

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + (n-1)\cdot n = (1+2+\ldots+n) - \text{LEDIA } n$$

$$(3)\ 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + (n-1)\cdot n = \frac{1}{3}n^3 - \frac{1}{3}n$$

or

$$(3a)\qquad = \frac{1}{3}(n-1)n(n+1)$$

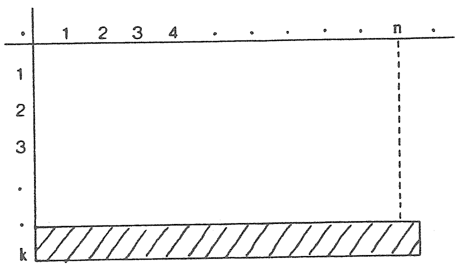
4. More sums from the multiplication table

It is not the knowing but the learning that gives the greatest pleasure.

Gauss

In complete analogy with our pattern example we consider

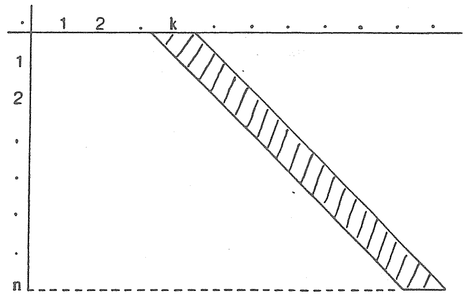
4.1. *Line sums*



$$(4)\ k \text{ LIN } n = k\cdot 1 + k\cdot 2 + k\cdot 3 + \ldots + k\cdot n \quad \text{(def)}$$

$$(5)\ k \text{ LIN } n = \frac{1}{2}kn(n+1)$$

4.2. *Right-running diagonal sums*



(6) k RIDIA $n = 1 \cdot k + 2(k+1) + 3(k+2) + \ldots + n(k+n-1)$ (def)

$$= \frac{1}{2}n(n+1)k + (1 \cdot 2 + 2 \cdot 3 + \ldots + (n-1)n)$$

and with (3a) this gives

(6a) k RIDIA $n = \frac{1}{6}n(n+1)(3k+2n-2)$

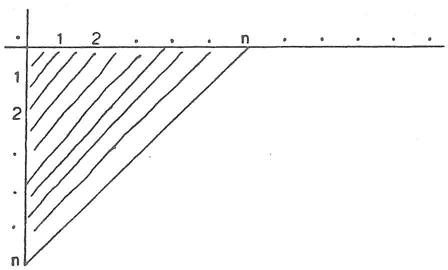
Especially we have of course

(6b) 1 RIDIA $n = 1^2 + 2^2 + 3^2 + \ldots + n^2$

$$= \frac{1}{6}n(n+1)(2n+1)$$

(We observe, that by this deduction of (6b), the snake swallows its own tail!)

4.3. *Triagonal sums.*



(7) TRI n = LEDIA 1 + LEDIA 2 + ... + LEDIA n (def)

$$= \left(\frac{1}{6}\cdot 1^3 + \frac{1}{2}\cdot 1^2 + \frac{1}{3}\cdot 1\right) + \left(\frac{1}{6}\cdot 2^3 + \frac{1}{2}\cdot 2^2 + \frac{1}{3}\cdot 2\right) + \ldots + \left(\frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n\right)$$

$$= \frac{1}{6}(1^3 + 2^3 + \ldots + n^3) + \frac{1}{2}(1^2 + 2^2 + \ldots + n^2) + \frac{1}{3}(1 + 2 + n)$$

With the help of the actual Bernoulli polynomials it follows

(7a) TRI $n = \frac{1}{24}n^4 + \frac{1}{4}n^3 + \frac{11}{24}n^2 + \frac{1}{4}n$

The factorization of the polynomial is done by the ordinary technique. One integral zero point is quickly obtained: -1. Then instead of using a method of division (no longer taught?), a display of unknown coefficients can be made

$$n^3 + 6n^2 + 11n + 6 = (n^2 + \alpha n + \beta)(n + 1)$$
$$= n^3 + (\alpha + 1)n^2 + (\alpha + \beta)n + \beta$$

By comparing, this makes

$$n^3 + 6n^2 + 11n + 6 = (n^2 + 5n + 6)(n + 1)$$

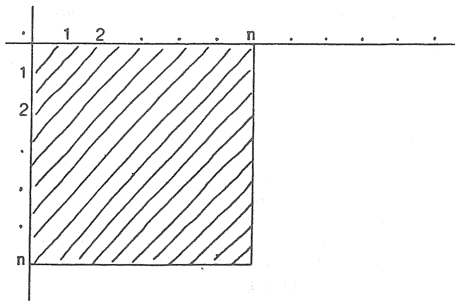
With the same method, or with quadratic completion, it follows at last

(7b) TRI $n = \frac{1}{24}n(n + 1)(n + 2)(n + 3)$

5. The Bernoulli polynomial for cubes falls into our hands

I love mathematics, not only because of its usefulness in technology, but also because it is beautiful
Rozsa

We also calculate the sums of all the numbers in a square of the multiplication table. This gives a little surprise.



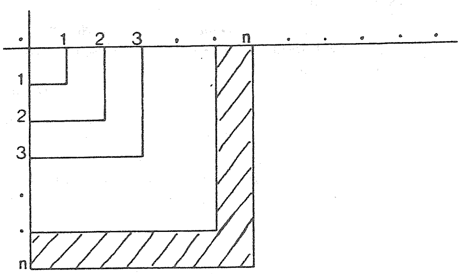
While we have short expressions for the line sums, we calculate

(8) SQUA $n = 1 \cdot$ LIN $n + 2 \cdot$ LIN $n + \ldots + n \cdot$ LIN n (def)

$$= (1 + 2 + \ldots + n) \cdot \frac{1}{2}n(n + 1)$$

(8a) SQUA $n = [\frac{1}{2}n(n + 1)]^2$

The square, however, can easily be split up into its marginals ("gnomons").



(9) $\text{MARG}\, n = 1\cdot n + 2\cdot n + \ldots + n\cdot n + n(n-1) + n(n-2) + \ldots + n\cdot 1$ def
$= 2n(1 + 2 + \ldots + n) - n^2$
$= n^2(n + 1 - 1)$

(9a) $\text{MARG}\, n = n^3$

The marginal sums are just cubic numbers, and here is the little surprise: with some algebra, nearly without effort, we get the Bernoulli polynomial for the sum of the cubic numbers

(10) $1^3 + 2^3 + \ldots + n^3 = [\frac{1}{2}n(n + 1)]^2 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}$

6. The Bernoulli polynomials for Σj and Σj^2 respectively

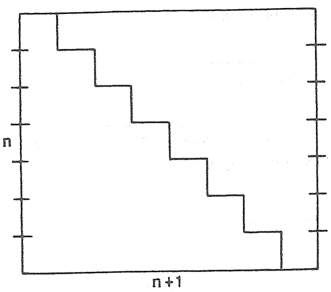
Don't swear by the name of your teacher but bring forth a proof!

Proverb from antiquity

Until now we have tested them with a few numbers, but used them without proving. Jakob Bernoulli has given directions for proof. Now we try to deduce these Bernoulli polynomials.

$$\sum_{j=1}^{n} j = 1 + 2 + \ldots + n = \frac{1}{2}n(n + 1) = \frac{1}{2}n^2 + \frac{1}{2}n$$

will be understood by the stair picture



The sum of squares is a harder nut than the corresponding sum of cubes. We are not ashamed to confess to the pupils that we don't see any straightforward way. So out of the need we make a virtue!

The pupils should also learn to read written mathematics with understanding, shouldn't they? For this kind of work the teacher should find some selected passages as a reading section, with special emphasis on their being understandable. Or better — several sections on the same theme, calling on different skills.

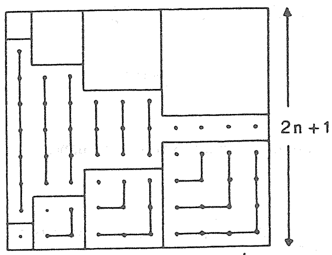
One reading section can, for instance, develop the Bernoulli polynomial from the starting point

$$\sum_{j=1}^{n} (j + 1)^3 = \sum_{j=1}^{n} j^3 + 3\sum_{j=1}^{n} j^2 + 3\sum_{j=1}^{n} j + n.$$

Another will bring in the recursive argumentation

$$\sum_{j=1}^{n+1} j^2 = \frac{1}{6}n(n + 1)(2n + 1) + (n + 1)^2$$
$$= \frac{1}{6}(n + 1)(2n^2 + 7n + 6)$$
$$= \frac{1}{6}(n + 1)(n + 2)(2n + 3)$$

A third starting point is the picture



$3\cdot(1^2 + 2^2 + 3^2 + \ldots + n^2) = \frac{1}{2}n(n + 1)(2n + 1)$

7. A reading passage from John Wallis

Even in the mathematical sciences, our principal instruments for discovering the truth are induction and analogy.

Laplace

A fourth reading passage on this same theme (simple Bernoulli polynomials) emphasizes less the finite power sums, but more the connecting limit values. We will guide our *ordinary pupils* some steps along the way of John Wallis (1616-1713). On the way they discover through well-planned numerical calculation some patterns: the simple Bernoulli polynomials, and quite easily still more: the usable approximation

$$\sum_{j=1}^{n} j^k \approx \frac{1}{k+1} n^{k+1} + \frac{1}{2}n^k$$

We let our *ordinary pupils* read a page from John Wallis' *Arithmetica infinitorum* [1665]. It may be found in Struik's source book. (See Appendix). This text gives enough material for a conversation among other things about the circumstances of the sums of cubes (Prop.40). Would not our pupils — accustomed to formula language (?!) — easily note

$$\frac{0^3 + 1^3 + 2^3 + \ldots + n^3}{n^3 + n^3 + n^3 + \ldots + n^3} = \frac{1}{4} + \frac{1}{4n}$$

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4n}(n+1)(n+1)n^3$$

(10a) $1^3 + 2^3 + \ldots + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \ldots$ (lower powers of n)

Do they really understand the Wallis text? Given the task of handling the sums $1+2+\ldots+n$ and $1^2+2^2+\ldots+n^2$ respectively according to his pattern we put them on trial.

From the results

$$\frac{0 + 1 + 2 + \ldots + n}{n + n + n + \ldots + n} = \frac{1}{2} \quad ; \quad \sum_{j=1}^{n} j = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\frac{0^2 + 1^2 + 2^2 + \ldots + n^2}{n^2 + n^2 + n^2 + \ldots + n^2} = \frac{1}{3} + \frac{1}{6n} \; ;$$

$$\sum_{j=1}^{n} j^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \ldots \text{ (lower powers of } n\text{)}$$

our ordinary pupils rise with Wallis even higher

$$\sum_{k=1}^{n} j^k = \frac{1}{k+1} n^{k+1} + \frac{1}{2}n^k + \ldots \text{ (lower powers of } n\text{)}$$

This gives useful approximations for power sums, important results for the coming integral calculus.

Using this result for the sum

$$1^{10} + 2^{10} + \ldots + 1000^{10}$$

we need, instead of Jakob Bernoulli's half a quarter of an hour, only a fraction, and we get

$$\sum_{j=1}^{1000} j^{10} \approx \frac{1}{11} \cdot 10^{33} + \frac{1}{2} \cdot 10^{30}$$

$$= \frac{2011}{22000} \cdot 10^{33}$$

= 91 409 100 000 000 000 000 000 000 000 000 correct to 5 digits

This approach to approximation, emphasizing limits of sequences, without doubt will prepare the ground for the calculus course.

8. A historical remark

Archimedes will be remembered when Aeschylus is forgotten because languages die and mathematical ideas do not.

G.H.Hardy

Trying to discover the truth with the help of induction and analogy, you need—by small conjectures and harder patterns as well—a good intuition, a good nose. Wallis in his *Arithmetica infinitorum* goes quickly from the limit value just mentioned to the utmost generality of the Cavalieri result.

$$\int_0^1 x^k \, dx = \frac{1}{k+1} \quad \text{for all } \textit{real} \text{ numbers } k \neq 1$$

"Thereupon, he plunged into a maelstrom of numerical work, and with fine mathematical intuition to guide him in his interpolations, arrived at the infinite product for π that bears his name" (Struik):

$$\frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n+1)(2n+1)}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \ldots \cdot 2n \cdot (2n+2)} \cdot \sqrt{1 + \frac{1}{2n+1}} < \frac{4}{\pi}$$

$$< \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n+1)(2n+1)}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \ldots \cdot 2n \cdot (2n+2)} \sqrt{1 + \frac{1}{n+2}}$$

What we now try to sketch with a few strokes is a paradigm showing that induction is an exceptional tool for discovering relationships.

Wallis knew $\int_0^1 (x - x^2)^{1/2} dx = \frac{\pi}{8}$, from the area of the circle.

His nose immediately picked up the scent:

$$\int_0^1 (x - x^2)^n dx = ?$$

Calculate, for $n = 0, 1, 2, 3, \ldots$ (?). Generalize.

$$\int_0^1 (x - x^2)^n dx = \frac{(n!)^2}{(2n)!(2n+1)}$$

With a mental jump

$$\frac{\pi}{8} = \frac{(\frac{1}{2}!)^2}{(2 \cdot \frac{1}{2})!(2 \cdot \frac{1}{2} + 1)}$$

To understand $n!$ as $1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ is a too narrow setting. Widen it! But how??? With a "free" variable k, which in turn will be killed by a limit process

$$n! = \frac{k! \, k^n \, (1+\frac{1}{k}) \cdot (1+\frac{2}{k}) \cdot \ldots \cdot (1+\frac{n}{k})}{(1+n)(2+n) \ldots (k+n)}$$

$$n! = \lim_{k \to \infty} \frac{k! \, k^n}{(1+n)(2+n) \ldots (k+n)}$$

The rest is not worth mentioning; calculate it!

The French, especially Fermat, criticized Wallis. They claimed an unfailing proof of this result which was so easily obtained. They claimed a complete mathematical induction (or a variant of this). Jakob Bernoulli worked through Wallis' *Arithmetica infinitorum* and gave such a proof of the power sums.

The use of mathematical induction in a simple case can be understood by *normal pupils* too, as long as we avoid its confusing logical subtleties.

9. A steeper path for a working community of interested pupils

> *A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.*
> *Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime.*
>
> Polya

The teacher will not let "the young mathematicians" fade out but invites them to more demanding work on the same theme. First let them study a text considering Pascal numbers and recursive methods. In German this can be found in the textbook Stowasser/Mohry *Rekursive Verfahren*, Schroedel Verlag 1978, a book which has the stimulation of interest as its aim.

Afterwards the pupils will be able to use recursive methods in simple cases, know the combinatorial meaning of the Pascal numbers and more than its two basic properties

(11) $\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$

(12) $\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1}$

Our interested pupils will, after some calculating, find themselves discovering that certain sums in the multiplication table are connected with the Pascal numbers. The steep path goes by some trials to the conjecture

$$\sum_{i+j=n+1} i\cdot j = \binom{n+2}{3}$$

The recursive thinking must be supported by examples. The heart of the matter can be made clear without variables

$$\text{LEDIA } 7 = \sum_{i+j=7}^{n} i\cdot j = 1\cdot 6 + 2\cdot 5 + 3\cdot 4 + 4\cdot 3 + 5\cdot 2 + 6\cdot 1$$
$$= (6+5+4+3+2+1) + (1\cdot 5 + 2\cdot 4 + 3\cdot 3 + 4\cdot 2 + 5\cdot 1)$$
$$= \binom{7}{2} + \binom{7}{3} \text{ (induction hypothesis)}$$
$$= \binom{8}{3}$$

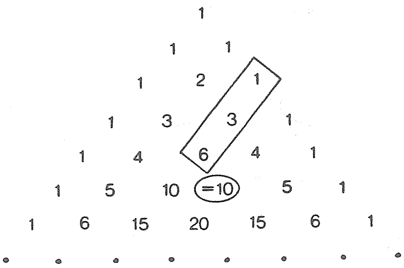
Here we have taken one step upwards. Further steps gives

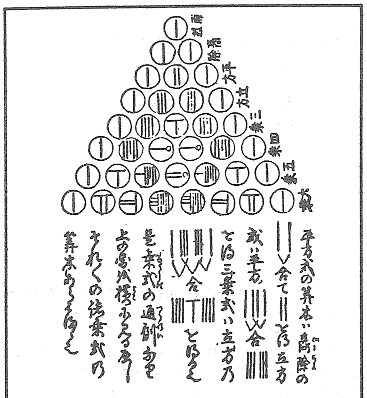
$$\binom{8}{3} = \binom{7}{2} + \binom{6}{2} + \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}$$

This will soon tempt us to generalize

(13) $\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1}$

This is however the *main point in Jakob Bernoulli's deduction of the polynomials for power sums and also for the higher arithmetical sequences in general.* Bernoulli's main theorem follows immediately from (13) and vice versa. The relation of (13) to the Pascal triangle is illustrated by the following example.





Pascal triangle in Japan

From Murai Chūzen's Sampō Dōshi-mon (1781), which also shows the Sangi forms of the digits.

The recursive proof of (13) is trivial. We have

$$\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1} + \binom{n+1}{k}$$

according to the induction hypothesis

$$= \binom{n+2}{k+1}$$

according to (12).

The main point (13) was also stated by Blaise Pascal in his famous paper "Traité du triangle arithmétique...", Paris 1665, in an elegant way. Therefore we are surprised that Jakob Bernoulli had so much trouble to prove his main theorem: 5 lemmas, 4 pages.

Just like Bernoulli we fetch the Bernoulli polynomials from (13):

k = 1:

$$\binom{n+1}{2} = \sum_{j=1}^{n} \binom{j}{1} = \sum_{j=1}^{n} j$$

k = 2:

$$\binom{n+1}{2} = \sum_{j=1}^{n} \binom{j}{2} = \frac{1}{2} \sum_{l=1}^{n} j\,(j-1) = \frac{1}{2} \sum_{j=1}^{n} j^2 - \frac{1}{2} \sum_{j=1}^{n} j$$

From this it follows

$$\sum_{j=1}^{n} j^2 = \frac{2}{6}(n+1)\,n\,(n-1) - \frac{1}{2}(n+1)\,n$$
$$= \frac{1}{3}\binom{n+1}{2}(2n+1)$$

In the same way it follows for k=3, from

$$\binom{n+1}{4} = \sum_{j=1}^{n} \binom{j}{3}$$

$$\sum_{j=1}^{n} j^3 = \binom{n+1}{2}^2$$

etc.

Armed with the summation sign, the further sums in the multiplication table should offer no resistance to our young mathematicians.

"One has to care that the symbols are well suited to discoveries. Usually this is obtained when the symbols express some elements of the intrinsic nature of the concept - *then the mental work is reduced in a remarkable way.*"[*Newton, quoted in Wussing, Isaac Newton,* Leipzig 1977)

The obstacles met in converting, for example, LEDIA *n* (Section 3) will be overcome easily.

$$\sum_{i+j=n+1} i \cdot j = \binom{n+2}{3}$$

$$\sum_{i+1 \leqq n+1} i \cdot j = \binom{n+3}{4} = \sum_{k=1}^{n} \sum_{j=1}^{n} j\,(k-j+1)$$

$$\sum_{i,j \leqq n} i \cdot j = \binom{n+1}{2}^2 = \sum_{k=1}^{n} \sum_{j=1}^{n} k \cdot j = \sum_{j=1}^{n} j^3$$

etc.

To our young mathematicians we offer the "quadratic multiplication table" for a similar exploration.

·	1	4	9	16	.	.	.
1	1	4	9	16	.	.	.
4	4	16	36	64	.	.	.
9	9	36	81	144	.	.	.
16	16	64	144	256	.	.	.
.	.	.	.	.			
.	.	.	.	.			

The comparison of these two tables can raise interesting problems, as for example: Under what conditions will LEDIA *n* divide the corresponding diagonal sum in the "quadratic multiplication table"?

Afterwards the original text by Jakob Bernoulli from 1713, the chapter on Pascal numbers, should be excellent reading for our young mathematicians. Hopefully both the work and the quotations have highly motivated them. It is rewarding to look into the workshop of a master to find encouragement and ideas for further work.

To end, Jakob Bernoulli's beautiful results of the sums of higher arithmetical sequences, the ideas in a system of exercises on figurate numbers, may be appropriate. The chapter "Figurierte Zahlen" in the already mentioned book *Rekursive Verfahren* gives such a system.

The journal *Mathematiklehrer* (1/82) gave under the heading "Forschungsaufgaben" some starting paths into a great mountain landscape of problems: figurate numbers, Pascal and Leibniz numbers, Pell triangle, etc. Readers were challenged to journey into this landscape. This article can be seen as such a trip, trying to take the objective of "multirelated mathematics" (Freudenthal) seriously. We hope that readers will put the content of the article on trial, asking "Does it show some interrelationships between the different parts of mathematics?"