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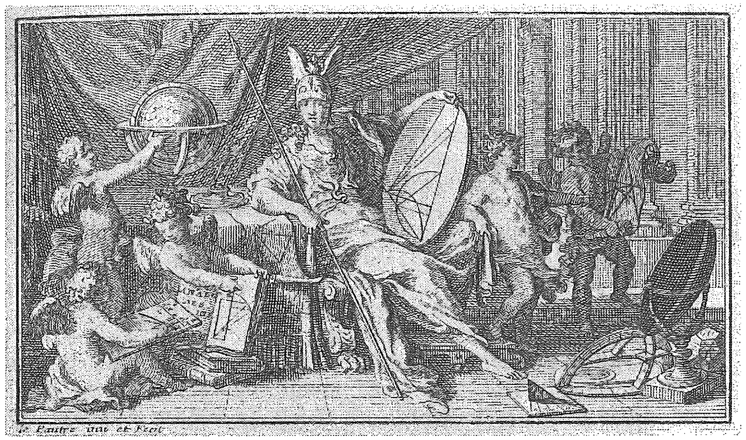

Exploring Fregean perspectives in mathematics education

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I shall persevere until I find something that is certain - or, at least, until I find for certain that nothing is certain.
(Descartes)

Abstract

Over the past two decades there has been a major upsurge of interest in the ideas of Gotlob FREGE (1848-1925). Two important and related themes in FREGE's writings are the logical foundations of mathematics and the importance of an appropriate conceptual notation in deriving mathematics from logic. While his thesis that mathematics is a branch of logic and his conceptual notation were deemed to fail, FREGE's perspectives concerning logic were to have a profound impact in both logical and mathematical developments in the 20th century. The goal of this lecture is to review a few of FREGE's ideas, not so much in terms of their contribution to either logic or mathematics, but rather, as the title of this lecture suggests, to try and explore Fregean perspectives for mathematics education. In particular, we explore three aspects of the link between FREGE's work and the way mathematics is taught in schools and universities : 1) the medium through which mathematics is communicated, 2) the nature of mathematical entities; and 3) the distinctiveness of the methodology of mathematics - based as it is, not on everyday ideas, but on abstract objects, exact calculation and proof.

1 Introduction

Three important and related themes in FREGE's (1948-1925) writings are: (i) the logical foundations of arithmetic; (ii) the importance of an appropriate symbolic language adequate both for any mathematical theory and simultaneously to embody all proofs within such a theory; and (iii) the need for an analysis of the meaning of words and sentences in ordinary language.

Largely ignored during his lifetime, FREGE's work has in the past three decades received considerable attention and recognition. FREGE is recognised as the father of 'linguistic philosophy', as the first philosopher to make a sharp distinction between analysis of the meaning of expressions and establishing what is true and the grounds for accepting it (DUMMETT 1981). FREGE is also celebrated as the founder of modern logic: the analysis of a proposition into function and argument(s), the truth-functional calculus, and the theory of quantification are among his fundamental contributions to the field.

FREGE's investigations into the foundations of arithmetic have both practical and theoretical significance. For example, PARSONS (1965), in assessing FREGE's theory of numbers, writes:

> It is impossible to compare Frege's *Foundations of arithmetic* with the writings on the philosophy of mathematics of Frege's predecessors - even with such great philosophers as Kant - without concluding that Frege's work represents as enormous advance in clarity and rigour. It is also hard to avoid the conclusion that Frege's analysis increases our understanding of the elementary ideas of arithmetic and that there are fundamental points that his predecessors grasped very dimly, if at all, which Frege is clear about. (p. 182).

In a similar spirit, DUMMETT (1991) claims:

> That [Frege's] philosophy of arithmetic was, indeed, fatally flawed; but it had an incontestable clarity, so that, even where it was mistaken, it pointed very precisely to where the problems lay. But it did much more than that. Frege's polemic against formalism contained a definitive refutation of that deadening philosophical interpretation of mathematics. To important questions in the philosophy of mathematics, above all those concerning the application of mathematics, the fruitfulness of deductive reasoning and the nature of mathematical necessity, his work provided, if not full-dress answers, at least sketches of what must be correct answers; later philosophers have come nowhere near his partial success in answering those questions, and have frequently failed to address them. Above all, Frege provided the most plausible general answer yet proposed to the fundamental question, 'What is mathematics', even if his answer cannot be unarguably vindicated. For all his mistakes and omissions, he was the greatest philosopher of mathematics yet to be written. (p. 321).

The goal of this paper is to review a few of FREGE's ideas, not so much in terms of their contribution to logic, or philosophy of mathematics, or philosophy of language, but rather, as its title suggests, to try and explore Fregean perspectives for mathematics education.

2 An outline of Frege's logicist program

By tracing the evolution of FREGE's logicist program (to show that arithmetic is part of logic) one might expect to gain some understanding of the processes by which mathematical theories develop. Indeed, FREGE provides an eloquent illustration of the creative mathematician.

DUMMETT (1981) considers FREGE's career divided into six periods, of which the first three correspond to setting out his project. Inspired by HADAMARD's (1945) views presented in his

Psychology of Invention in the Mathematical Field, we propose to add an initial period to the first three suggested by DUMMETT, drawing attention to the fundamental stage of unconscious work of incubation of new ideas. We have termed these four periods as *1) Incubation; 2) Preparation; 3) Illumination; and 4) Formalisation.*

2.1 Incubation

FREGE started his career as a mathematician with the presentation of his doctoral dissertation, *'On a Geometrical Representation of Imaginary Forms in the Plane'*, in 1873. In the following year he wrote his *Habilitationschrift, 'Methods of Calculation based on an Extension of the Concept of Magnitude'*, which would allow him the post of Privatdozent at the University of Jena. It is clear that this latter work contains the germs of FREGE's logicist insights. On the one hand, we see FREGE's underlying idea that whereas geometry can be intuited, 'quantity' cannot be an intuitive notion: "Bounded straight lines and planes enclosed by curves can certainly be intuited, but what is quantitative about them, what is common to lengths and surfaces escape our intuition". On the other hand, there is in this work an expanded notion of function related to the one he would later use extensively in his project.

In 1874, FREGE published a review of a book on arithmetic. He was largely unsympathetic to this work: he could not understand how it was that the propositions which formed the foundation of arithmetic were "lumped together without proof", while "theorems of a much more limited importance are distinguished with particular names and proved in detail". BYNUM (1972) suggests that it was probably his disappointment with this book which gave birth to FREGE's decision of setting out to explore the foundations of arithmetic.

2.2 Preparation

FREGE's investigation of the foundations of arithmetic made it necessary for him to engage in two different tasks. First, as one might expect, he focused his attention on views of other scholars on the topic: from Euclid to Newton, from Thomae to Cantor, from Leibniz to Spinosa, from Kant to Mill. In so doing, FREGE became convinced that previous approaches were untenable, and that there was a real need to find a new way out.

Second, FREGE set out to make logic more rigorous. Though attempts in this direction were already beginning to be made, namely by De Morgan and Boole in England, FREGE felt that they were entirely inadequate for his own purpose. This led him in a natural way to the development of his logical system which culminated with the publication of the *Begriffsschrift* in 1879.

The significance of this book can hardly be overlooked nowadays. It marks the beginning of modern logic, standing out as a major contribution to the field of logic and no longer as a preparatory work. But at the time, the ideas and notation that FREGE presented in this book were so revolutionary that very few people were able to appreciate them. To answer his critics, FREGE went on to defend his conceptual notation and ideas, comparing them with those of Boole, but without much success. He even saw a couple of the papers he submitted for publication rejected.

2.3 Illumination

Although disappointed and frustrated, FREGE continued to be motivated to find a solution for his original problem of the foundations of arithmetic. In the process of reviewing other authors, and finding sufficient grounds to criticise them, FREGE began to formulate his own ideas - an entirely new perspective, inspired in part by Leibniz (as it was indeed the case of his logical system).

His incisive critique of alternative views about number and his own approach were made public, in 1884, in his *Grundlagen der Arithmetic*, one of FREGE's best-known works. In this book, FREGE succeeded in lending plausibility to both his definition of number and showing that the series of natural numbers was endless. At this stage, however, FREGE presented his own project in a totally informal manner. His next step would be to provide a formal version of it.

Once more, the reaction to FREGE's work was disappointingly poor, but once more FREGE was motivated to continue to work on his project. It was during this time that FREGE presented some of his most brilliant insights concerning language. Having recognised that some of the notions he had used needed clarification, he published three influential papers, *Function and Concept, On Sense and Reference*, and *On Concept and Object*, in which he elaborated upon: (i) the notion of function, and the possibility of admitting functions of various levels, (ii) the distinction between sense and reference, and (iii) the distinction between concept and object.

2.4 Formalisation

Finally, in 1893, FREGE published the first volume of *Grundgesetze der Arithmetik*. In it, FREGE presented a reformulated version of his logical system, and carried out within such a system the construction of arithmetic sketched in the *Grundlagen*. He was now absolutely convinced that he had achieved his goal, but this work had no more success than his previous writings in terms of its reception and acceptance.

Despite this further disappointment, FREGE went on to write a large number of papers. It was a period of consolidation. His efforts to make his ideas known and accepted drove him to correspond with other scholars, such as Ballue, Couturat, Pasch, Peano, Hilbert, and Husserl.

Finally, in 1902, the second volume of the *Grundgesetze* was published. Here, he continued the work of formally deriving arithmetic from logic. There is also an attempt to derive the whole theory of real numbers from the same source, but this work is left uncompleted.

It is obvious that FREGE still intended to publish a third volume of the *Grundgesetze*. However, just before his second volume was published, Russell presented him with a paradox which showed that his program was inconsistent. For some years, FREGE kept up a correspondence with Russell who was visibly interested in his ideas. FREGE had finally found somebody who could appreciate his work. Until 1905, the year in which his wife died, FREGE still attempted to salvage his program, but it was in vain. Two devastating attacks on his work in the following year left him convinced that he had lost his battle. It was the beginning of the end.

3 What are these things called numbers?

One has only to read the Introduction to *Grundlagen*: to realise how important it was for FREGE to define in a precise way what numbers are:

> is it not a scandal that our science should be so unclear about the first and foremost among its objects, and one which is apparently so simple? Small hope, then, that we shall be able to say what number is. If a concept fundamental to mighty science gives rise to difficulties, then it is surely an imperative task to investigate it more closely until those difficulties are overcome. (p. IIe)

It is well to remember that FREGE was not alone in this kind of preoccupation. At roughly the same time, famous mathematicians (e.g. Dedekind, Cantor) expressed similar concerns. Their concerns are embedded in a broader movement which included the rigorisation of mathematics. In FREGE's words "that mighty academic positivistic scepticism which now prevails in Germany... has finally reached arithmetic".

Like Cantor, but unlike Dedekind, FREGE made philosophy an important partner to his discussion. According to him: "any thorough investigation of the concept of number is bound always to turn rather philosophical. It is a task which is common to mathematics and philosophy". But as BENACERRAF (1981) concedes, *Grundlagen* is first and foremost a mathematical enterprise. Philosophy comes in only as a convenient way to emphasise FREGE's point that arithmetical propositions must be proved "with the utmost rigour".

FREGE's starting point for solving the problem of what numbers are was to review the positions about the matter most commonly encountered among both philosophers and mathematicians. His analysis and refutation of other authors' views about number and numerical propositions is in three parts and amounts to over 60 pages. Here, we can do little more than restate his words in summarising his review:

> Number is not abstracted from things in the way that colour, weight and hardness are nor is it a property of things in the sense that they are. [...] Number is not anything physical, but nor is it anything subjective (an idea). Number does not result from the annexing of thing to thing. [...] The terms 'multitude', 'set' and 'plurality' are unsuitable, owing to their vagueness, for use in defining number. (1953, p. 58e)

What then is a number? FREGE turned the problem around by answering a non-linguistic question with a linguistic answer: "the content of a statement of number is an assertion about a concept". This was precisely what he meant when he stated one of the three principles - the one known as the *Context Principle* - on which he centred his investigation: *never ask for the meaning of a word in isolation, but only in the context of a proposition. (The other two principles are: (i) always separate sharply the psychological from the logical, the subjective from the objective; and (ii) never lose sight of the distinction between concept and object).*

FREGE's answer to the question is given in Part IV of the *Grundlagen*. This is divided into four subsections, each serving as a frame within which his thinking develops: (a) *every individual number is a self-subsistent object*; (b) *to obtain the concept of Number, we must fix the sense of a numerical identity*; (c) *our definition completed and its worth proved*; and (d) *infinite Numbers*. Here, we proceed by summarising his discussion in two parts, one corresponding to finite numbers and the other to infinite numbers.

3.1 Finite numbers

It is clear that in constructing his theory of natural numbers FREGE ran into considerable difficulties, and that he wanted to tell the reader about them. Hence, in his presentation, FREGE chose to incorporate how ideas and goals were formulated and refined in the course of action. For example, he began the first subsection (§55) by suggesting definitions of the expressions *the number 0 belongs to a concept, the number 1 belongs to a concept, and the number* $(n+1)$ *belongs to a concept*, but in §56, FREGE rejected them on the grounds (in part unconvincing) that in this way numbers would not be recognised as self-subsistent objects.

Next, at the beginning of the second subsection (§62), after restating his *Context Principle*, FREGE turned to the question of defining the sense in which two numbers are the same. In this he attempted to follow the road opened by Hume according to whom "when two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them equal". But once more he found this way not completely satisfactory: identity ought to be a general notion, and not just one which is applied to numbers. His second idea was to draw on Leibniz' definition of identity: "things are the same as each other, of which one can be substituted for the other without loss of truth". At his stage, FREGE resorted to a comparison with the idea of parallel lines and of direction of a line: "the direction of a is identical with the direction of b is to mean the same as line a is parallel to line b". But again, he noted a difficulty with such an approach: "that says nothing as to whether the proposition 'the direction of a is identical with q' should be affirmed or denied, except for the one case where q is given in the form of 'the direction of b'". He then chose a different but related path that led him to present, in §68, an explicit definition of direction, namely by defining directions as equivalence classes of lines, or as he put it as the extension of the concept parallel to a particular line.

Finally, in §68, too, he presented an analogous definition of number: "the Number which belongs to the concept F is the extension of the concept 'equinumerous to the concept F'".

Two observations are worth making at this stage. First, FREGE used the term 'extension' without having introduced it previously. In a footnote he observed that it was assumed to be known what the term meant. Whether or not he felt that there was something problematic with the notion of 'extension' is another story. The matter of fact is that, in *Grundlagen*, after showing (in §73) that "the extension of the concept 'equinumerous to the concept F' is identical with the extension of the concept 'equinumerous to the concept G' is true if and only if the proposition 'the same number belongs to the concept F as to the concept G' is also true", FREGE never used the notion again.

The second observation concerns the notion of 'equinumerous'. FREGE justly considered that such a notion may be defined in terms of one-one correlation. Moreover, he examined at some length the problem of whether or not this latter notion has anything to do with intuition, resolving the issue in favour of the doctrine of 'relation-concepts', which are part of pure logic. On the basis of the above definition, FREGE then proceeded to define the expression 'n is a Number' by stating that it is to mean the same as "there exists a concept such that n is the number which belongs to it". Then FREGE introduced the notion of the Number 0, by stating that "0 is the Number which belongs to the concept 'not identical with itself'". In describing his train of thought, FREGE mentioned that he could have used for the definition of 0 any other concept under which no objects falls, and showed further that every concept under which no object falls is equinumerous to every other concept under which no objects falls.

One would expect that FREGE would next define the Number 1. But he chose first to consider the general case of two adjacent numbers of the series of natural numbers, by introducing the definition of the expression 'n follows in the series of natural numbers directly after m' to mean the same as

> there exists a concept F, and an object falling under it x, such that the Number which belongs to the concept F is n and the number which belongs to the concept "falling under F but not identical with x" is m.

He also observed that he was not using the expression 'n is *the* Number following next after m' since he had shown neither that such an object existed nor that there was only one such an object.

Using this latter definition, FREGE went on to show that there exists a Number that follows directly after 0, namely the Number which belongs to the concept 'identical with 0', which by definition he took as being the Number 1. With this definition it is not clear that 1 is the successor of 0, but FREGE added a series of four propositions concerning the Number 1 which show that this is effectively the case.

The following, no less important, step consisted in showing that every natural number has a successor. This FREGE did by outlining the proof that "the Number which belongs to the concept 'member of the series of natural numbers ending with n' follows directly after n, given that n is a member of the series of natural numbers beginning with 0". The concept of 'member of series of natural numbers ending with a' was given in terms of 'the following of an object y after an object x in a general series', essentially the same notion that he had already defined in purely logical terms in the *Begriffsschrift*. FREGE used the expression 'n is a finite Number' to mean the same as 'n is a member of the series of natural numbers beginning with 0'.

To show that there is no last member in the series of natural numbers beginning with 0, FREGE stated that it was necessary to show that no finite number follows itself, and indicated how to prove this fact.

3.2 Infinite numbers

In contrast with his discussion of finite numbers, his presentation of infinite numbers is very short and direct. In line with the terminology he had used to define finite Numbers, he stated "the Number which belongs to the concept 'finite Number' is an infinite Number", and he used the symbol ∞_1 to denote it. Specifically, he observed that such a Number was not a finite one since it could be shown that its was a successor of itself. And he remarked:

> About the infinite Number ∞_1 there is nothing mysterious or wonderful. "The Number which belongs to the concept F is ∞_1" means no more and no less than this: that there exists a relation which correlates one to one the objects falling under the concept F with the finite Numbers. In terms of our definitions this has a perfectly clear and unambiguous sense; and that is enough to justify the symbol ∞_1 and to assure it of a meaning. [...] Any name or symbol that been introduced in a logically unexceptionable manner can be used in our enquiries without hesitation, and here our Number ∞_1 is as sound as 2 or 3. (1953, p. 96e-97e)

The title of the subsection, *infinite numbers*, and the fact that he used the symbol ∞_1 justify the suspicion that FREGE was prepared to define further infinite numbers. And yet he did not do that. On this same subject, one further point is worthy of note. As BOOLOS (1987) writes, it is somehow strange that FREGE did not define the number belonging to the concept 'identical with itself'. Hence he did not deal with the question of whether such a number would be the same as ∞_1. BOOLOS' point is that FREGE could not possibly have failed to consider such a number, and suggests that he would have regarded them as different.

Note, however, that FREGE devoted a considerable part of his subsection on infinite numbers in the *Grundlagen* to comment on the work that Cantor had published in the previous year. He praised Cantor's aims and spoke unmistakably in favour of transfinite numbers, although he openly criticised Cantor's indefinite and unclear use of the notions 'following in the succession' and 'Number' based on 'inner intuition'. He went on to say that he could anticipate how these two concepts could be made precise, but he did not offer any additional explanations.

In his later work, the *Grundgesetze*, FREGE did not take the analysis of infinite numbers much further. He even wrote in the Introduction that the propositions concerning the infinite number could have been omitted since they were not necessary for the foundation of arithmetic. Clearly, infinite numbers were not at the centre of FREGE's mathematical interests.

4 Remarks on later developments

FREGE's comments at the end of the *Grundlagen* show that he was convinced that the truths of arithmetic can be derived from logic and from logic alone. Yet he was still determined to raise this conviction to the level of absolute certainty. He regarded it as fundamental to his enterprise to provide formal proofs of all the arithmetical results he had presented in the *Grundlagen* - something he already had in mind when he developed his conceptual notation in the *Begriffsschrift*. It is worth noting that his idea was that the aim of proof is "not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another".

In the *Grundgesetze*, FREGE carried out his project in the way he had foreseen: "every 'axiom', every 'assumption', 'hypothesis', or whatever you wish to call it, upon which a proof is based is brought to light", so that there are no gaps in the chains of inference. In order to be able to meet such standards, FREGE felt it necessary to lay down previously all the primitive terms (it may be worth noting that one of the central primitive notions in FREGE's system is that of *function*) and symbols he would use, as well as all the 'axioms' to which these terms conform.

Likewise, FREGE presented the rules of inference that he would utilise to derive new results and the rules to introduce new names. Moreover, every single notion needed to define numbers was introduced previously in symbolic terms. Indeed, it is striking how meticulous FREGE was in confirming that his definitions were in agreement with the rules that he had laid out, in order to secure that the new symbols do have a referent. The crucial point for FREGE was that no symbolism brings into life a new being. Definitions are just abbreviations and one could well manage without them.

But on the whole, the path he followed to define numbers is similar to that he pursued in the *Grundlagen*. His comments in the Introduction to the *Grundgesetze* show that, while he could foresee some problems with regard to the acceptance of his Axiom V, he was convinced that nobody could erect a more durable edifice than the one he had built, nor show that his principles would lead to false conclusions.

Russell's letter informing FREGE that he had found a contradiction which could be derived from FREGE's system, namely from Axiom V, was a tremendous blow. FREGE still tried to salvage his system by presenting a different version of that axiom. But this and other possible solutions he attempted afterwards were unsuccessful. In the last years of his life FREGE judged his own

project - namely to show that numbers can be defined in purely logical terms - a complete failure. Throughout his life FREGE had dismissed Kant's view that arithmetic was synthetic *a priori*, but had accepted such a view with regard to geometry. Interestingly, he later stated that "arithmetic and geometry have developed on the same basis - a geometrical one in fact -so that mathematics in its entirety is really geometry". However, no time was left to him to explore such a radical view.

5 Implications for mathematics education

While FREGE's work is of considerable interest, its primary appeal is to philosophers, mathematicians and historians of mathematics. Our original interest in his work arose in the context of the nature of proof from precisely such a perspective. And though this was part of a larger program concerned with the position of proof in school mathematics (GARDINER & MOREIRA 1999a, 1999b; MOREIRA & GARDINER 1999) FREGE's work is so focused on subtle details that we realised only slowly the lessons for mathematics education which are implicit in his work, and in the way it was received.

The quote from DUMMETT (1991) in the Introduction acknowledges that, though FREGE's program failed, "it pointed very precisely to where the problems lay [and] did much more than that" For our purposes here, the details are in some ways less important than

- FREGE's general goal of identifying and characterising those concepts which constitute the basis of mathematics, and
- the reception which his efforts received.

Our remarks are presented under three headings, which represent important themes both for FREGE and for mathematics education.

5.1 Mathematics and language

FREGE wanted to establish mathematics on a foundation which avoided the ambiguities of ordinary language. The first (and perhaps simplest) part of his program was to establish a precise, formal language in terms of which everything else could be expressed.

FREGE struggled to make a clear distinction between *Zeichen, Sinn, and Bedeutung* - that is between a *sign* or *symbol*, its *sense* (what one understands from it), and its reference (what it denotes). Such distinctions are relevant not only in philosophy, but at all stages of education. Mathematics educators recognise the importance of deciding precisely which such distinctions need to be made, and when and how they should be made (e.g. ARZARELLO et al. 1995; BAZZINI 1998; FERRANDO 1998).

FREGE realised that Boole's symbols were too limited for a symbolic logic which included variable and quantifiers, so he had no choice but to develop his own notation. Given the profound nature of his insights into the foundations of logic and of mathematics, one could argue that the language he chose to use was largely irrelevant to the success or otherwise of his program. However, as with mathematics textbooks and courses, the language he used constituted a superficial barrier, which restricted access and sapped the determination of those who might otherwise have made sufficient progress to get to the heart of the questions FREGE was trying to address. In other words, while FREGE's commitment to the need for a precise language and symbolism was entirely sound, the language and symbolism he used hindered the understanding and acceptance by his contemporaries of his more significant logical ideas.

Today we recognise that many of the basic concepts and structures in modern logic have their origins in FREGE's work. However, the symbols we use to denote them derive from Peano - not directly from FREGE. Unlike Peano, the logical symbols which FREGE introduced ignored the linear tradition of European printing and reading, and adopted instead a 2-dimensional format. It is generally accepted that this made it more difficult for his logical ideas to be accepted: the difficulties which FREGE's notation in the *Grundgesetze* presents to the reader may explain why few scholars appear to have looked closely at his theory of real numbers. However, these difficulties arose not only because of the symbols FREGE used, but also because he failed to win over influential members of the mathematical community. Established tradition is not without its justification. New symbols and styles to overcome a naturally conservative view of what is appropriate: the ultimate fate of new ideas often depends on whether they are adopted by senior members of the relevant community, who then mediate their use to others (GAUVAIN 1998). For example, William Jones introduced the symbol π for the ratio circumference : diameter of a circle in 1706; but it was not until Euler adopted the symbol in the 1730s that its use became widespread. The radical nature of FREGE's work, and his rejection of much established practice, placed him firmly on the fringe, rather than in the centre, of contemporary mathematical culture.

The fact that FREGE wrote the *Grundlagen* in a relatively informal prose style indicates that he was aware of the difficulty. Indeed, if he had not written the *Grundlagen* in this more accessible manner, his other works may well have received even less attention than they did. But FREGE knew that, if he was to do justice to his overall goals, he had no choice but to use a precise, formal language. Whereas he compared ordinary language to the human eye, he saw his conceptual notation as like microscope, which may not help one to see "the big picture", but which is essential if one wants to uncover important, yet previously unsuspected, details.

In the context of mathematics teaching, the language we use determines how our subject is perceived and learned. We cannot (and should not try to) avoid symbols and precise language. Yet we have to recognise that learners face a substantial challenge in seeking to master the unfamiliar language of mathematics - a language that is scarcely used outside the classroom, and whose symbols require the user to observe special conventions (MORGAN 1999). Learners should not be left to stagnate in a primitive world of baby-language and naive concepts; but neither should they be prematurely swamped by meaningless terminology and rules. Hence, when a new mathematical topic, method or convention is introduced, we first seek to explain things in familiar language - initially using everyday language, then in terms of previously learned mathematical language. Learners must eventually understand that the advantages of mathematical exactness and precision are available only to those who are prepared to make this transition, who learn to respect and use its language and to breathe freely in its rarefied air. But we as teachers have to remain sensitive to the difficulties they face, and to the fact that those who fail to make this transition are likely to remain alienated from the mathematical ideas and methods which the mathematical language and symbolism encapsulates.

Reflecting on the difficulties faced by anyone who tries to understand FREGE's conceptual notation lends weight to the traditional idea that it can help to break down the process of accessing highly unfamiliar material into four stages.

- First, it is important to understand that what mathematics has to offer depends on the precise character of its terminology and language; that is, learners must accept that precise ideas and exact thinking are only possible if we restrict attention to a *mathematical world* with its associated unfamiliar language, notation and rules.

- Second, although the terminology and language may be unfamiliar, it is always important for human beings to relate the unfamiliar to the familiar, to make sense of, or grasp the meaning of, what it is that is being encapsulated in unfamiliar language or notation. In particular, examples should be given which indicate the extent to which the new and unfamiliar language sharpens important distinctions and allows one to describe and to work more accurately with important ideas (that is, distinctions and ideas whose importance can be appreciated by the learner).

- Third, one needs time and plenty of opportunity for *routine*, meaningful practice, so that the new methods and language can be incorporated into existing cognitive structures, and can be used to consolidate previous ideas and to extend one's powers of analysis and calculation.

- Finally, it is important to provide opportunities for learners to use, and to talk about, new methods and ideas in non-routine settings, so that they and their teachers become aware of outstanding conflicts, misconceptions and limitations.

5.2 The nature of mathematical objects

At the root of all FREGE's work is the fundamental question: What is the exact nature of mathematical objects? Though the way he pursued this question may not be directly relevant to the classroom, mathematics teachers face a closely related question every day. What is the status of the entities we work with in mathematics textbooks, in pupils' workbooks, and on ordinary school blackboards and white boards?

> These [and other related questions] are not just ruminations of philosophers hidden in ivory towers. The answers we give to them have a profound impact on our educational policies and research programs. Piaget's constructivism and Bourbaki's austere rigour have left their marks on our schools. (DEHAENE 1997, p. 232)

What teachers think (perhaps unconsciously) about the nature of mathematics affects the way they teach, and hence the way their pupils learn. By struggling with the kind of epistemological questions addressed by FREGE, teachers can become more aware of how questions reveal themselves in the classroom, and can begin the long process of developing a view of their subject which is consistent with the nature of the discipline.

FREGE asserted that numbers exist as "objects". By this he meant that they are *objective* - not that they are "real". FREGE rejected the empiricist view (associated with John Stuart Mill) that numbers are somehow "abstracted from experience". He also rejected the formalist conception (associated with Thomae and, later, with Hilbert) of "numbers as mere signs manipulated in conformity with certain rules specified by mathematicians" - that is, that numbers are mere creations of the human mind.

FREGE's view that numbers are objective may seem as implausible as the views he rejected. When the torpedo of Russell's paradox revealed the flawed nature of the *Grundgesetze*, FREGE accepted that his efforts to establish the objective nature of numbers had failed. Yet to the end of his life he refused to equate numbers with mere symbols. And he continued to reject the empiricist fallacy that numbers can somehow be extracted from experience. In particular, he argued that if numbers were based on sense perception, then the series of natural numbers could never be endless: this, FREGE remarked, "is not just false, it is absurd".

A final answer to the underlying question "What exactly are numbers?" continues to elude us. Yet FREGE's analysis, and subsequent work during the twentieth century, mean that we are much clearer about the nature of the question and of the extent to which we can give partial answers. The logicist approach may be epistemologically and ontologically different from the formalist and the intuitionist approaches; but each has something to teach us about the nature of mathematics. Moreover, the strengths and limitations of each approach are now more clearly understood. By studying such fundamental questions one comes to appreciate that mathematics is not quite the monolithic construction it is often thought to be. The logical structure of mathematics is indeed hierarchical, cumulative and linear: but it is more like a tree - developing downwards as well as upwards, with roots as well as branches taking part in the process of development.

5.3 Methodology of mathematics

In recent years there have been numerous claims that school mathematics should more closely reflect the adult discipline of mathematics. For example, HILTON, HOLTON & PEDERSON (1997) assert that "mathematics must be taught so that students comprehend how and why mathematics is done by those who do it successfully". To assess whether this a realistic goal -and to understand better the nature of the discipline they profess- future teachers need to confront the question: "What is it that mathematicians do when they do when they do mathematics?".

This confronts us with a dilemma. In the context of modern mathematics education one can no longer simply declare that school mathematics constitutes an "apprenticeship", and that mere apprentices are quite different from mature mathematicians; thus, the proposition that we should teach in a way that provides high school graduates with some insight into what mathematicians do has a superficial appeal. On the other hand, the attempt in England during the late 1980s and 1990s to make "investigation" a significant part of the official curriculum only served to underline the dangers.

School curricula have to survive the process of institutionalisation; that is, curriculum content has to retain its essential form and its value even when subjected to the limitations, distortions and stereotyping of textbooks, examinations and the whole gamut of pupils, teachers and schools. Though traditional curricula are often criticised, the kind of standard topics they contain have proved themselves over the centuries to be remarkably "stable" in the face of such challenges. In contrast, the art of doing mathematics is not - and may never be - sufficiently well understood to allow us to develop a universal approach at school level which is comparably stable.

Mathematical practices - as HADAMARD (1945) showed - cannot be described in a uniform way. POLYA's classic books further illustrate the difficulty of showing what it is that mathematicians do. On the one hand, his major works (1954, 1962) are far too demanding for most potential teachers; on the other hand, his popular version (1957) has effectively encouraged many to present the process of solving mathematical problems in embarrassingly simplistic terms. Attempts to capture the spirit of mathematical problem solving in a manner which remains faithful both to mathematics and to adolescent psychology (e.g. GARDINER 1987a, 1987b) have been

largely ignored in favour of styles of exploration at school level which are antithetical to mathematics.

FREGE could scarcely be described as a typical working mathematician. But he was a very creative and productive scholar. The fact that much of his work concerns something as basic as natural numbers makes him a valuable case study for anyone who wants to understand the difficulties faced by those who struggle to construct and to communicate a novel mathematical theory. The details of FREGE's work may be appropriate and accessible only to those interested teachers (for whom we recommend the English translation of the *Grundlagen* (1959) as a starting point). But one feature of his program highlights an essential ingredient in any school mathematics curriculum which wishes to convey to high school graduates what it is that makes mathematics special.

FREGE's logicist program led to a distinctive approach to *proof*. FREGE insisted that arithmetic had not yet been properly mathematised, and he wanted to show that arithmetical facts can be derived unequivocally from basic *logical* laws, using only logical rules of inference. In one sense, his plan was to extend Euclid's vision from geometry to arithmetic. His goal was to demonstrate how arithmetical propositions may be derived from a few axioms and primitive terms, and to make explicit the logical rule which are used at each stage.

Within such a program the central focus shifts from the apparent subject matter (arithmetic) to the underlying methodology - namely *proof* itself. FREGE was concerned not so much to establish the "truth" of arithmetical propositions, as to identify the logical dependencies between propositions, and between propositions and the axioms. A simplified version of this shift of focus occurs in traditional school geometry - even if it is best experienced directly and subconsciously by the pupil rather than being made explicit at that stage.

- The initial goal is to set up a small number of geometric principles in order to analyse geometrical problems from the world around us.
- To carry this through one moves from the real world into a world of mathematical objects (ideal, *mental* triangles, rather than cardboard cut-outs), together with a hierarchy of results, and accepted rules of inference.
- The problems one then solves need to be sufficiently meaningful for pupils to engage with the task. But the details of each problem are quickly forgotten. What remains is the sense of having played the game of deducing the required result from earlier known results within the hierarchy, according the accepted rules.

If we seek to convey something of the flavour of real mathematics to those in schools, then all learners need to experience and internalise the difficulties and delights of this process - through solving problems in elementary arithmetic in a structured way, through solving equations, through calculating estimates, through reducing the solution of hard problems to easier known facts, and through angle-chasing and the derivation of important results in elementary geometry - such as the formula for the area of a triangle, or the fact that the angles in any triangle sum to two right angles (see GARDINER & MOREIRA 1999a, 1999b; MOREIRA & GARDINER 1999). There is then a chance that pupils will appreciate not only that mathematical reasoning is very different from everyday reasoning, but how it differs, and why it has to differ if it is to provide us with reliable results.

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