FIGURE 1

How did candidates pass the state examination in mathematics in the Tang dynasty (618-917)? - Myth of the "Confucian-Heritage-Culture" classroom

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Abstract

Towards the late 80s some educators began to pay attention to cultural differences that may affect the process of learning and teaching. This interest is further strengthened by the results coming out of the two recent studies sponsored by the International Association for the Study of Educational Achievement (IEA). In particular, the learning process of Asian students brought up in the tradition of the Confucian-Heritage culture (CHC) has become a much discussed issue in the past decade. In this talk we try to look at the issue from the historical angle : to investigate the curriculum in mathematics at the state university in the Tang Dynasty and to piece together a rational reconstruction of the state examination in mathematics from official records in the ancient chronicles. It is hoped that such a study will shed light on the Paradox of the Asian Learner : CHC students are perceived as using low-level, rote-based strategies in a classroom environment which should be conducive to low achievement; yet CHC students report a preference for high-level, meaning-based learning strategies and they achieve significantly better in IEA studies!

1 Prologue

In a paper on medical education, G.E. MILLER says,

> We say we want sensitive, thoughtful, analytic, independent scholars, then treat them like Belgian geese[1] being stuffed for pâté de foie gras. We reward them for compliance, rather than independence; for giving the answers we have taught them rather than for challenging the conclusions we have reached; for admiring the brilliance of purely scientific advances rather than developing greater sensitivity to the inequities in health care we have too often ignored. (MILLER 1978)

This passage, its allusion to medical education notwithstanding, sets the tone for the theme of this paper.

2 CHC classroom culture

In the past decade, with the rising number of multiethnic classrooms in many countries and with increasing international cooperation in assessing the curriculum and the academic performance of students around the world, some researchers in education take cultural differences into account in their studies. The cultural background of the students and of the teacher is seen to be a factor affecting the process of learning and teaching. See for example (CAI 1995, STEVENSON & STIGLER 1992, WATKINS & BIGGS 1996, WONG 1998).

Of particular pertinence to our present discussion is the theory put forward by a group of researchers including J.B. BIGGS (1994, 1996), F. MARTON (1976, 1996) and D.A. WATKINS (1991). What kind of teaching environment is conducive to good learning, or to what is known as the "deep" approach to learning as contrasted with the "surface" approach (MARTON 1976)? In (BIGGS 1993) the following factors are singled out: (i) varied teaching methods, student-centred activities, (ii) content presented in a meaningful context, (iii) small classes, (iv) warm classroom climate, (v) high cognitive level outcomes expected and assessed, (vi) classroom-based assessment in a non-threatening atmosphere. In their research this group came into contact with students at different levels, of different cultural backgrounds and in different parts of the world (eg. Australia, Hong Kong, Korea, Nepal, the Philippines, Sweden, and the United States). What they observed in the classrooms in the so-called "Confucian-heritage culture" is just the opposite from (i) to (vi) cited above.[2] In CHC classrooms, it appears that classes are conducted in a traditional teacher-talk/student-listen manner within a class of large size, with the austere teacher commanding an authoritative role. Assessment is carried out through strict formal examinations which exert significant influence on the student's future career. It seems that such examinations reward good memorization and industrious drilling. Therefore, the obvious conclusions drawn by Western observers of the CHC classrooms are: (1) CHC classrooms should be conducive to low quality outcome, viz. rote learning and low achievement; (2) CHC students are perceived as using low-level, rote-based learning strategies.

[1] This paper is the full text of a presentation delivered in July 1999 at the troisième université d'été européene sur l'histoire et l'épistémologie dans l'éducation mathématique held at Louvain-la-Neuve and Leuven in Belgium. *La citation ci-dessus n'est qu'une allusion bienveillante, il n'y a pas de malveillance envers notre hôte!* I take this opportunity to thank our Belgian hosts for their hospitality.

[2] The nebulous term "Confucian-heritage culture", abbreviated as CHC, is not to be taken here as a term with a precise definition which clearly delineates a specific genre of culture. The term is used in a general sense to cover the cultural background of communities in mainland China, Hong Kong, Japan, Korea, Singapore, Taiwan and Vietnam. It would take many books and papers to explicate in detail and depth what CHC really means, with all its subtlety and historical evolution. That is beyond the scope of this paper and beyond the capacity of this author. I would entreat readers to take CHC classrooms to mean classrooms in the communities mentioned above, although differences already exist between them.

3 The Asian Learner Paradox : some clarifications

However, reality reveals a totally different picture! In the several international studies sponsored by the IEA (International Association for the Evaluation of Educational Achievement) — the FIMS in the 60s, the SIMS in 1976-1983 and the most recent TIMSS in 1993-1994 — CHC students have scored significantly higher levels of achievement than those of Western students. In his research J.B. BIGGS (1996) found that CHC students report a preference for high-level, meaning-based learning strategies.

This discrepancy between what one expects and what one witnesses has come to be known as the Asian Learner Paradox or the CHC Learner Paradox (BIGGS 1994).[3]

It is strange that a popular view is to equate Confucian learning with rote learning and with submissive learning. Let us look at some samples of what the Masters themselves had to say.

In the *Confucian Analects* (5th century B.C.) we find,

> Learning without thought is labour lost; thought without learning is perilous. (LEGGE 1960)

In the *Doctrine of the Mean* (6th-5th century B.C.) we find,

> He who attains to sincerity, is he who chooses what is good, and firmly holds it fast. To this attainment there are requisite the extensive study of what is good, accurate inquiry about it, careful reflection on it, the clear discrimination of it, and the earnest practice of it. (LEGGE 1960)

In the books by the leading neo-Confucian scholar ZHU Xi (1130-1200) we find,

> In reading, if you have no doubts, encourage doubts. And if you do have doubts, resolve doubts. Only when you've reached this point have you made progress. (GARDNER 1990)

Would one call this rote-learning? submissive learning?

By reading more extensively in the books by Zhu Xi, we can perhaps understand better what appears to Western observers as rote-learning actually consists of. Zhu Xi said,

> Generally speaking, in reading, we must first become intimately familiar with the text so that its words seem to come from our own mouths. We should then continue to reflect on it so that its ideas seem to come from our own minds. Only then can there be real understanding. Still, once our intimate reading of it and careful reflection on it have led to a clear understanding of it, we must continue to question. Then there might be additional progress. If we cease questioning, in the end there'll be no additional progress. (GARDNER 1990)

[3] But personally I have certain reservations about the two outcomes described above, judging from the teaching experience I have with undergraduates in mathematics at the university. It is a paradox within a paradox! But we will put that aside in the present discussion and concentrate only on the Asian Learner Paradox.

He also elaborated further,

> Learning is reciting. If we recite it then think it over, think it over then recite it, naturally it'll become meaningful to us. If we recite it but don't think it over, we still won't appreciate its meaning. If we think it over but don't recite it, even though we might understand it, our understanding will be precarious. ... Should we recite it to the point of intimate familiarity, and moreover think about it in detail, naturally ourmind and principle will become one and never shall we forget what we have read. (GARDNER 1990)

This is an unmistakable differentiation between repetitive learning and rote learning. Contemporary researchers explain the Asian Learner Paradox based on this differentiation (BIGGS 1996, MARTON 1996).

On the other hand, modern day education in the Western world which arose in the 19th century along with the Industrial Revolution started by emphasizing the 3Rs — reading, writing and arithmetic. In a code issued by R. Lowe of the Education Department of England in 1862, specific standards for each R were explicitly stated (e.g. Standard I in Reading: Narrative monosyllables; Standard II in Writing: Copy in manuscript character a line of print; Standard IV in Arithmetic: A sum in compound rules (money)) (CURTIS 1967). The emphasis on mechanical rote learning is captured vividly in the opening sentences (which were intended as a satirical exaggeration) of the 1854 novel *Hard Times* by Charles Dickens (as words uttered by Mr. Gradgrind of Coketown)

> Now, what I want is, Facts. Teach these boys and girls nothing but Facts.
> Facts alone are wanted in life.... This is the principle on which I bring up
> my own children, and this is the principle on which I bring up these children.
> Stick to Facts, sir!"

4 Examination culture in China

4.1 Sources

Even if one can explain the Asian Learner Paradox, it remains a fact that in CHC there is a strong tradition of examination. Some even label CHC classroom as an examination-oriented classroom culture! To what extent is this true? It is commonly believed that an examination-oriented culture will hinder the learning process. Is it really that bad? Or is it a necessary evil? Or is it even beneficial to the learning process in some sense? These questions urge me to look at the issue from a historical perspective. I intend to look at an ancient period when official examination in mathematics was in its most established form — in the Tang Dynasty (618-907) — and see if it can enlighten me in my classroom teaching.

In collaboration with A. Volkov, an historian in mathematics, we attempt to piece together a picture of the state examination in mathematics in the Tang Dynasty (SIU 1999). This we do by gleaning what we can from the official records contained in certain chronicles, among which the main ones are:

(a) *Jiu Tangshu* (Old History of the Tang Dynasty), c. 941-945,

(b) *Xin Tangshu* (New History of the Tang Dynasty), c. 1044-1058,

(c) *Tang Liudian* (Six Codes of the Tang Dynasty), 738,

(d) *Tongdian* (Complete Structure of Government), c. 770-801,

(e) *Tang Huiyao* (Collection of Important Documents of the Tang Dynasty), 961.

One informative secondary source which contains the main excerpts of relevant interest in the chronicles listed above and a lot of interesting information, including lists of successful candidates in each round of examination (in the classics and literary composition rather than in mathematics) is *Dengke Jikao* (Journal on the Examinations in the Tang Dynasty) compiled by the Qing scholar XU Song in 1839.

I should at the start state clearly the attitude I adopt when consulting those ancient chronicles. I subscribe to a wider (but somewhat controversial) view of studying history as propounded by the British philosopher-historian R.G. Collingwood in *The Idea of History* (1946), "History is thus the self-knowledge of the living mind. ... For history is not contained in books or documents; it lives only, as a present interest and pursuit, in the mind of the historian when he criticizes and interprets those documents, and by so doing relives for himself the states of mind into which he inquires." Collingwood echoed the view held by the Italian philosopher B. Croce who said in *History: Its Theory and Practice* (1915), "History is living chronicle, chronicle is dead history; history is contemporary history, chronicle is past history; history is principally an act of thought, chronicle an act of will. Every history becomes chronicle when it is no longer thought, but only recorded in abstract words, which were once upon a time concrete and expressive."

4.2 Chinese Examination System inspired by the West

Let us look at the examination system in its historical context. Despite the shortcomings the system later developed, it is praised for the role it once played. Examination employed as a way to select is a very Chinese institution. According to P. Monroe in *Cyclopaedia of Education* (1931), "Written examination was probably unknown in Europe until 1702. ... Practical examinations had been employed for a long time in the medieval universities in such a subject as medicine." Dr. Sun Yat-Sen, founder of the Chinese Republic in 1911, said in *The Five-Power Constitution*,

> At present, the civil service examinations in the (Western) nations are copied largely from England. But when we trace the history further, we find that the civil service of England was copied from China. We have very good reason to believe that the Chinese examination system was the earliest and the most elaborate system in the world. (TENG 1942-43)

Indeed, Dr. Sun instituted the division of the government structure into five-power, viz. the Legislative Yuan, the Executive Yuan, the Judicial Yuan, the Examination Yuan and the Censorate. E.A. KRACKE has said,

> One of China's most significant contributions to the world has been the creation of her system of civil service administration, and of the examinations which from 622 to 1905 served as the core of the system. (KRACKE 1947)

As early as in the beginning of the 17th century, the Jesuit Father Matteo Ricci[4] reported with commendation in his journal "the progress the Chinese have made in literature and in the sciences, and of the nature of the academic degrees which they are accustomed to confer." About one and a half centuries later, another illustrious European, Voltaire (F.M. Arouet)[5], made a similar observation, "The human mind certainly cannot imagine a government better than this one where everything is to be decided by the large tribunals, subordinated to each other, of which the members are received only after several severe examinations. Everything in China regulates itself by these tribunals."

4.3 Different Types of Examination in China

The Chinese term for state examination is "keju". Literally, "ke" means "subject" and "ju" means "recommend". Combined together it means recommendation of suitable candidates (for taking up official positions) through examinations in different subjects. Some historians date the beginning of the keju system to the Sui Dynasty (581-618) when the emperor convened a state examination by decree. But some historians maintain that it started in 622 when the first Tang emperor decreed that any qualified candidates could sit for the state examination without having to be recommended by a provincial magistrate. The keju system was abolished in 1905 by an imperial edict towards the end of the last imperial dynasty in China, the Qing Dynasty (1644-1911).

In *Xin Tangshu* a section on recruitment by examinations records that there were two kinds of state examinations: (1) regular examinations held annually in the first or second lunar month for graduates of colleges and universities or for provincial candidates, (2) special examinations held by imperial decree. The second kind depended on the need at the time or on the whim of the emperor, so it covered a wide range of expertise, but could also sound rather strange. In official records one can find about a hundred of such special examinations. Just to cite a few, there were: examination on "vast erudition and great composition", examination on "deep knowledge of the ancient books and great talents in the art of teaching", examination on "having military plans with foresight and well qualifiedas a general", examination on "wisdom and good nature, rectitude and righteousness, and speaking honestly and remonstrating insistently", examination on "remarkable understanding of the art of government and suitability for administering people". A most amusing item is examination on "leading an hermetic life at Qiuyuan, not seeking fame", since logically speaking one should be awarded a degree in that if and only if one should not be! (In fact, it was recorded in *Dengke Jikao* that somebody was awarded the degree *in absentia* in 794 as he refused to receive it!) For the first kind there were initially seven subjects: examination on perfect talent, examination on classics, examination on distinguished man of letters, examination on accomplished man of letters, examination on law, examination on calligraphy and examination on mathematics. Examination on perfect talent was soon abolished, while examination on accomplished man of letters became in time the main focus enjoying the highest prestige. It was recorded in *Tongdian* that by 752, of a thousand candidates who sat for the annual examination on accomplished man of letters only one or two were awarded

[4]The quotation by Matteo Ricci can be found in his journal later compiled by Nicolas Trigault in 1615. There is a modern English translation of the journals titled "China in the Sixteenth Century : The Journals of Matthew Ricci, 1583-1610", translated by L.J. Gallagher, published by Random House, New York in 1953. There is also a modern French translation titled "Histoire de l'expédition chrétienne au royaume de la Chine, 1583-1610" translated by G. Bessiere, published by Bellarmin in 1978.

[5]"Œuvres complètes de Voltaire", t.13, 163, published by Nedeln, Lichtenstein (in the 18th century?), with a Kraus reprint in 1967.

the degree, while for instance, successful candidates for the examination on classics numbered in the tens. A source of the time said that one who passed the examination on accomplished man of letters at fifty (perhaps after many repeated attempts) was still regarded as outstanding, while one who passed the examination on classics at thirty was considered too old already! No similar data or remark is found for examination on mathematics, which serves to indicate that mathematics was accorded a lower prestige among the various subjects, only on a par with calligraphy. This becomes even more apparent when we look at the number of students enrolled at the state university. Tang institutions of higher education were divided into hierarchies. The highest institution was the School for the Sons of the State which accepted only sons of noblemen or officials from a certain rank upward. Next came the National University which accepted a similar crop with the official rank somewhat lowered. Then came the School of Four Gates which accepted besides sons of officials also a small number of sons of ordinary citizens. The three Schools of Law, Calligraphy and Mathematics accepted sons of officials of low rank and of ordinary citizens. In the early Tang Dynasty, according to *Xin Tangshu*, there were 300 students in the School for the Sons of the State, 500 students in the National University, 1300 students in the School of Four Gates, 50 students in the Law School, 30 students in the Calligraphy School and 30 students in the Mathematics School. At one time, throughout the whole empire, including the provincial colleges, there were 8000 students pursuing higher education with foreign students coming from nearby countries as well. The whole edifice of state higher education was very well-established in the Tang Dynasty.

4.4 The Annual State Examination

The culminating apex of this edifice, the annual state examination, was a gruelling experience for many. Some authors in the Tang Dynasty wrote about how candidates stood in a long queue, carrying their own stationery, supply of food and water, candles and charcoal (for preparing meals and for getting warm), waiting to be admitted to their cells, only to be searched and shouted at by guards who were stationed by the thorny hedge (an ancient analogue of barbed-wire fence) which encompassed the examination venue. Candidates were clad in flimsy clothes and shivered in the freezing weather, for they were not allowed thick clothing to prevent concealment of manuscripts. Throughout the long hours they worked on their examination scripts, the candidates were confined to their cells, in which they would prepare their own meals and take care of their own personal hygiene. In the case of failure in the examination, which was not uncommon, this gruelling experience would have to be repeated in another year, and perhaps in yet another year, ... WEI Chengyi, who was awarded the degree of accomplished man of letters in 867, once sneaked into the office of the Ministry of Rites called Nangong, which was in charge of examination affairs, and composed a poem on the wall: "Like a thousand white lotus petals, /The candles lit up the hall, /Which was filled with the peaceful rhymes /Of the Ya and the Song. /As the flame of the third candle/Flickered towards its end, /One realized it meant failure /To complete the scene of Nangong." This poem, with its trace of resignation, depicted vividly those assiduous candidates racing against time with their examination scripts by the light of the three candles allowed them to last through the night. (See Figure 1 for a humorous rendering of the scene).

Modern examinations are definitely much less gruelling than that. It would be unfair to my ancestors in the Tang Dynasty if I fail to point out that even over a thousand years ago some good modern examination procedures were already in place. In 759 the Chief Examiner LI Kui said, "The empire selects its officials for their talent. The requisite classics are displayed

here. Candidates are welcome to consult them at will." This was perhaps the earliest open-book examination! In 742 the Chief Examiner WEI Zhi said, "The performance of a candidate in one single examination may not reflect his true potential, hence his previous essays should also be consulted." This was perhaps the earliest instance of assessment by project work!

5 Curriculum in mathematics in the Tang Dynasty

It is recorded in *Xin Tangshu* that the mathematics curriculum at the state university consisted of two programmes, hereafter denoted by A and B for short. For details see (SIU 1995, 1999). It suffices at this point simply to note that each programme lasted for seven years of study, with Programme A covering eight of the ten books in *Suanjing Shishu* (Ten Mathematical Manuals) and Programme B covering the remaining two. (Therefore, Programme B was a more advanced course of study. *Suanjing Shishu* was the collation of ten existing mathematical classics by LI Chunfeng at an imperial edict, and adopted as the official textbook in 656.) In each programme students must also study two more books, *Shushu Jiyi* (Memoir on Some Traditions of the Mathematical Art) and *Sandeng Shu* (Three Hierarchies of Numbers). We will come back to these two books later. Regular examinations were held throughout the seven years of study, and at the end of each year an annual examination was held. Any student who failed thrice or who had spent nine years at the Mathematics School would be discontinued. Judging from the age of admission at 14 to 19 years-old, we know that a mathematics student would sit for the state examination at around 22.

In the state examination for either programme, the candidate was examined on two types of question. The first type was described in *Xin Tangshu* as: "[The candidates should] write [a composition on] the general meaning, taking as the basic/original task a 'problem and answer'. [They should] elucidate the numbers/computations, [and] construct an algorithm. [They should] elucidate the structure/principle of the algorithm in detail." (For Programme B there was added the remark, "If there is no commentary, [the candidates should] make the numbers/computations correspond [to the right ones?] in constructing the algorithm." For an attempt to explain the latter remark, see (SIU 1999)). We will say more about this type of question in the next section. The second type was testing on quotations. Candidates had shown a line taken from either *Shushu Jiyi* or *Sandeng Shu*, with three characters covered up. Candidates had to answer what those three characters were. In to-day's terminology, this type of questions is called "fill in the blank". It is interesting to note that *Shushu Jiyi* (credited to the authorship of XU Yuein the early part of the 2nd century and commented on by ZHEN Luan in the late 6th century, although the extant version might perhaps been forged by Zhen Luan himself) is a short text with only 934 characters, which could be committed to memory with reasonable ease (not to mention that a candidate had seven years to do it!). There may well be other reasons for singling out this book for the purpose of testing on quotations, but that would be the subject of another paper. (See (VOLKOV 1994) for an interesting discussion on the content of *Shushu Jiyi*.) The book *Sandeng Shu* was lost by the Song Dynasty (960-1279). We can only surmise that it might be a text similar to *Shushu Jiyi* in this respect.

By the way, there was a reason for instituting the practice of testing on quotations. The practice was proposed by the Chief Examiner LIU Sili in 681 (in all subjects) to rectify the deficiency of a prevalent habit of candidates who only studied "model answers" to past questions instead of studying the original classics. Testing on quotations forced candidates to read (at least some) original classics. However, examination being what it is, it is prone to abuse. The setting of questions on quotations got more and more difficult and unreasonable, testing candidates on

obscure phrases, sometimes even setting up traps to confound the candidate intentionally. In response, candidates collected such obscurities and memorized them for the sole purpose of passing those unreasonable tests! The original purpose of encouraging candidates to read the original classics was totally defeated. In 728 it was decreed that quotations should be set within reasonable bounds. There is a good lesson to be learnt here about making use of examination to direct the curriculum.

6 "Re-constructed" examination questions

Let us come back to the first type of question. What are these tasks on elucidation and construction of algorithms about? Since no trace of any examination question is extant, we can only attempt to "re-construct" an examination based on the scanty and sketchy official account on the state examination in mathematics gleaned from the ancient chronicles.

Before giving such examples, it is helpful to look at a textbook and see how the author did the mathematics. We choose the prime textbook *Jiuzhang Suanshu* (Nine Chapters on the Mathematical Art, compiled between 100 B.C. and A.D. 100)[6] with commentaries by the 2nd century mathematician LIU Hui. This will also add to the stock of "circumstantial evidence" for our attempted "re-construction".

In Chapter 5 some formulae for the volume of various solids are given. In particular, Problem 17 is about that of a tunnel at the entrance of a tomb (xianchu). Mathematically speaking, a xianchu is the solid bounded by three trapeziums and two triangles on the two sides. The three trapeziums have opposite parallel sides of length a, b; a, c and b, c, the depth is h and the trapezium on top has length ℓ. (See Figure 2)

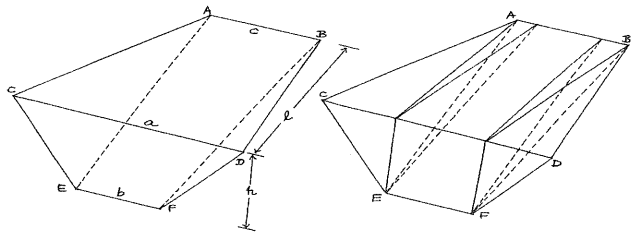

FIGURE 2

The formula for the volume of the xianchu is given in the text as

$$V = \frac{1}{6}(a+b+c)h\ell.$$

[6]The earliest translation of the JIUZHANG SUANSHU in a Western language was the German translation titled "Neun Bücher Arithmetische Technik" by Kurt Vogel, published by Friedr. Vieweg & Sohn, Braunschweig in 1968. There are two recent translations, one in English and one in French. The English translation has appeared and is titled "The Nine Chapters On the Mathematical Art : Companion and Commentary", by Kangsheng Shen, John N. Crossley and Anthony W.C. Lun, published by the Oxford University Press, Oxford in 1999. The French translation will appear in 2000 and is titled Édition critque, traduction et présentation des Neuf chapitres sur les procédures mathématiques (les débuts de l'ère commune) ainsi que des commentaires de Liu Hui ($3^{ème}$ siècle) et de Li Chunfeng ($7^{ème}$ siècle), by Karine Chemla and Shuchun Guo, in preparation by Diderot Multimedia. For the Chinese edition I follow the corrected and edited version of JIUZHANG SUANSHU by Shuchun Guo, published by Liaoning ducational Press in 1990. The original version dated back to of course much earlier times.

(In the text, numerical data are given in place of a, b, c, but the numerical data are actually generic rather than special.) Liu Hui explains in his commentaries how the volume is calculated. He dissected the xianchu into smaller pieces, each of some standard shape such as a triangular prism (qiandu), a tetrahedron of a particular type (bienao), or a pyramid with a square base (yangma). If you try to do that by yourself, you will find out that the way of dissection is different for different relations between a, b, c. For instance, if $a > c > b$, then you obtain two bienaos each of volume $(a-b)h\ell/12$, two bienaos each of volume $(c-b)h\ell/12$ and one qiandu of volume $bh\ell/2$. (See Figure 2) They add up to $(a+b+c)h\ell/6$. But if $a > b > c$, then you obtain two bienaos each of volume $(a-b)h\ell/12$, two yangmas each of volume $(b-c)h\ell/6$ and one qiandu of volume $ch\ell/2$. They also add up to $(a+b+c)h\ell/6$. In fact, Liu Hui in his commentaries treats all eight different cases except the one case $b > a = c$. The calculation is different for different ways of dissection, but the basic underlying idea is the same. Probably candidates in the examination were asked to carry out a similar explanation for other formulae on area and volume, possibly with given numerical data. Once the basic idea is understood, such a demand for elucidation is reasonable.

In the same chapter, Problem 10 is about the volume of a pavilion (fangting) with square bases. (See Figure 3)

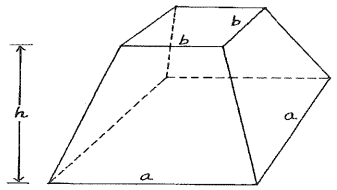

FIGURE 3

Mathematically speaking, a fangting is a truncated pyramid with square base. If a, b are the sides of the bottom and top squares respectively and h is the height, then the volume is given in the text as

$$V = \frac{1}{3}(a^2 + b^2 + ab)h.$$

Again, Liu Hui in his commentaries explains how to obtain the formula by an ingenious method of assembling blocks of standard shape (called by him "qi"). There are three kinds of "qi": cube of side a with volume a^3 (lifang, LF); pyramid of square base of side a and one vertical side of length a perpendicular to the base, with volume $\frac{1}{3}a^3$ (yangma, YM); triangular prism with isosceles right triangle of side a as base and height a, with volume $\frac{1}{2}a^3$ (qiandu, QD). He observed that the truncated pyramid is made up of one LF, four YM and four QD. (Careful readers will notice that here we require $h = b$, so that we are talking about blocks of a standard shape). He then observed that one LF makes up a cube of volume b^2h; one LF and four QD make up a rectangular block of volume abh; one LF, eight QD and twelve YM make up a rectangular block of volume a^2h. (Careful readers will notice that here we require $h = b$ and $a = 3b$ so that each corner piece is a cube formed from three YM.) In problem 15, Liu Hui further explains how to obtain the more general formula of the volume of a pyramid of rectangular base with an

arbitrary height by an infinitesimal argument (WAGNER 1979)). Altogether, three LF, twelve QD and twelve YM make up a volume $b^2h + abh + a^2h$. Hence the volume of the truncated pyramid is $\frac{1}{3}(a^2 + b^2 + ab)h$. (See Figure 4)

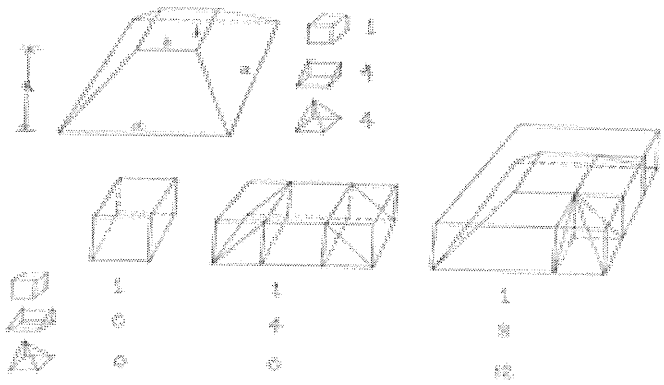

FIGURE 4

Liu Hui gave an alternative formula

$$V = \frac{1}{3}(a-b)^2h + abh$$

by another way of dissection. (See Figure 5)

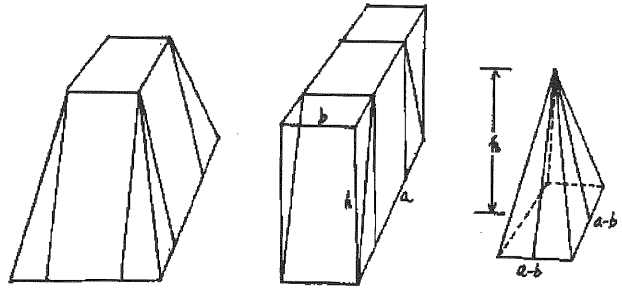

FIGURE 5

In the second explanation, there is no need to assume $h = b, a = 3b$. But it works only when the bottom and top pieces are squares.

Now, let me suggest a fictitious examination question: Compute the volume of an "oblong pavilion" of height h with bottom and top being rectangles of sides a_1, a_2 and b_1, b_2 respectively $(a_1 \neq a_2,\ b_1 \neq b_2)$. If one understands the argument by Liu Hui, one can easily modify either

method to arrive at the answer, which is left as an exercise for the readers. (Readers may also wish to solve the problem in a way commonly known to school pupils of today, viz. by making use of similar triangles.) The answer turns out to be

$$V = \frac{1}{3}[a_1a_2 + b_1b_2 + \frac{1}{2}(a_1b_2 + a_2b_1)]h.$$

If one merely memorizes the formula given in the textbook by heart, it is not easy to hit upon the correct formula. This is probably what is meant by "constructing a (new) algorithm". Again, such a demand is reasonable, especially when these candidates might well be facing in their subsequent career problems which are variations (e.g. with parameters changed) of the problems they learnt in the textbooks.

7 Conclusion

I have attempted to "re-construct" an examination in mathematics in the Tang Dynasty to persuade readers that the curriculum in that period was not so elementary nor was it learnt by rote. It is hard to imagine that a group of young men spent seven of their golden years in simply memorizing the mathematical classics one by one without understanding at all!

Granting that an examination is not to be passed through rote learning, what good will an examination bring?

Let us first compare the ancient Chinese examination format with a modern theory on assessment by B.S. BLOOM (BLOOM et al. 1956). The modern viewpoint includes both the formative and the summative aspects of assessment, while the ancient Chinese examination focused only on the latter function. The six major classes of taxonomy of Bloom can be matched up with the four different types of question in the ancient Chinese examination, viz. (i) testing on quotations is about knowledge, (ii) short questions are about comprehension and application, (iii) long questions (on contemporary affairs) are about analysis and synthesis, (iv) composition and poems are about evaluation.

With these varied objectives, examination can have a beneficial influence on both the student and the teacher. For the student it is good for consolidation of knowledge, enhancement of comprehension, planning of schedule of study, judgement on what is important to learn, development of learning strategies and motivation and self-perception of competence. For the teacher, besides what has been said above, it is good for monitoring the progress of the class, as a gauge of the receptivity and assimilation of the class and evaluation of the teaching. In this sense, "examination-oriented education" and "quality education" need not be a dichotomy. T.J. CROOKS says,

> As educators we must ensure that we give appropriate emphasis in our evaluations to the skills, knowledge, and attitudes that we perceive to be most important. (CROOKS 1988)

Viewed in the summative aspect, examination is a necessary evil. But viewed in the formative aspect, examination can be a useful part of the learning process. The important thing to keep in mind is not to let the assessment tail wag the educational dog! (TANG & BIGGS 1996). The demise of the examination system in Imperial China, even with its initial good intention and with its long life span of 1287 years, is a lesson to be learnt from.

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DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
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Arithmetic and algebra : can history help to close the cognitive gap ? A proposed learning trajectory on early algebra from an historical perspective

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Abstract

Is it possible for students to self-handedly gain access to early algebra starting from informal strategies embedded in arithmetic? Can apparently fundamental differences between arithmetical and algebraic conceptions of mathematical problems be (partly) surmounted? The historical development of algebraic problem solving and algebraic symbolic language has inspired the author to develop a prototype pre-algebra learning strand on reasoning and equation solving. This article sketches the project background and gives some examples of classroom activities.