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Some reflections on the illusion of linearity

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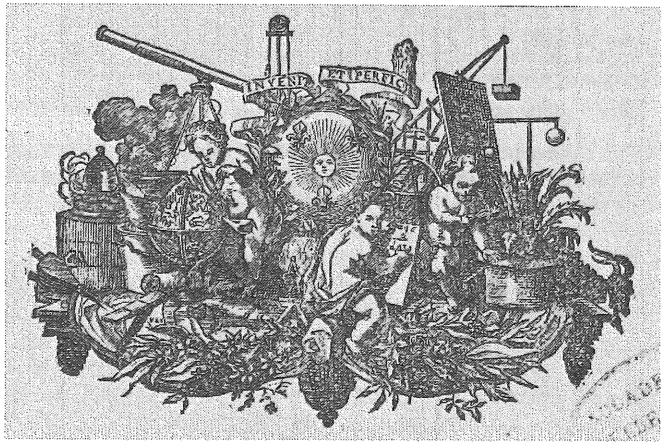
Abstract

Linear (proportional) functions are undoubtedly one of the most common models for representing and solving both pure and applied problems in mathematics education. But according to several authors, different aspects of the current culture and practice of school mathematics develop in students a tendency to use these linear models also in situations in which they are not applicable. In this paper, we first present some historical examples of and comments on this "illusion of linearity". Second, we briefly discuss the results of five recent empirical studies about the occurrence of this phenomenon in 12-16-year old students working on problems about the relation between the linear measurements, the area and/or the volume of similar geometrical figures, as well as about the effect of several task variables on this improper use of linearity. Finally, we analyse the connection between this linear illusion and other intuitive rules and erroneous ways of thinking in mathematics education.



1 Introduction

Linear (or directly) proportional relationships are a major topic in elementary mathematics education. The attention given to linear relationships is mainly due to the fact that they are the underlying basic model for a lot of problems in pure and applied mathematics. Unfortunately, students' growing familiarity and experience with linear models may have a serious drawback: it may lead to the misbelief that these models have a "universal" applicability and to a tendency to deal with each numerical relation as though it were linear (FREUDENTHAL 1983). In the literature, this phenomenon is referred to as the "illusion of linearity", "linear misconception", "linear obstacle" or "linear trap" (see, e.g., BERTÉ 1993; FREUDENTHAL 1973, 1983; ROUCHE 1989). Although these terms may have subtly different meanings (for instance, the term "linear illusion" carries a connotation of visual perception which seems absent in the other terms), we will not differentiate between them and use them intermixedly in the rest of this article. The illusion of linearity in students' reasoning has been frequently described and illustrated with respect to different domains of mathematics in this literature, but it has elicited little systematic empirical research. This is quite remarkable taking into account the large amount of research on proportional reasoning in the last 10-15 years (for reviews of this research see, e.g., BEHR, HAREL, POST & LESH 1992; TOURNIAIRE & PULOS 1985), including research on students' misconceptions about and primitive strategies for solving proportional tasks.

2 Examples of the illusion of linearity

A first series of examples relates to elementary school pupils' modelling of word problems in school arithmetic.

"It takes 15 minutes to dry 1 shirt outside on a clothesline. How long will it take to dry 3 shirts outside?"

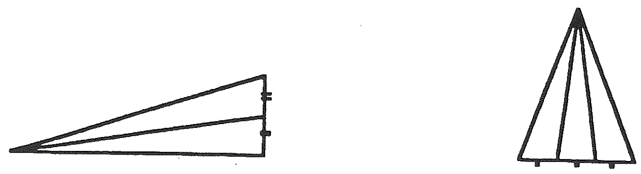
"It costs 2 EUR to send a parcel weighing 500 g. How much will it cost to send a parcel weighing 1500 g?"

"John's running record for 100 m is 12 seconds. How long will it take him to run 1 km?"

Pupils who fall into the trap of proportional reasoning on the first example, obviously neglect the realistic consideration that the drying time doesn't depend on the number of clothes on the line. Rather than using their real-world knowledge about the situation described in the problem - namely: drying clothes - they simply play the "game of school word problems" (VERSCHAFFEL, GREER & DE CORTE, in press), in which the players are assumed not to attend too much to the realities of the situation described in the problem statement and to identify the arithmetic operation(s) with the given numbers that yields the correct answer. In the example of the mail costs, it is less sure that pupils giving the incorrect, directly proportional response neglect to apply their knowledge of the context or meaning of the problem. Probably, they are just ignorant of the specific mathematical model linking mailcost to weight, a "staircase" instead of a linear function (VERSCHAFFEL et al., in press). Apart from that, even if they knew the adequate mathematical model, this knowledge wouldn't enable them to answer the problem correctly; therefore, they have to know the appropriate costs for sending parcels or at least have the possibility to look them up. In a way, the problem about the running time is even more delicate because, strictly speaking, it cannot be solved correctly based on the information given, on the one hand, and the solver's real-world and mathematical knowledge, on the other hand.

An adequate mathematical model for this problem situation, enabling to determine precisely the running time, and thus taking into account the time the runner needs to accelerate at the beginning as well as his tiredness latterly, isn't obvious at all. A student who realizes that the distance and the running time in this "problem" are not proportional, is -in some sense- placed in a dilemma: either acknowledging that the realities of the problem context prohibit that it is solved by means of proportional reasoning and therefore refusing to give a (precise) answer (which is a violation of one of the basic rules of the "game of word problems", namely that word problems always have to be solved by means of a numerical answer), or playing the game by doing as if one had not detected that mathematical modelling complexity and responding with the outcome of a routine proportional reasoning process.

A mathematical subject-matter in which many people (frequently) fall in the proportional trap is *geometry*. For instance, it appears that some pupils regularly apply an incorrect linear reasoning in problems involving the relationships between angles and sides of plane figures. Figure 1, borrowed from ROUCHE (1992a), suggests incorrect methods for bisecting (drawing left) and trisecting (drawing right) an angle.



The first construction assumes a linear relation between an acute angle and its opposite side in a right-angled triangle. The second construction, which is correct for the bisection but not for the trisection of an angle, assumes a linear relation between the angle at the top and the base of an isosceles triangle (or between an angle in a circle and its corresponding chord).

Some other interesting geometrical examples of the linearity illusion can be found in the doctoral dissertation of DE BLOCK-DOCQ (1992). In the margin of her "epistemological comparative analysis of two teaching methods for plane geometry with pupils of the age of twelve", she mentions some typical erroneous reasoning processes, based on an inappropriate application of direct (or inverse) proportionality between non-proportional quantities.

"The angle of a regular dodecagon can be obtained by dividing the angle of a regular hexagon by six and multiplying this result by twelve."

"To construct an equilateral triangle inscribed in a circle, one has to pace the diameter on the circumference; an inscribed regular dodecagon can be constructed by pacing the half of the radius on the circumference."

"If one can split a heptagon into five triangles (by joining one corner to all the other corners), one can split up a 14-sided polygon into ten triangles."

The field of probabilistic thinking provides also a lot of nice examples of improper applications of proportionality. Confronted with the problem:

> *"In a game of chance the probability of success is one tenth. What is the probability of at least one success if one plays the game of chance three times?"*

many pupils will trap into proportional reasoning.

Very famous in this domain are the two historical problems Chevalier de Méré posed to his friend Pascal (see, e.g., FREUDENTHAL 1973). De Méré knew (by experience?) the advantage of betting on the event "at least one six in 4 rolls of one fair die" and he deduced that it must be equally advantageous to bet on "at least one double-six in 24 rolls of two fair dice". One specific outcome among 6 possibilities in 4 trials must occur equally frequent (and thus justify the same stake) as one specific outcome among 36 possibilities in 24 trials, because 6/4 = 36/24. Later on, because he experienced that, notwithstanding his reasoning, bets on the latter event didn't yield the hoped-for financial gain, he consulted his friend Pascal. Pascal's (correct) answer was as follows: the probability of no six in one roll of a fair die is 5/6, thus the probability of no six in 4 trials is $(5/6)^4$ and thus the probability of at least one six is $1 - (5/6)^4 = 0.5177$, a bit more than a half. The probability of no double-six in 24 rolls of two fair dice is $(35/36)^{24}$ and, consequently, the probability of at least one double-six is $1 - (35/36)^{24} = 0.4914$, a bit less than a half.

The second problem de Méré propounded to Pascal is the so-called "problème des partis". Two persons, let's say A and B, stake the same amount in a game of chance consisting of at most 9 sets, each with the same chance of profit for A and B. Due to circumstances without their consent, they must break the game after 7 sets. At that moment, A has won 4 sets and B 3. How can the stake be honestly shared out? Chevalier de Méré proposes a proportional reasoning based on the three numbers given in the problem (3, 4 and 5), but hesitates between a proportion of 4 over 3 and (5 - 3) over (5 - 4). What is correct? None of both, Pascal judges! Suppose both players would play two more sets, than there would be four possibilities:

A wins, A wins
A wins, B wins
B wins, A wins
B wins, B wins.

In three of these four cases, A would receive the whole stake while B would receive it in only one case. Thus, A has three chances versus the one chance of B. In consequence, the stake should be shared out in a proportion of 3 to 1.

Finally, also in the fields of algebra and calculus one can find several examples that can be qualified as linearity illusions. In this domain, what pupils actually misuse in most cases is not the linear model itself, but rather its properties, especially the preservation of the addition and the multiplication by scalars. Every high school teacher knows examples of students applying "properties" like: "the square root of a sum is the sum of the square roots", "the logarithm of a multiple is the multiple of the logarithm",... BERTÉ (1992) takes up the question how this "linear obstacle" can be removed to open pupils' mind for the acquisition of new mathematical models and their proper range of application. The improper use of linearity in the solution of extremum problems has been described in DE BOCK (1992).

In addition to the numerous examples of pupils' misuse of the linear model in a wide range of situations, the mathematics education literature also contains some reflections and comments

on this phenomenon. Some authors suggest that maybe the simplicity and self-evidence of the linear model are at the root of the illusion of linearity.

> Linearity is such a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as though it were linear. (FREUDENTHAL 1983, p. 267)

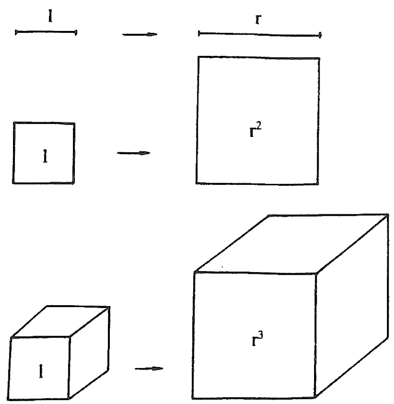
> C'est l'idée de proportionnalité qui vient d'abord à l'esprit, parce qu'il n'y a sans doute pas de fonctions plus simples que les linéaires. (ROUCHE 1989, p. 17).

The faulty solution strategies by de Méré based on a straightforward application of linearity mentioned above, elicit the following (sharp) comment by FREUDENTHAL (1973, p. 585) with respect to mathematical instruction:

> He [de Méré] applied the mathematics he knew, the kind of mathematics which in my childhood was called the rule of three... Maybe he would have performed better if he had never learned mathematics at all! Then there would have been some chance that he would have applied not the mathematics he had learned but the mathematics that he would have to create himself.

3 The effect of a linear enlargement (reduction) on area and volume

Our own research focuses on applied mathematical problems about the relation between the linear measurements, the area and/or the volume of similar geometrical figures. The principle governing that kind of application problems is well-known: an enlargement or reduction by a factor <u>r</u>, multiplies lengths by factor <u>r</u>, areas by factor $\underline{r}^2$ and volumes by factor $\underline{r}^3$ (Figure 2).



A crucial aspect of understanding this principle is the insight that these factors depend only on the magnitudes involved (length, area, and/or volume), and not on the particularities of the figures (whether these figures are squares, circles, etc.). According to FREUDENTHAL (1983, p. 401), this principle is mathematically so fundamental that it must come first, both from a phenomenological and didactical point of view.

> This principle deserves, as far as the moment of constitution and the stress are concerned, priority above algorithmic computations and applications of formulae because it deepens the insight and the rich context in the naive, scientific, and social reality where it operates.

The improper use of a linear proportional model in enlarging and reduction operations is a classical mistake - probably one of the oldest in the history of mathematical thought. The most often quoted example can be found in Plato's dialogue <u>Meno</u> (see, e.g., BERTÉ 1993, ROUCHE 1992b) in which a slave, when asked by Socrates, Plato's master, to draw a square having two times the area of a given square, firstly proposes to <u>double the side</u> of the square. So, the slave spontaneously applies the idea of linear proportionality (between length and area) and changes his mind only when Socrates helps him in diagnosing and correcting the error in his reasoning confronting him with a drawing.
Since then many other authors have argued that getting insight in the above-mentioned relationships between lengths, areas and volumes of similar figures usually is a slow and difficult process. In the American <u>Standards</u>, for instance, it is stated:

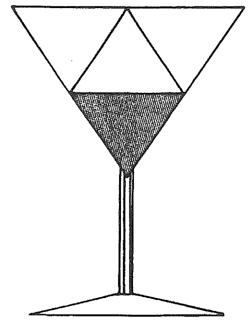
> . . . most students in grades 5-8 incorrectly believe that if the sides of a figure are doubled to produce a similar figure, the area and volume also will be doubled. (NCTM, 1989, pp. 114-115).

But scholars of the Freudenthal Institute, who have explored the influence of linear enlargement on area and volume in realistic contexts (like "Gulliver" in TREFFERS 1987, or "With the giant's regards" in STREEFLAND 1984), have claimed and provided some evidence that -at least in the context of realistic mathematics education- this misbelief can rather easily be overcome even in the primary school. However, this assertion is -as far as we know- not supported by systematic empirical research data.

Remarkable and contrary to some other domains in which pupils (frequently) fall in the linear trap, is the ascertainment that real-word or commonsense knowledge doesn't always suffice to grasp correctly the influence of a linear scaling on length, area and volume. In this respect, FEYS (1995, p. 123) describes an experience with his pre-service teachers as follows:

> We ask pre-service teachers what will happen when they lay out two A4 pages side by side on a copier in order to reduce them on one A4 page. Regularly, they answer that the text will be no longer readable because the height and width of the characters and of the drawings should be halved.

Also typical for this phenomenon is the fact that pupils are often surprised that by an enlargement, the area and certainly the volume is enlarged <u>thus much</u>; contrarily, by a reduction, they are often surprised that the area and certainly the volume is reduced <u>thus much</u> (a giant being ten times as tall as an adult man of 70 kg, weighs 70 ton; a goblin ten times smaller than this adult, only weighs 70 g!). A well-known example of pupils' misjudgement of the very strong effect of a reduction on volume is the conic glass that is filled half (of the height) of a full glass: pupils most often realise that its volume is less than the half of a full glass, but when asking them to estimate more precisely which part it is, their answers are mostly more than one eighth. (Pupils' tendency to overestimate this volume probably relates to visual perception: in front view, one can see in fact a triangle whose area is reduced to one fourth, see Figure 3.)



Finally, it is reality itself that, in a certain sense, puts the pupils on the wrong track. The mathematical idea of a "linear enlargement" doesn't always fall together with the physical and biological reality of scaling. Old trees are more plump than younger specimens; tigers have relatively thicker paws than cats. These examples of enlargements "taken from nature" are not similar enlargements and, consequently, the relationships between linear measurements, area and volume, as described above, are not applicable in these situations. The reasons why are not mathematical, but have a physical or biological origin. Let us explain, for instance, why higher trees (must) have relatively thicker trunks than smaller species. Suppose the trunk of a tree being twice as high, should be twice as thick, then, the higher tree's volume should be increased by a factor 8! In fact, the bearing-power of a trunk (pillar, paw,. . .) is directly proportional to the cross-section of the trunk, that, in the case of a linear enlargement by a factor 2 (in all dimensions), only would quadruple. In order to bear a tree being 8 times as heavy, the diameter of the trunk should increase by a factor $\sqrt{8}$ (which is nearly a triplication!). Also in this respect, we notice that babies are not "linearly reduced" adults: their head and bones have a relatively bigger portion in their weight which makes them relatively heavier than adults.

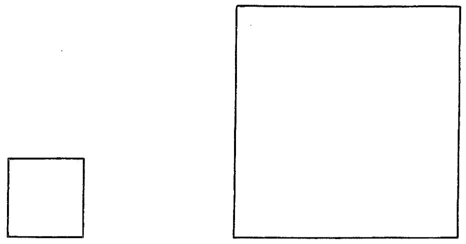
In the next part of this article we report on five closely related ascertaining studies on the illusion of linearity with respect to problems involving length, area and volume of similar figures presented in a school context.

4 Five ascertaining studies on the illusion of linearity

We first summarize the objectives and major results of these five empirical studies (DE BOCK, VERSCHAFFEL & JANSSENS 1998a, 1998b; DE BOCK, VERSCHAFFEL, JANSSENS & ROMMELAERE 1999; DE BOCK, VERSCHAFFEL & CLAES 1999). Study 1 investigates the illusion of linearity in 12-13-years old pupils working on word problems involving length and area of similar plane figures of different kinds of shapes. To answer the question whether self-made or ready-made drawings are helpful in breaking pupils' tendency to overgeneralize the applicability of linear reasoning for that kind of problems, testing took place under various conditions. Study 2 is basically a replication of the first one with 15-16-year old pupils. Both Study 1 and 2 convincingly demonstrated the strength of the illusion of linearity in pupils solving (non-linear) scaling problems presented in a school context. With a view to arrive at a better understanding of the results observed in these studies, three follow-up studies were executed focusing on the influence of different aspects of the testing context on pupils' solutions. Study 3 was set up to examine the resistance to change of the illusion of linearity by providing pupils adequate metacognitive and adequate visual support. Subsequently, Study 4 investigates to what extent the tendency towards improper proportional reasoning is caused by particularities of the problem formulation, more specifically by the missing-value type of problem that pupils learned to associate with proportional reasoning throughout their school career. At last, Study 5 explores yet another possible explanation for pupils' misuse of the linear model, namely the inauthentic or unrealistic nature of the problem context.

4.1 Study 1

Hundred-and-twenty 12-13-year old pupils, divided in three equal groups, participated in this study. The experiment consisted of two phases. During the first phase all pupils were administered the same paper-and-pencil test consisting of 12 experimental items and several buffer items. No hints or special instructions were given. All 12 experimental items involved similar plane figures, and belonged to either one of three categories: 4 items about squares, 4 about circles, and 4 about irregular figures. Within each category of figures, there were 2 proportional items (e.g. "Farmer Gus needs approximately 4 days to dig a ditch around his square pasture with a side of 100 m. How many days would he need to dig a ditch around a square pasture with a side of 300 m?") and two non-proportional items (e.g. "Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a square piece of land with a side of 600 m?"). Two weeks after the first test the three groups of pupils were confronted with a parallel version of the first test. The problems in this second test were the same for all three groups, but the instructions were different. In Group I, which functioned as a control group, the testing conditions were exactly the same as during the first test. The students of Group II were explicitly instructed to make a drawing or a sketch of the problem situation before computing their answer. In Group III, finally, every problem was accompanied by a correct drawing (like the one given in Figure 4). The influence of the task variables on pupils' performance was determined by means of an analysis of variance and a posteriori Tukey tests.



The results confirmed the hypothesis that the predominance of the linear model would be a serious obstacle for the vast majority of the pupils. Indeed, the analysis of variance revealed an extremely strong main effect of the task variable "proportionality": while the proportional items elicited 92% of correct responses, only 2% of the non-proportional items was answered correctly. Second, we unexpectedly did not find any beneficial effect of the self-made or given

drawings, nor in general, nor for the non-proportional items in particular. Third, the type of figure had a significant effect. Percentages of correct responses were also in the expected direction (problems about squares > circles > irregular figures).

For more than one reason, we found it appropriate to set up a follow-up study with an older target group. First, the small number of correct responses on non-proportional items made us wonder how strong the predominance of the linear model would be for pupils who were older and -therefore- mathematically better equipped for overcoming the obstacle of unlimited linear proportional reasoning. Second, because the illusion of linearity proved to be so strong with 12-13-year olds, the first study did not yield adequate information about the possible influence of self-made or given drawings on the occurrence of errors based on inappropriate proportional reasoning.

4.2 Study 2

Two-hundred-and-twenty-two 15-16-year old pupils participated in the follow-up study. Contrary to the first study, we did not administer the test twice to all pupils, but worked with three groups that were rigorously matched based on several subject characteristics. Group I (in which no special help or instructions were given), Group II (in which the pupils were instructed to make a drawing) and Group III (in which every item came with a correct drawing). We used the same 12 experimental items and the same procedure for test administration as in the first study.

The analyses of variance showed once again an extremely strong main effect of the task variable "proportionality": the overall percentages of correct responses on the proportional and non-proportional items were 93% and 17%, respectively. The hypothesis about the positive influence of the self-made or given drawings was, once again, not confirmed. Finally, as in the first study, the type of figure involved played a significant role. Pupils performed better on the non-proportional items when the figure involved was regular (a square or a circle), but as a drawback they performed worse on the proportional items about these regular figures because they sometimes started to apply non-proportional reasoning on the proportional items too.

So, both Study 1 and 2 revealed the expected alarmingly strong tendency among pupils to apply linear proportional reasoning in problem situations for which it was not suited, but they did not show the anticipated facilitating drawing effect on students' performance. It could be argued that the extremely weak results on the non-proportional items and the absence of a positive drawing effect were due to the fact that the students had approached the test with the expectation that it would consist of routine tasks only (as is frequently the case in current mathematics education). A possible additional explanation for the weak results could be that the response sheets used in these studies were not suited for measuring lengths and areas of (given) plane figures because of the lack of useful reference points for this activity on these sheets. Based on these two arguments, one could predict that the accuracy rates for the non-proportional items would increase significantly (1) if pupils would get at the beginning of the test some kind of explicit warning that not all problems in the test were standard problems, and (2) if we would make use of response sheets with (drawings made on) squared paper instead of blank paper. The third study investigates the possible influence of these two forms of "scaffolding" on pupils' solution processes and outcomes.

4.3 Study 3

Two-hundred-and-sixty 12-13-year olds and hundred-and-twenty-five 15-16-year olds participated in the study. In both age-groups students were matched in four equivalent subgroups which received one and the same paper-and-pencil test, consisting of 6 proportional and 6 non-proportional items, but the administration of the test was different in the four subgroups of pupils. In Group I -the control group- no special help was given. In Group II -the metacognitive scaffold group- the test was preceded by an introductory task that confronted pupils with a correct and an incorrect solution to a representative non-proportional item and asked them which one was the correct. In Group III -the visual scaffold group- every item came with an appropriate drawing of the problem situation made on squared paper. Finally, in Group IV both kinds of help were combined.

The study yielded small but significant effects of both kinds of scaffolds. As a result of the metacognitive scaffold, the percentage of correct responses on the non-proportional problems increased slightly from 12% to 18%. With respect to the visual support, the increase was even smaller: from 13% in the groups without visual scaffold to 17% in the groups with the visual scaffold. As a drawback of these better results on the non-proportional items in the scaffolded conditions, the pupils' results on the proportional items decreased. Apparently, the scaffolds made it easier -at least for some pupils- to discover the non-proportional nature of a problem, but as a result they sometimes began to question the correctness of the linear model for problem situations in which that model was appropriate. However, the most important result of the study is that the positive effects of the two scaffolds on pupils' solutions of the non-proportional items remained remarkably small, suggesting that students' tendency towards linear modelling is very strong, deep-rooted and resistant to change.

4.4 Study 4

While all these three studies revealed pupils' almost irresistible tendency to apply proportional reasoning in problem situations for which it was totally inappropriate, the question remains why so many pupils fell into this "proportionality trap", even after receiving visual and/or metacognitive support. In Study 4, we investigated the effect of another attempt to overcome the linearity illusion by changing the experimental setting, namely the problem formulation. In the three previous studies, all proportional and non-proportional items were presented as missing-value problems. In this problem type, three numbers ($\underline{a}$, $\underline{b}$ and $\underline{c}$) are given and the problem solver is asked to determine an unknown number $\underline{x}$. In a proportional missing-problem, the unknown $\underline{x}$ is the solution of an equation of the form $\underline{a}/\underline{b} = \underline{c}/\underline{x}$. It is clear that the vast majority of the missing-value problems pupils encounter in the upper grades of the elementary school and the lower grades of secondary school, are problems for which the linear model suits perfectly. Therefore, it could be argued that pupils' extremely weak results on the non-proportional items may not be due to intrinsic difficulties with the mathematical concept involved in these problems -namely understanding the effect of a linear enlargement on area- but are merely the result of a misleading problem formulation which is associated with and therefore calls up the overlearned solution scheme and procedure of proportional reasoning. To find out this, we set up a new study in which the formulation of the problems was experimentally manipulated while keeping the intrinsic conceptual difficulties constant.

Hundred-and-sixty-four 12-13-year old pupils and hundred-and-fifty-one 15-16-year old pupils participated in the study. All pupils were administered the same paper-and-pencil test consisting

of different kinds of proportional and non-proportional items, - just as in the previous studies. In both age-groups we worked with two equivalent subgroups of pupils that were matched on an individual basis and that were given a different version of the test. For half of the pupils (the so-called missing value-group), all items were presented as missing-value problems (see, e.g., the examples given in the report of Study 1), while in the other half (the comparison group), the items were formulated as comparison problems (e.g. "Farmer Carl manured a square piece of land. Tomorrow, he has to manure a square piece of land with a side being three times as big. How much more time would he approximately need to manure this piece of land?").

The results of the analysis of variance with respect to the effects of proportionality and age on pupils' performance, confirmed the findings from the earlier studies. More interestingly, however, is that while the analysis of variance did not reveal a main influence of the problem formulation variable, a highly significant interaction effect of problem formulation and proportionality was found. As expected, the group who received the comparison problems performed significantly better on the non-proportional items than the group who received the missing-value problems (41% and 23% correct answers, respectively), but this better performance of the comparison group on the non-proportional items was parallelled with a worse score on the proportional items (i.e. 68% versus 87% correct answers in the missing-value group). Apparently, the formulation of the items used in the comparison group prevented pupils for falling into the proportionality trap, but as a result these pupils sometimes began to question the correctness of the proportional model for problem situations in which that model was appropriate - a finding that is very similar to the one obtained in our previous studies and that has been observed in several other studies about strategic and conceptual change.

So, the significantly better results of the comparison group on the non-proportional items made it clear that a significant number of pupils fail on traditionally presented non-proportional items not because of their belief in the omni-applicability of the linear model, but rather because of the association of that model with a particular type of problem formulation, in this case the missing-value type. While the effect of problem formulation was significant, it was again still rather small, as still more than half of the pupils in the comparison condition failed on the non-proportional items. This also raises the question what other aspects of the testing context affected pupils' incorrect reasoning process.

4.5 Study 5

In the last follow-up study, we investigated another possible explanation for pupils' misuse of the linear model, namely the inauthentic and unrealistic nature of the problem situations. Some evidence for this hypothetical explanation can be found in TREFFERS (1987), who realized a design experiment with sixth graders on the influence of linear enlargement on area and volume that was built around the context of "Gulliver's travels", and claimed that, in this realistic mathematics education approach, pupils have no difficulty with a problem like "How many Lilliputian handkerchiefs make one for Gulliver if you know that the length of a Lilliputian is 12 times smaller than that of Gulliver?". To test the facilitating power of making the problem context more realistic and motivating, we executed a new study.

In this study, hundred-and-fifty-two 13-14-year olds and hundred-and-sixty-one 15-16-year olds were matched in two equivalent subgroups. In both groups, a paper-and-pencil test, consisting of proportional and non-proportional scaling problems was administered. In the first group, the test was preceded by an assembly of well-chosen fragments of a film version of Gulliver's visit to the isle of the Lilliputians and all experimental items were linked to these film fragments. For instance, inspired by a fragment showing a Lilliputian occupied with filling Gulliver's wineglass, we asked "Gulliver's wineglass has a volume of 172 800 mm^3. What's the volume of a Lilliputian wineglass?". In the second group, an equal number of mathematically isomorphic problems was presented in the form of a series of non-related traditional school problems, without any contextual support.

Contrary to our expectation, there was no positive effect of the authenticity factor on pupils' performance on the test as a whole, and on their scores on the non-proportional items in particular. On the contrary, pupils who watched the video and who received the video-related items performed even significantly worse than the other group (25% for the video versus 42% for the non-video condition). At this moment we are planning a replication study to investigate whether this unexpected finding was an artefact of our operationalization of the experimental variable (e.g., the fact that the pupils who watched the video had less time to solve the test) or if it was due to the fact that these pupils' involvement in an attractive and rich context had led them away from the necessary in depth analysis of the mathematical problem structure, instead of helping them to find it.

5 Discussion

A couple of years ago, we executed a study showing that the vast majority of pupils failed on word problems about the length and area of similar plane figures. We have conducted follow-up studies showing that several attempts to overcome these failures by changing the experimental setting did not work or had only minor success: providing drawings, providing metacognitive support, rephrasing the problem and increasing its authenticity. In our future work, we will turn from ascertaining studies to individual interviews with selected groups of pupils and then to design experiments wherein we will develop and test new instructional materials and techniques aimed at improving pupils' necessary conceptual tools and cognitive strategies to overcome this very strong and resistant illusion of linearity.

This more process-oriented research could also yield a better understanding of the relations between the illusion of linearity and the broad domain of pupils' "misconceptions" (or "preconceptions" or "alternative conceptions") and "illusions" in the learning of both mathematics and science. A related, but in a way more primitive misconception than the illusion of linearity, is the one of additive reasoning in situations wherein (rather) a multiplicative reasoning is applicable (for instance, in the context of a linear enlargement of a figure, pupils argue that all lengths are added to instead of multiplied by a constant. This "additive illusion" is ascertained in numerous studies and appears to affect especially elementary school pupils passing on from additive to multiplicative structures (see, e.g., HART 1981; KARPLUS, PULOS & STAGE 1983; LIN 1991; STREEFLAND 1988). A similarity with the linearity illusion is that in both cases the error results from pupils' overgeneralizations of a previously learned model beyond its proper range of application.

There exists also a similarity between the illusion of linearity and the so-called "illusion of constancy" or the intuitive rule "Same A - same B" (see, e.g., Tsamir, Tirosh & Stavy, 1998) - a rule emerging regularly when pupils formulate properties of geometrical figures (e.g. "triangles with equal angles have equal sides", "quadrilaterals with equal sides have equal angles", etc.). In the same way, many pupils spontaneously think that plane figures with the same perimeter have the same area or solids with the same surface area have the same volume. One can expect that

those pupils neither make any distinction between the scale factors appropriate for these different dimensional quantities. CASTELNUOVO & BARRA (1980) describe two classical examples of this phenomenon. The first one deals with crushing a square or rectangle made of straws. By crushing the figure only slightly, most pupils think that the area remains the same. Only when confronting them with an extreme situation (the figure is crushed almost completely), everybody can see that the area decreases by crushing the figure. A second one is the famous cylinder-problem of Galilei. Using a rectangular piece of stuff (with different length and width), one can make cylindrical (corn)sack: a wide but low one (by connecting the smallest sides of the rectangle) and a narrow but high one (by connecting the biggest sides of the rectangle). The wide but low cylinder does have the greatest volume, but many people ("except the farmers in Galilei's time") think they have the same volume because they have the same surface area.

In the field of probabilistic thinking, many well-known erroneous reasonings of pupils can be both explained in terms of the "Same A - same B" intuitive rule and of a proportionality illusion. Let's first remind the first problem de Méré propounded to Pascal: de Méré's incorrect argumentation for the equiprobability of two events was based on an equivalence of ratios, in Freudenthal's view a straightforward application of the "rule of three" de Méré learned at school. Yet another explanation for de Méré's faulty approach to this problem can be found in the theory of intuitive rules, more specially, as a manifestation of the "Same A - same B" scheme (de Méré argues: "Same proportion, thus same probability").

Recently TIROSH & STAVY (1999, p. 190) reported a quite similar example:

> The Carmel family has two children, and the Levin family has four children. Is the probability that the Carmels have one son and one daughter larger/equal to/ smaller than/ the probability that the Levins have two sons and two daughters?

The probabilities of these events could be reached by a relatively simple counting and calculation. In fact, the probability of "one boy, one girl" in the Carmel family is 1/2 while for the Levin family, the probability of "two boys, two girls" is 3/8. Thus the probability that the Carmels have one son and one daughter is larger than the probability that there are two sons and two daughters in the Levin family. The researchers presented this problem to about 40 students in grades 7 to 12. The majority of these students incorrectly argued that both probabilities are equal because "the ratio is the same, therefore the probability is the same". Remarkably, the distribution by grade showed a (slightly) increasing trend in the percentages of incorrect responses with age.

This result is very similar to the one obtained by FISCHBEIN & SCHNARCH (1996, pp. 355-356), with respect to the problem:

> In a certain town, there are two hospitals, a small one in which there are, on the average, about 15 births a day and a big one in which there are, on the average, about 45 births a day.
> The likelihood of giving birth to a boy is about 50%. (Nevertheless, there were days in which more than 50% of babies born were boys and there were days in which less than 50% if babies born were boys).
> In the small hospital one has kept a record during a year of the days in which the number of boys born was greater than 9, which represents more than 60% of the total of births in the respective hospital.
> In the big hospital, one has kept a record during a year, of the days in which there were born more than 27 boys which represented more than 60% of the births.
> In which of the two hospitals there were more such days?

In fact, the small hospital will record more days where more than 60% boys were born. The stochastic law one has to consider is the law of large numbers. As a sample size (or the number of trials) increases, the relative frequencies tend to come closer to the theoretical probability. And, on the contrary, if one considers a small sample, the relative frequencies of expected outcomes may deviate largely from the theoretical probability. Fischbein and Schnarch presented this problem to groups of 20 pupils of grades 5, 7, 9 and 11. The main misconception, in this case, "the number of days in a year on which one has recorded the birth of more than 60% boys does not depend on the sample size", increased with age in a surprisingly regular manner. Starting from 10% in grade five, it reached about 80% in grade 11. The erroneous answer was usually justified by the equality of ratios: "9/15 = 27/45 - they express both the same ratio".

Most likely, the "scheme in action" in pupils working on this kind of problems, the intuitive rule "Same A - same B" or the proportionality illusion, depends on pupils' age and school career. The first scheme is more general and even observed in very young children (for instance when solving Piagetian tasks), the second is more specific and strongly affected by schooling. Further research is needed to unravel the interaction between both schemes, its evolution with age and how this interaction can be influenced by mathematics education.

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