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Arithmetic and algebra : can history help to close the cognitive gap ?
A proposed learning trajectory on early algebra from an historical perspective

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Abstract

Is it possible for students to self-handedly gain access to early algebra starting from informal strategies embedded in arithmetic? Can apparently fundamental differences between arithmetical and algebraic conceptions of mathematical problems be (partly) surmounted? The historical development of algebraic problem solving and algebraic symbolic language has inspired the author to develop a prototype pre-algebra learning strand on reasoning and equation solving. This article sketches the project background and gives some examples of classroom activities.

Introduction

Several algebra research projects of the last decade report on poor student performance when it comes to solving linear equations (KIERAN 1989, 1992; FILLOY & ROJANO 1989; SFARD 1991, 1996; HERSCOVICS & LINCHEVSKI 1994, 1996; BEDNARZ et al. 1996). Secondary school students often have trouble learning how to construct equations from arithmetical word problems, and how to rewrite, simplify and interpret algebraic expressions. It is conjectured that part of the problem is caused by fundamental differences between arithmetic and algebra (FILLOY & ROJANO 1989; HERSCOVICS & LINCHEVSKI 1994; BEDNARZ & JANVIER 1996; MASON 1996). Arithmetical problems, for instance, involve straightforward calculations with known numbers, whereas algebra requires reasoning about unknown or variable quantities and recognizing the difference between specific and general situations. In the transition from arithmetic to algebra there is claimed to be a discrepancy called *cognitive gap* (HERSCOVICS & LINCHEVSKI 1994) or *didactic cut* (FILLOY & ROJANO 1989), hampering manipulations of algebraic expressions.

A good starting point for an investigation into this matter could be a return to the roots. In this project we shall try to gain insight into the differences and similarities between arithmetic and algebra by looking into the historical development of algebra and learning from past experiences. Recent research on the advantages and possibilities of using and implementing history of mathematics in the classroom has led to a growing interest in the role of history of mathematics in the learning and teaching of mathematics[1]. Inspired by the HIMED (History in Mathematics Education) movement, a developmental research project called 'Reinvention of Algebra' was started at the Freudenthal Institute in 1995 to investigate which didactical means will enable students to make a smooth transition from arithmetic to early algebra. Specifically, the 'invention' of algebra from a historical perspective will be compared with possibilities of 're-invention' by the students. This paper first sketches the background of the project (part 1) and then gives a brief outline of the learning strand and some classroom impressions (part 2).

1 Background of the project

1.1 Research motive

The American Middle School Project (VAN REEUWIJK 1995) and small experiments in The Netherlands (ABELS 1994, STREEFLAND 1995) revealed a great number of accesses into algebra for relatively young learners. Ten- and eleven-year-olds have shown that they can reason algebraically in problem situations that are familiar and meaningful to them. The level of knowledge, skills and abilities of the children, and in some cases the mathematics itself, are the driving forces of the teaching-learning process. A similar observation can be made for the historical development of algebra, where both practical needs in society and internal motivation led to further progress. Given the fact that historical developments play an increasingly important role in the teaching and learning of mathematics, one of the project's aims is to investigate if history of mathematics can be a useful didactical tool. The application will be twofold: history as a guide for the hypothetical learning trajectory, and history as a rich source of mathematical problems and learning moments. Moreover, the historical development of algebra can shed light on the ruptures between arithmetical and algebraic modes of thinking.

[1]For example, work by Fauvel, Van Maanen, Kool, Arcavi, Eagle and many more; special issues *For the Learning of Mathematics 11-2* (1991) and *Mathematical Gazette 76* (1992); discussion document for an ICMI study by Fauvel and Van Maanen, *Educational Studies in Mathematics 34-3*, 255-259.

A decade ago, the algebra working group of the W12-16 project designed a new approach of algebra for the first three years in Dutch secondary schools (Algebragroep W12-16, 1990, 1991; W12 -16 COW, 1992). In this new algebra, algebraic relations play a very important role. Students develop algebraic conceptions and skills very gradually from concrete situations by switching between different forms of representation: descriptions of situations, tables, graphs and formulas. However, since the implementation of the new program it has become increasingly clear that the learning of algebraic skills like manipulation of formulas and equations still needs to be improved. Consequently, we have decided to attempt another approach. Inspired by the historical development of algebra, we will investigate accesses to algebra within the context of story problems and solving equations. Developmental research will be carried out on the teaching-learning process of the teachers as well as the groups of students involved, to determine whether the discrepancy between arithmetic and algebra can be minimized. But before going into more detail, a brief description is called for of two standpoints -on mathematics education and educational research- which are at the heart of this project.

1.2 Developmental research and Realistic Mathematics Education

Developmental research is a type of educational research whereby design of instructional material is an integrated part of the research method. In a cyclic process of anticipating and testing, new ideas on teaching and learning mathematics are developed and tried out in classroom experiments. In order to construct a hypothetical learning trajectory, educational designers can make use of heuristics such as the reinvention principle and didactical phenomenology (FREUDENTHAL 1983, 1991). Analysis of the classroom results leads to the formation of theory, which in turn is used to improve the instructional design. The completion of various cycles -in this project there have been three- will result in a product which is theoretically and empirically founded. So developmental research yields not only a new learning strand on a certain topic, but also a theory on the preferred way in which the topic should be taught and learnt. 'The preferred way' in our opinion is one according to the didactical vision of Realistic Mathematics Education (RME), which propagates the teaching and learning of mathematics as a human activity.

In agreement with the tradition of RME, the founding principles of the early algebra learning strand are:

- to create rich problem situations that are meaningful to students, either in the real world or in their mathematical experience
- to construct activities that offer opportunities for mathematizing, modeling and schematizing, not only as problem solving tools but also as a means to formalize mathematical thinking
- to choose contexts that students are familiar with to serve as frameworks of reference
- to enable students to construct their own mathematics, starting from informal knowledge and strategies and progressively building up a more formal mathematical understanding
- to instigate interactive reflection (student-student and student-teacher) and student participation in establishing algebraic conventions.

(For more information on developmental research and the RME tradition, see FREUDENTHAL 1983, 1991; TREFFERS 1978; GRAVEMEIJER 1994; VAN DEN HEUVEL-PANHUIZEN 1996).

1.3 Subject matter: algebra and arithmetic

Algebra has many faces and is therefore difficult to define. But for the sake of practicality, it is useful to distinguish four basic perspectives of school algebra: algebra as generalized arithmetic, algebra as a problem-solving tool, algebra as the study of relationships, and algebra as the study of structures. Each of these operates in a different medium, where for example letters have a specific meaning and role (USISKIN 1988). In this research project we have decided to restrict ourselves to linear relationships, formulas and equation solving. The proposed learning activities belong to the first three perspectives of school algebra as mentioned, and assume a dialectic relationship between algebra and arithmetic.

A closer look at the similarities and differences between algebra and arithmetic can help us understand some of the problems that students have with learning algebra. In bold terms, arithmetic deals with numbers and algebra with letters - letters that can stand for numbers. But the essential difference lies deeper. Several researchers (Booth 1988; KIERAN 1989, 1992; SFARD 1991, 1996) have studied problems related to the recognition of mathematical structures in algebraic expressions. Kieran speaks of two conceptions of mathematical expressions: *procedural* (concerned with operations on numbers, working towards an outcome) and *structural* (concerned with operations on mathematical objects) (or *operational* and *structural* respectively, SFARD 1996). The contrasting natures of algebra and arithmetic in this respect will be discussed in connection with the theoretical conjectures later in this paper.

And yet there is a definite interdependency: algebra relies heavily on arithmetic operations and arithmetic expressions are sometimes treated algebraically. And word problems have always been and still are a part of mathematics that algebra and arithmetic have in common. A summary of the historical development of algebra[2] can shed more light on how algebra has its roots in arithmetic.

1.4 Historical development of algebra

It is generally accepted to distinguish three periods in the development of algebra (oversimplifying, of course, the complex history in doing so!), according to the different forms of notation: rhetorical, syncopated and symbolic (see also table 1)[3]. From ancient times until about 500 years ago, with the exception of Diophantus and a number of other mathematicians who used abbreviations and symbols, both the problem itself and the solution process were mostly written in only words (*rhetorical* notation). Early algebra was a more or less sophisticated way of solving word problems. A typical rule used by the Egyptians and Babylonians for solving problems on proportions is the Rule of Three: given three numbers, find the fourth. Such problems are commonly classified as arithmetic, but in situations where numbers do not represent specific concrete objects and where operations are required on unknown quantities, we can speak of algebraic problems. Another commonly used method for solving word problems is called the Rule of False Position, first used systematically by Diophantus (TROPFKE 1980). According to

[2]The historical overview is confined to the research topic and therefore "algebra" will be limited here to early algebra, in particular the field of algebraic notation, word problems and linear equations.

[3]The classification of algebra into rhetorical, syncopated and symbolic algebra first appeared in G.H.F. Nesselmann, *Die Algebra der Griechen*, Berlijn 1842 (STRUIK 1990, p. 78).

this rule one is to assume a certain value for the solution, perform the operations stated in the problem, and depending on the error in the answer, adjust the initial value using proportions. Although the Rule of False Position is generally not said to be an algebraic algorithm, its wide acceptance and perseverance even after the invention of symbolic algebra indicate it was and can still be a very effective problem solving tool.

	rhetoric	syncopated	symbolic
written form of the problem	only words	words and numbers	words and numbers
written form in the solution method	only words	words and numbers; abbreviations and mathematical symbols for operations and exponents	words and numbers; abbreviations and mathematical symbols for operations and exponents
representation of the unknown	word	symbol or letter	letter
representation of given numbers	specific numbers	specific numbers	letters

Table 1: characteristics of the 3 types of algebraic notation

Depending on the number concept of each civilization as well as the mathematical problem, the unknown could be a quantity or a measure and was denoted by words like "heap" (Egyptian), "length" or "area" (Babylonian, Greek), "thing" or "root" (Arabic), "cosa", "res" or "ding(k)" (Western). The solution was given in terms of instructions and calculations, with no explanation or mention of rules. The unknowns were treated as if they were known; reasoning about an undetermined quantity apparently did not form a conceptual barrier. For instance, in the case of problems that we would nowadays represent by linear equations of type $x + \frac{1}{n}x = a$, the unknown quantity x was conveniently split up into n equal parts.

Diophantus (ca. 250 AD) invented shortened notations (*syncopated* algebra) which enabled him to rewrite a mathematical problem into an 'equation' (abbreviated form). He systematically used abbreviations for powers of numbers and for relations and operations. In his equations he used the symbol ζ to denote the unknown and additional unknowns were derived from it. TROPFKE (1980) explains that this change from representing the unknown by words to symbols really persevered only once the symbols were also used in the calculations. He gives two arguments to indicate that Diophantus appears to have been the first mathematician to do so. Firstly, Diophantus performed arithmetic operations on powers of the unknowns, carrying out additions and subtractions of like terms self-evidently without explicitly stating any rules. And secondly, he explained the method and purpose of adding and subtracting like terms on both sides of an equation. (TROPFKE 1980, p. 378).

After Diophantus there were other practitioners of syncopated algebra. In India (7^{th} century AD) words for the unknown and its powers -which were extended in a systematic way- were abbreviated to the first or the first two letters of the word. Additional unknowns were named after different colors. In Arabic algebra (9^{th} century AD) powers of the unknown were also built up consecutively, using the terms for the second and third power of the unknown as base. In abbreviated form, the first letter of these words was written above the coefficient. In Western Europe (13^{th} century) there were minor differences in the technical terms between Italy and Germany, and only in the second half of the 14^{th} century the words "res" and "cosa" were shortened to r and s respectively. In the middle of the 16th century Stifel introduced consecutive letters for unknowns and stated arithmetical rules using these letters. From there Buteo, Bombelli, Stevin, Recorde (see figure 1) and many others developed a system to symbolize powers of unknowns and formulate equations. (TROPFKE 1980, pp. 377-378). Recorde introduced the equals-sign in print, saying: "And to avoid the tedious repetition of these words: is equal to: I will set as I do often in work use, a pair of parallels, or Gemowe lines of one length, thus: =, because no 2 things, can be more equal." [4] (EAGLE 1995, p. 82).

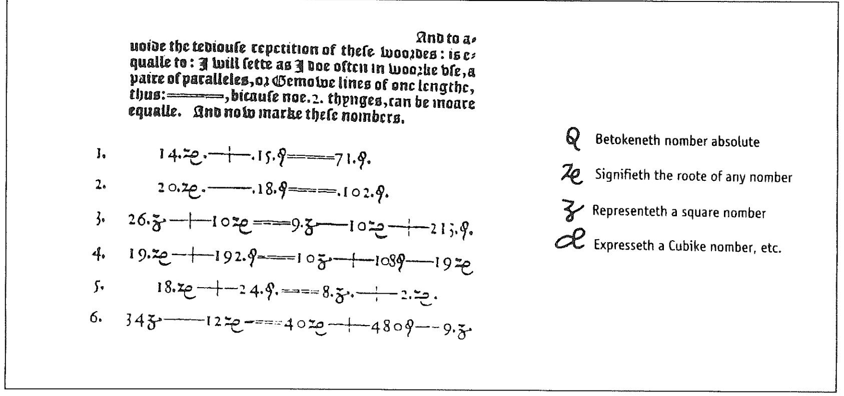

And to avoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus: ═══, bicause noe. 2. thynges, can be moare equalle. And now marke these nombers.

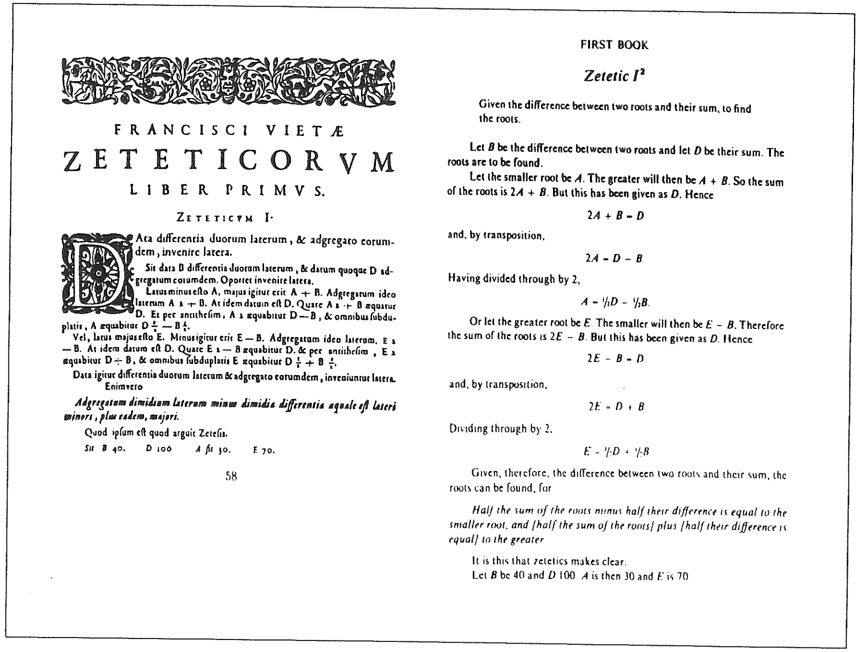
Figure 1: Algebraic notations in Western Europe
Date: Recorde (1557), *The Whetstone of Witte*
Source: EAGLE (1995), *Exploring Mathematics through History*

In the rhetorical and syncopated periods we see a certain degree of standardization. Routine solving procedures were based on the specific numerical properties of standard problems. Diophantus, Arabic mathematicians and the mathematicians in Western Europe contributed a variety of general methods of solving indeterminate, quadratic and cubic equations. But with the lack of a suitable language to represent the *given* numbers in the problem, it was still a difficult task to write the procedures down legibly. In a few isolated cases geometrical identities were expressed algebraically (with variables instead of numbers) but nonetheless written in full sentences. Syncopated notation did not (yet) enable mathematicians to take algebra to a higher level: the level of generality. It is important that students experience this limitation themselves in order to appreciate the value and power of modern mathematical notation.

The development of algebraic notation in the 16^{th} century was a process still instigated by problem solving (see also RADFORD 1995). In 1591 Viète introduced a system for denoting the unknown as well as given numbers by capital letters, resulting in a new number concept: "algebraic number concept" (HARPER 1987). The signs and symbols became separated from that what they represent (a context-bound number) and symbolic algebra became a mathematical object in its own right. For a Vietan solution to a typical Diophantine problem, see figure 2 below. A few decades later Descartes proposed the use of small letters as we do nowadays: letters early in the alphabet for given numbers, and letters at the end of the alphabet for unknowns. With the creation of this new language system, earlier notions of the "unknown" had to be adjusted. The first objective had always been to uncover the value of the unknown, but in the new symbolic algebra the unknown served a higher purpose, namely to express generality.

[4]Gemowe lines mean twin lines, as in Gemini (EAGLE 1995).

Algebra as generalized arithmetic was a fact, and in its new role algebra detached itself from arithmetic.


FRANCISCI VIETÆ

ZETETICORVM

LIBER PRIMVS.

ZETETICVM I.

DAta differentia duorum laterum, & adgregato eorumdem, invenire latera.

Sit data B differentia duorum laterum, & datum quoque D adgregatum eorumdem. Oportet invenire latera.

Latus minus esto A, majus igitur erit A + B. Adgregatum ideo laterum A 2 + B. At idem datum est D. Quare A 2 + B æquatur D. Et per antithesim, A 2 æquabitur D — B, & omnibus subduplatis, A æquabitur D ½ — B ½.

Vel, latus majus esto E. Minus igitur erit E — B. Adgregatum ideo laterum, E 2 — B. At idem datum est D. Quare E 2 — B æquabitur D. & per antithesim, E 2 æquabitur D + B, & omnibus subduplatis E æquabitur D ½ + B ½.

Data igitur differentia duorum laterum & adgregato eorumdem, inveniuntur latera. Enimvero

Adgregatum dimidium laterum minus dimidia differentia æquale est lateri minori, plus eadem, majori.

Quod ipsum est quod arguit Zetesis.

Sit B 40. D 100 *A fit* 30. E 70.

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FIRST BOOK

Zetetic I[2]

Given the difference between two roots and their sum, to find the roots.

Let *B* be the difference between two roots and let *D* be their sum. The roots are to be found.

Let the smaller root be *A*. The greater will then be *A* + *B*. So the sum of the roots is 2*A* + *B*. But this has been given as *D*. Hence

2*A* + *B* = *D*

and, by transposition,

2*A* = *D* − *B*.

Having divided through by 2,

A = ½*D* − ½*B*.

Or let the greater root be *E*. The smaller will then be *E* − *B*. Therefore the sum of the roots is 2*E* − *B*. But this has been given as *D*. Hence

2*E* − *B* = *D*

and, by transposition,

2*E* = *D* + *B*

Dividing through by 2,

E = ½*D* + ½*B*

Given, therefore, the difference between two roots and their sum, the roots can be found, for

Half the sum of the roots minus half their difference is equal to the smaller root, and [half the sum of the roots] plus [half their difference is equal] to the greater

It is this that zetetics makes clear.
Let *B* be 40 and *D* 100. *A* is then 30 and *E* is 70

Figure 2: Symbolic algebra
Date, left: Viète (1593), *Zeteticorum Liri Quinque*; date, right: Witmer (1983)
Source, left: Latin text from F. van Schooten's edition, p. 42 (Leiden, 1646, reprint in Hofman, 1970); source, right: English translation in Witmer (1983), p. 83-84.

The historical development of equations in particular shows that, no matter how revolutionary, symbolic algebra was not a necessity for the existence of equations. That is, if we allow other forms of notation than the conventional symbolic one. As a matter of fact, linear equations were very common in Egypt, and the Babylonians already knew how to solve equations of the first, second and third degree. In order to solve with the method of elimination the following system of equations (given in modern notation):

$$\begin{aligned} x + \frac{1}{4}y &= 7 \\ x + y &= 10 \end{aligned}$$

the first equation was multiplied by 4 and the second equation was then subtracted from the first, which gave $3x = 18$. Hence $x = 6$, and from the second equation it followed that $y = 4$.

In a very different part of the world a systematic treatment of solving equations developed in ancient China. Just like the ancient civilizations, the Chinese lacked a notational system of writing problems down in terms of the unknowns, but the computational facilities of the rod numeral system enabled them to surpass the rest of the world in equation solving. The *Jiu zhang suanshu* or 'Nine Chapters on the Mathematical Art' (206 BC to 220 AD) is the oldest book known until now that contains a method of solving any system of n simultaneous linear equations with n unknowns, with worked-out examples for n = 2,3,4 and 5. This was done using the method *fang cheng* (calculation by tabulation), writing the coefficients down or organizing them on the counting board in a tabular form and then performing column operations on it (much like the Gauss elimination method of a matrix). The general application of the *fang cheng* method led quite naturally to negative numbers and some rules on how to deal with them, which is in great contrast with the late acceptance of negative numbers in other parts of the world.

Diophantus certainly demonstrated a pursuit of generality of method, but his first concern was to find a (single) solution for each problem. The *Arithmetica* (ca. 250 AD) is a collection of about 150 specific numerical problems that exemplify a variety of techniques for problem solving. Diophantus distinguished different categories and systematically worked through all the possibilities, reducing each problem to a standard form. Negative solutions were not accepted, and if there was more than one solution, only the largest was stated. He solved linear equations in one unknown by expressing the unknown and the given numbers in terms of their sum, difference and proportion. If a problem contained several unknowns, he expressed all the unknowns in terms of only one of them, thereby dealing with successive instead of simultaneous conditions. Diophantus is also known for his treatment of indeterminate equations: equations of the second degree and higher with an unlimited amount of rational solutions. Once again the general method involved reducing the problem to one unknown and finding a single solution.

The Arabs also played an important role in the historical development of equation solving. Although the boundaries of this research project have been set at (systems of) linear equations, their achievements on quadratic and cubic equations deserve mentioning. An influential book on Arabic algebra is al-Khwarizmi's *Hisab al-gabr wa-l-muqabala* (early 10^{th} century). It contains a clear exposition of the solutions of six standard equations, followed by a collection of problems to illustrate how all linear and quadratic equations can be reduced to these standard forms. Al-Khwarizmi also gave geometric proofs and rules for operations on expressions, including those for signed numbers, even though negative solutions were not accepted at that time. But as far as the difficulty of the problems and the notations are concerned, the book remained behind compared to the work of Diophantus; everything was written in words, even the numbers. The Arabs did not succeed at solving cubic equations algebraically, but in the 11^{th} century AD Omar Khayyam presented a well-known yet incomplete treatise on solving cubic equations with geometric means.

Arabic algebra became known in the Western world in the 12^{th} century, when al-Khwarizmi's work was translated by Robert of Chester. Two centuries later, mathematical textbooks on arithmetic and algebra were very common in certain parts of Europe, and equation solving (even of the third and fourth degree) had become a regular subject in the Italian abbacus schools. In 1545 Cardano presented the solution of the general cubic equations by means of radicals. After the invention of symbolic algebra, equation solving developed very rapidly and soon found new applications in other areas of mathematics.

1.5 History in mathematics education

We would not plead for the use of history of mathematics in mathematics education if we did not believe that history has something extra to offer. It can benefit students, teachers, curriculum developers and researchers in different ways. Students can see the subject in a new light, they will have a notion of processes and progress, they will learn about social and cultural influences, to name just a few advantages (FAUVEL 1991). Teachers may find that information on the development of a mathematical topic makes it easier to tell, explain or give an example to students. It also helps to sustain the teacher's interest in mathematics. And history of mathematics can give the educational developer or researcher more insight into the subject matter and perhaps even the learning process.

Another argument for using history in education is the so-called Biogenetic Law popular at the beginning of this century. The Biogenetic Law states that mathematical learning in the individual (philogenesis) follows the same course as the historical development of mathematics itself (ontogenesis). However, it has become more and more clear since then that such a strong statement cannot be sustained. A short study of mathematical history is sufficient to conclude that its development is not as consistent as this law would require. Freudenthal also warns against unthinkingly accepting the Biogenetic Law in the following passage on 'guided reinvention':

> Urging that ideas are taught genetically does not mean that they should be presented in the order in which they arose, not even with all the deadlocks closed and all the detours cut out. What the blind invented and discovered, the sighted afterwards can tell how it should have been discovered if there had been teachers who had known what we know now [. . . . . .] It is not the historical footprints of the inventor we should follow but an improved and better guided course of history. (FREUDENTHAL 1973, pp. 101, 103).

In other words, we can still find history helpful in designing a hypothetical learning trajectory and use parts of it as a guideline. HARPER (1987), for example, argues that algebra students pass through consecutive stages of equation solving, using more sophisticated strategies as they become older, in a progression similar to the historical evolvement of equation solving. He pleads for more awareness of these levels of algebraic formalism in algebra teaching.

There are different ways of implementing history in educational design. First of all, history can be used as a designer guide. Milestones in the development of mathematics are indications of conceptual obstacles. We can learn from the ways in which these obstacles were conquered, sometimes by attempting to travel the same course but at other times by deliberately using a different approach. 'Reinvention' does not mean following the path blindly. On the contrary, it means that developers need to be selective and should attempt to set out a learning trajectory in which learning obstacles and smooth progress are in balance. History can set an example but also a non-example. And secondly, we can choose between a direct and an indirect approach, bringing history into the open or not. Learning material can be greatly enriched by integrating historical solution methods and pictures and fragments taken from original sources, but in some situations it may be more appropriate that only the teacher knows the historical background.

Having decided to use history of mathematics as a source of inspiration for both the researcher and the students, it has become an important issue to find out in this project what the effect is. We aim to determine:

- how the historical development of algebra compares with the individual learning process of the student following the proposed learning program;
- whether or not historical problems and texts indeed help students to learn algebraic problem solving skills.

1.6 'Reinvention of algebra'

In order to facilitate the 'reinvention of algebra' in the classroom, we need to find out where the historical development of algebra indicates accesses from arithmetic into algebra. Historically, word problems form an obvious link between arithmetic and algebra. Although algebra has made it much simpler to solve word problems in general, it is remarkable how well specific cases of such mathematical problems were dealt with before the invention of algebra, using arithmetical procedures. Some types of problems are even more easily solved without algebra! One important characteristic of algebra, the ability to reason with unknown or variable quantities, can be trained within an arithmetical context. Another possible access is based on notation use, for instance by comparing the historical progress in symbolization and schematization with that of modern students. Thirdly, we could study old textbooks on early algebra in order to learn more about how algebra was understood and applied just after it became accepted.

Despite the clear bond between algebra and arithmetic shown by the historical development of algebra, one look at a schoolbook is enough to realize that they still seem to be separate worlds. Decades ago it was already clear that inconsistencies between arithmetic and algebra can cause great difficulties in early algebra learning. The difficulty of algebraic language is often underestimated and certainly not self-explanatory: "Its syntax consists of a large number of rules based on principles which, partially, contradict those of everyday language and of the language of arithmetic, and which are even mutually contradictory." (FREUDENTHAL 1962, p. 35). He then says:

> The most striking divergence of algebra from arithmetic in linguistic habits is a semantical one with far-reaching syntactic implications. In arithmetic 3 + 4 means a problem. It has to be interpreted as a command: add 4 to 3. In algebra 3 + 4 means a number, viz. 7. This is a switch which proves essential as letters occur in the formulae. a + b cannot easily be interpreted as a problem. (FREUDENTHAL 1962, p. 35)

The two interpretations (arithmetical and algebraic) of the sum 3+4 in the citation above correspond with the terms procedural and structural used by Kieran.

We also need to consider how notation and concept formation are related. SFARD (1991) conjectures that symbolic algebra is equivalent to a structural conception of algebra and consequently more advanced in terms of concept development than rhetoric algebra, which corresponds with an operational approach. However, this view is not commonly shared. Radford argues that the categorization rhetoric –syncopated– symbolic is the result of our modern conception of how algebra developed, and that it is often mistaken for a gradation of mathematical abstraction (RADFORD 1997). When the development of algebra is seen from a socio-cultural perspective instead, syncopated algebra was not an intermediate stage of maturation but it was merely a technical matter. As Radford explains, the limitations of writing and lack of book printing quite naturally led to abbreviations and contractions of words. Perhaps modern day students do naturally shorten their notations (from a context-bound to a general mathematical language), but it

has yet to be decided whether this process implies a better understanding of letter use.

1.7 Cognitive gap

In recent years, much research has been done on difficulties that students have in translating word problems into algebraic equations, and it has produced an abundance of new conjectures. In the transition from arithmetic to algebra there is a discrepancy known as the *cognitive gap* (HERSCOVICS & LINCHEVSKI 1994) or the *didactic cut* (FILLOY & ROJANO 1989). There are differences regarding the interpretation of letters, symbols, expressions and the concept of equality. For instance, in arithmetic, letters are usually abbreviations or units, whereas algebraic letters are stand-ins for variable or unknown numbers. FILLOY & ROJANO (1989) as well as LINCHEVSKI & HERSCOVICS (1996) point out a rupture in the learning process of equation solving. Operating on an unknown requires another notion of equality. In the transfer from a word problem (arithmetic) to an equation (algebraic), the meaning of the equal sign changes from announcing a result to stating equivalence. And when the unknown appears on both sides of the equality sign instead of one side, the equation can no longer be solved arithmetically (by inverting the operations one by one). Matz (1979) and Davis (1975), for example, have done research on students' interpretation of the expression $x + 3$. Students see this as a process (adding 3) rather than a final result that stands by itself. They have called this difficulty the "process-product dilemma". SFARD (1996) has compared discontinuities in student conceptions of algebra with the historical development of algebra. She writes that syncopated algebra is linked to an operational conception of algebra, whereas symbolic algebra corresponds with a structural conception of algebra.

DA ROCHA FALCÃO (1996) suggests that the disruption between arithmetic and algebra is contained in the approach to problem-solving. Arithmetical problems can be solved directly, possibly with intermediate answers if necessary. Algebraic problems, on the other hand, need to be translated and written in formal representations first, after which they can be solved. MASON (1996, p. 23) formulates the problem as follows: 'Arithmetic proceeds directly from the known to the unknown using known computations; algebra proceeds indirectly from the unknown, via the known, to equations and inequalities which can then be solved using established techniques'.

Summarizing the theoretical background of the research project described above, we aim to determine how a bottom-up-approach (starting from informal methods that students already use) towards algebra can minimize the discrepancy between arithmetic and algebra. We will investigate which early algebra activities can help students to proceed more naturally from the arithmetic they are familiar with to new algebraic territories, and how procedural and structural properties in both algebra and arithmetic can become more connected. In our attempt to investigate possible accesses from arithmetic into algebra from a historical perspective, we will look into the past for contexts (topics), types of mathematical problems, mathematical ways of thinking, solving procedures, notations, and suitable sources. The historical development of algebra indicates certain courses of evolution that the individual learner can reinvent. Ideally, the student will acquire a new attitude towards problem solving by developing certain (pre-)algebraic tools: a good understanding of the basic operations and their inverses, an open mind to what letters and symbols mean in different situations, and the ability to reason about (compare and relate) (un)known quantities. The study will be based on data collected through lesson observations, two written assessment tests made by the students at the end of each booklet, student workbooks and student and teacher questionnaires.

The principal aim of the research project is to find answers to questions like:

- are there moments in the learning process when students overcome a part of the discrepancy between arithmetic and algebra, and why?
- what is the effect of integrating the history of algebra in the learning strand the students?
- which type of shortened notations do children use naturally, and how does it compare with the historical development of algebraic notations?
- is there an acceptable compromise between intuitive, inconsistent symbolizations and formal algebraic notations?
- how can students actively take part in the process of fine tuning notations and establishing (pre-) algebraic conventions?
- to what extent and in what way can students become aware of different meanings of letters and symbols?

The next part of the article gives an outline of the learning strand and reports on a few classroom results from the most recent try-out.

2 Learning strand and classroom results

2.1 Proposed learning program: an outline

The historical development of algebra has inspired us to base the core learning material on word- or story-problems. The early rhetorical phase of algebra finds itself in-between arithmetic and algebra, so to speak: an algebraic way of thinking about unknowns combined with an arithmetic conception of numbers and operations. Babylonian, Egyptian, Chinese and early Western algebra was primarily concerned with problem solving situated in every day life, but mathematical riddles and recreational problems were common too. Fair exchange, money, mathematical riddles and recreational puzzles are rich contexts for developing handy solution methods and notation systems, and they are also appealing and meaningful for students. The natural preference and aptitude for solving word problems arithmetically will form the basis for the first half of the learning strand, whereby students' own informal strategies will be adequately fit in. The barter context in particular appears to be a natural, suitable setting to develop (pre-) algebraic notations and tools such as a good understanding of the basic operations and their inverses, an open mind to what letters and symbols mean in different situations, and the ability to reason about (un)known quantities. The transfer to a more algebraic approach will be instigated by the guided development of algebraic notation, especially the change from rhetorical to syncopated notation, as well as a more algebraic way of thinking. It will be interesting to determine whether the evolvement of intuitive notations used by the learner show similarities with the historical development of algebraic notation. Several original texts will be integrated to illustrate the inconvenience of syncopated notations and the value of our modern symbols, and different historical sources will be used to let students compare ancient solving methods like the Rule of False Position with modern techniques.

Outline of the mathematical content:

- restriction problems: problems with two variables and one or two conditions

- reverse calculations: practicing with inverse operations and arrow diagrams
- comparing quantities: reasoning with given barter relations
- progressive formalization of symbol use: discussing conflicts of notations and the changing role of symbols (eg. letters, equal sign)
- informal algebra: Rule of Three, reasoning about unknowns, Rule of False Position
- linear equations in one unknown and two unknowns.

The learning strand currently consists of two consecutive booklets at primary school level (*Change* and *Barter trade*, totaling 25 lessons), and two consecutive booklets at secondary school level (*Fancy Fair* and *Time travelers*, totaling 15 lessons). The learning strand can be split up into two parts, but ideally it is treated as one complete lesson series.

2.2 Classroom impressions

In the spring of 1999, the booklets *Change* and *Barter trade* were tested in two primary school classes, grade 6, consisting of 18 and 23 students. The 25 pre-algebra lessons were given by the regular teacher, based on explanatory notes in the teacher guide and occasional talks with the researcher. Approximately one-third of the lessons was observed and recorded; some lessons were videotaped. In early summer 1999, the third booklet *Fancy Fair* was tried out in two first year classes of secondary school; one of these classes also tested the last booklet. The regular teacher gave all the lessons according to guidelines in the teacher guide. However, since the booklet *Time travelers* had never been tested before, the teacher was guided more closely during the last lesson series. The next few paragraphs are meant to give an idea of what kind of solutions and discussions occurred; there has not been time yet to analyze the data with reference to the research questions.

The first topic in the primary school part of the program is problems with restrictions. The students are given a list with prices of 20 different candy bars (ranging between 5 and 95 cents), and are asked to write down what can be bought for precisely 1 guilder (100 cents). Many students realize that the answer will require a lot of paper and decide to use abbreviations. Immediately there is an opportunity to talk about effective mathematical notation (letters, syllables, operator symbols, tabular forms). In the next question, students are asked to comment on a disagreement between two imaginary students: "I found all the possibilities for 1 guilder!" one says; "But you can never know that for sure!", the other says. In one of the classes this activity instigated a lively discussion on the total number of possibilities, along these lines:

> *Several students working on their own reckon it is possible to know for sure, but it will take a long time.*
> *Observer: 'How do you know you haven't missed one out?'*
> *A girl replies that in that case you are doing it wrong. Another girl replies that she would start at the top of the list, take one item and check all the possibilities, and then take the next item from the top of the list, and so on.*
> *Class discussion. The teacher asks for answers; some students give a numerical answer.*

Teacher: 'How do you know there are so many?'

Student: 'At some time there will be an end to all the possibilities'.

The class investigates all the possibilities in combination with potato chips; there are too many to write down.

Teacher: 'How many possibilities altogether, do you think?'

A boy replies: 400. He then explains: he compared the problem with a comment the teacher made a week earlier, that there are as many as 520 possible simple sums with the first 20 natural numbers! And so, he concludes, there must be at least 400 in this case.

Other students then suggest more than 1000 possibilities, but they would like to hear the exact number from the author of the booklet!

This example illustrates how an open problem can lead to higher level thinking (reasoning about solvability) and can invite students to strike up other mathematical knowledge.

Another typical restriction problem in the first paragraph is situated in a money context. It is split up into two parts:
1. *how many quarters and dimes do you get for a coin worth 2.5 guilders*
2. *if the total number of coins is 13, how many dimes and how many quarters are there.*

Figure 3 shows how one student thought up a useful strategy for part 1. In general, students use a trial-and-error method and do not think of supportive notations like a table to structure their attempts. It also does not occur to them or disturb them that they might miss out some solutions this way.

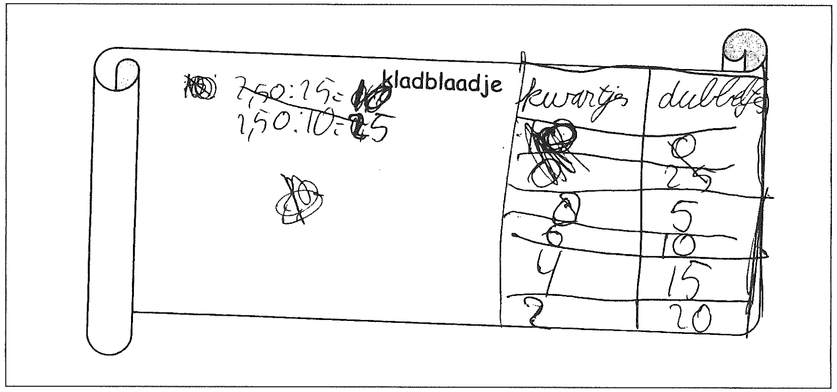

Figure 3: combinations of coins totaling 2.5 guilders

Restriction problems also appear in the third paragraph, for example:

1. riddles on age: *Mom is 5 times as old as John; but she is also 28 years older than him.*
2. Diophantine problems on sum and difference: *split the number 150 up into two numbers such that the difference between those numbers is 65* (Figure 4).

Figure 4a shows a spontaneous student strategy, where the given number is halved (75 and 75) and then the given difference is evenly allocated to the two numbers (75 + 32.5 and 75 - 32.5). In the other class the number line method as shown in figure 4b was introduced by the teacher as an alternative - more visual - strategy. Diophantine problems are handled again in the secondary school program, but this time with the intention to solve them using a linear equation in one unknown: *call the smaller number s, then the bigger one is $s + 30$, and the sum $2s + 30 = 100$.* In this way students get a chance to reflect on the effectiveness of an informal and formal strategy of problem solving.

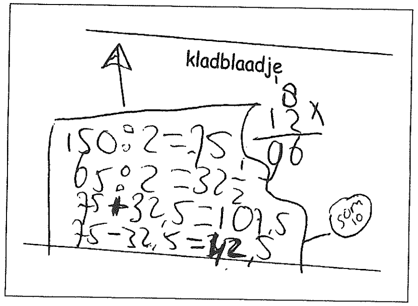

Figure 4a: halving the sum and the difference

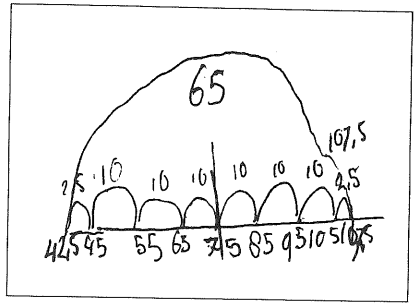

Figure 4b: mirrored jumps on the number line

Towards the end of the first booklet, there is a paragraph on reversing a string of calculations to find the initial number. 'Guess my number' goes as follows: one student thinks up a number and tells another student something like 'do it times 3, then add 5, subtract 2 and divide by 2, and you get 4; what was the number?' It is a successful activity: students enjoy it and they can do it at their own level and pace. Moreover, the teacher can make up many variations to practice even fractions and percentages in a playful way that takes little time. The last paragraph is a historical application of reverse calculations, organized around an original problem by Chistianus van Varenbraken (1532) (translated and summarized):

> A hermit prays to Saint Paul 'Double the amount of money in my purse and I will give you 6 pennies', and the saint complies. The hermit does the same when he comes to Saint Peter and Saint Francis. In the end, when his prayers have all been heard, the hermit has no money left. The question is, how much money did the hermit have at the start?[5]

The initial plan was to give students the original text along with some explanatory notes, and then ask them to solve the problem. This task turned out to be too complex even for a colleague designer, and irrelevant for the learning process besides. Looking for a way to visualize the problem and make it dynamic, we found the solution: instead of solving the problem on paper, the situation should be acted out in a short play. The teacher appoints 4 students to play the roles of the hermit and the three saints, and the other students in the class have to solve the problem. The play can be repeated indefinitely with different outcomes, enabling all students to catch on. The students then see the author's own solution in the booklet, where he merely gives the answer and checks that it is correct. Two higher order questions in this paragraph are:

[5]Source: KOOL (1988).

1. *suppose you have to solve a similar problem, whereby the hermit has 38 pennies in the end instead of none; does the author's solution help you solve it?*

2. *what is the minimum number of pennies that the hermit needs to have at the start in order to make a profit?*

One clear outcome of the questionnaire is that students really enjoy acting. By literally *doing* the problem, it comes alive. Considering the problem's original purpose, 'a matter of delight' as the Van Varenbraken says, this activity is a good example of an appropriate reproduction of history.

The content of *Fancy Fair*, the first secondary school booklet, is concerned primarily with solving systems of two equations in two unknowns. The fancy fair attractions are represented by iconic markers; some of these have a fixed price and others are not yet priced. In order to concur with the primary school program, the booklet begins with expressions (equations) for trading markers fairly and suitable notations to represent these trade expressions. In the third paragraph students perform reverse calculations to determine the price of the markers. The program then moves on to pairs of combinations of markers for a given price: an iconic system of equations (see figure 5a and 5b). The problems can all be solved informally, by comparing the numbers of markers and reasoning about them. The problem in figure 5a requires determining the difference between the two combinations of markers and the prices ('subtracting'), and comparing again.

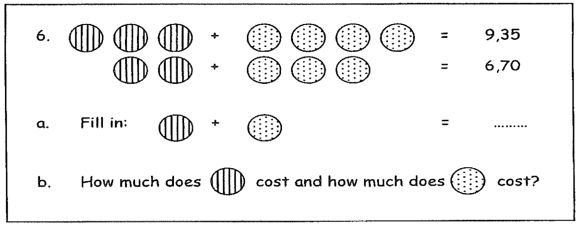

Figure 5a: solving an iconic system of equations by determining the difference

The problem in figure 5b is based on interchanging repeatedly one striped marker for a check-ered one - raising the price by 50 cents in doing so - until only one type of marker remains.

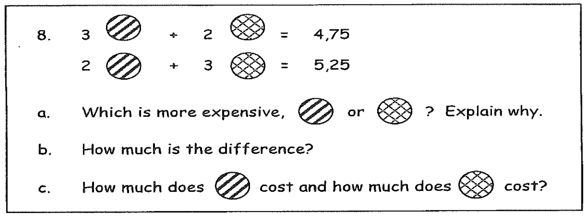

Figure 5b: solving an iconic system of equations by repeatedly interchanging one marker for another

In the final booklet historical problems are embedded in a story about two 13-year-old children who visit different countries in different eras and discover mathematics from the past. For

example, there is a paragraph on the Rule of Three, another on the Rule of False position and the last paragraph deals with Diophantine problems. The Rule of False position is initiated by a well-known fish problem by Calandri (1491): *The head of a fish weighs 1/3 of the whole fish, his tail weighs 1/4 and its body weighs 300 grams. How much does the whole fish weigh?* Students are asked to estimate the weight first and then solve it using a rectangular bar (see Figure 6), after which they study the solution method of Calandri (see Figure 7).

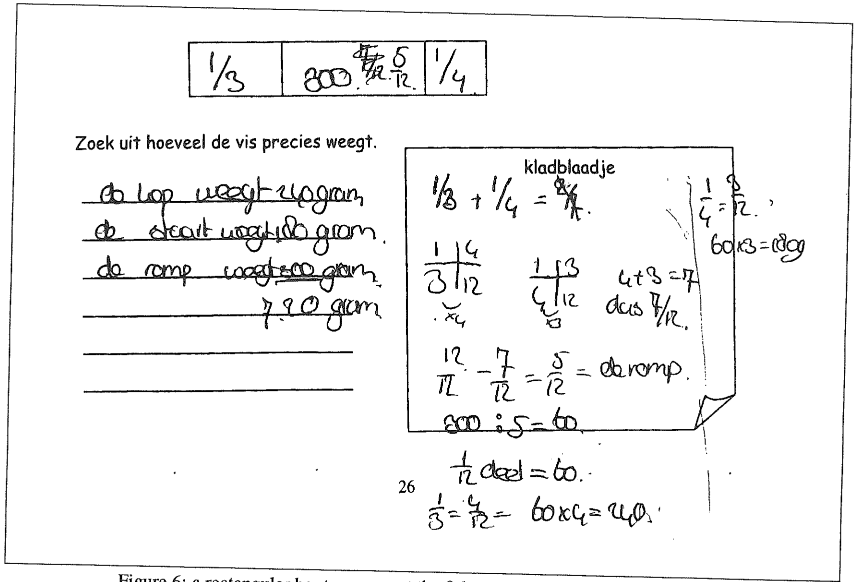

Figure 6: a rectangular bar to represent the fish, and the calculation of the weight

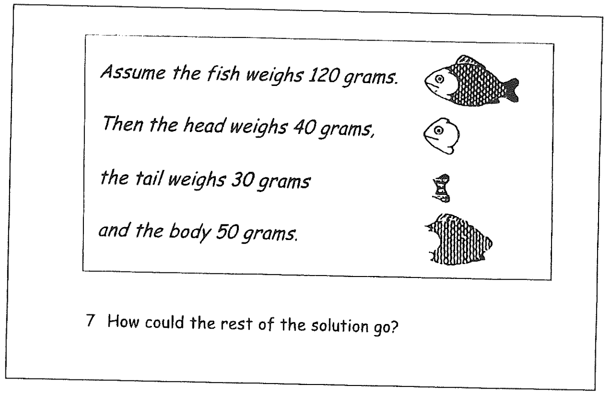

Figure 7: part of Calandri's solution[6]

The paragraph also includes some reflective questions:

[6]Problem and pictures originate from OFIR, R. & ARCAVI, A. (1992).

1. *why does Calandri choose 120 to start with?*
2. *what name would you give to this Rule of False Position?*
3. *what do you think of this method?*

It is really remarkable that not one student remarks that 120 is a ridiculous number to begin with, considering that the body of the fish already weighs 300 grams! The second question triggered very little response in the classroom but perhaps the student notebooks will reveal more.

2.3 Final remarks

In the next few months the research data will be searched for unequivocal, concrete indications that will help to answer the research questions. Nevertheless, at this time we would like to put forward a few conjectures regarding student attitude based on the observation of lessons. At primary school level, students are not trained to make notes or draft work as an aid to problem solving; they want to and try to solve even complex reasoning problems mentally. In secondary school they learn that they should distill the information, but they often don't. Students have trouble formulating their solution strategy; they are sometimes unwilling to write down an explanation to their answer, believing that the solution itself is more important how they got it. Especially at primary school we see a very passive attitude towards problem solving; students tend to wait for the teacher to give them a clue rather than investigating for themselves. The activities in the learning program seem to challenge the boys more than the girls. The effect of historical elements in the classroom at primary school level is disappointing; students are not as interested in the mathematical heritage as we expected and ancient solution strategies have not really stimulated students to attain a more critical attitude.

To finish off, here is just a note of precaution to the reader. The classroom results presented in this article serve to illustrate the kind of activities the proposed learning program can activate; they are by no means a representative selection of what the moderate student can accomplish. Similarly it must be clearly understood that the conjectures on student attitude are still subject to change if the final data analysis proves differently.

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"Si les mathématiques m'étaient contées..."

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Abstract

J'aborde ici la question suivante : peut-on rendre les seuils épistémologiques plus accessibles à la plupart des élèves ? Très souvent ceux-ci demeurent "pseudostructurels" (au sens de Sfard), c'est-à-dire qu'ils tendent à sous-évaluer les aspects sémantiques pour rester au niveau syntaxique. Ils perçoivent les mathématiques essentiellement comme un ensemble de symboles plus ou moins vides, qu'il faut savoir manier pour réussir à l'école et dans la vie. Mon travail, s'agissant d'élèves de 14 à 19 ans, vise à montrer qu'il est intéressant de proposer par moments, pour véhiculer certaines notions mathématiques de base, des langages moins structurés et symboliques que ceux utilisés le plus souvent. La proposition que je fais dans la suite consiste en une approche des ensembles infinis à l'aide d'une pièce de théâtre.