'Telling mathematics', an activity that integrates

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Abstract

The aim of this paper is to review the value of a classroom activity, in which a student tells a problem or story that she or he knows from hearsay. In this activity a variety of aspects come together : the problems and stories that enter the classroom in this manner often belong to the very long tradition of so-called 'recreational mathematics', they are challenging, generally they surpass the borders of the school curriculum, and they contribute to the self esteem of the person who tells the problem or story. In this sense it is an activity that integrates several aspects of mathematical work.

1 Just before the start

Formally the lesson began already some minutes ago, when the bell prompted the change of classes.

Pupils are still coming in, the teacher remains seated at his desk, pondering about the proper moment to raise his voice. One of the pupils, a boy, comes to him and says:

Boy Sir, I have a sum for you. Do you want to try it?

Teacher Yes, please, do tell me.

Boy It is a multiplication... It starts with

$$(x-a)(x-b)(x-c)$$

and goes on like that.

The teacher plays his role well, and starts to calculate. After a while he suddenly stops, he looks at the boy and says:

Teacher Oh I see, the result is zero; there is no need to continue the multiplication, since there is a factor $(x-x)$, and if in a multiplication one of the factors is zero, the whole product is zero.

The boy seems to be satisfied. The teacher has found the right answer, but at least he had to think hard, for a while. The teacher seems to be satisfied too. He has met the challenge.
By this time the class is complete and the lesson begins.

2 The problem set

The multiplication with outcome zero is maybe not the prototype problem for what I shall call 'telling mathematics'. Yet I have chosen to stage it before anything else.

The first reason for doing so is that the story really happened to me, when I was a secondary school teacher. I even heard the 26-factor problem on several occasions, sometimes with years in between, and my deep thoughts before I 'saw' the answer were theatre, except maybe for the first time it occurred. In this sense the problem is characteristic, since in this paper I shall deal almost exclusively with mathematics that people told me. 'Telling mathematics' of course goes beyond the observations of one single person. On the other hand the personal perspective is valuable, as a kind of proof that the tradition of 'telling mathematics' still goes on.

The second reason for starting with the multiplication problem is that it came up in an educational context, in which it had a clear function —challenging the teacher. Many of the cases that follow are set in the educational dimension, and the role of these 'oral' problems in the classroom is one of the main topics of the paper.

Many problems from the oral tradition have a long history. That does not hold for the multiplication problem, at least I have not found early records of it, so with respect to the historical dimension the problem is not representative. As we shall see, the historical dimension is well recognisable in my personal cross-section of the oral tradition.

After these preliminary remarks the central problem can be stated: how is the tradition of 'telling mathematics' reflected in the experience of one individual mathematician, and what possible

value can it have, especially its historical dimension, in mathematics education?

3 Hundred 'birds'

My father brought the oldest problem in my (re)collection home from work. His professional career was entirely with the Dutch Post, at that time still owned by the state and called PTT, where the T's stand for Telegraph and Telephone. He was a white collar employee who did a lot of paper work. Apart from some basic mathematics that he had done at school (secondary school with an emphasis on languages and bookkeeping and without a science branch) my father had no background in maths, nor had he any interest in it. So, it was rather peculiar that one evening he presented me with a problem that he had heard from a colleague at work. If I remember well it happened when I was in the upper half of secondary school (16 or 17 years old). I have asked my father if he remembers the episode, but he has completely forgotten it.The problem ran as follows:

> Someone has one hundred guilders, and wants to buy one hundred animals for it. He can choose between cows, pigs and chickens. A cow costs 5 guilders, a pig 1 guilder and a chicken 25 cents (a *kwartje*). How many of each animal should he take?

I went to my room, and still know that, before I wrote anything down, I was puzzled by the combination of two equations and three unknowns; and amazed when I realised that the extra condition of integer solutions reduced the infinite number of real solutions to only a few solutions. I was also amazed that I had to do some reasoning in order to find the solutions. At school mathematics was more a kind of imitating the teacher.

As to the mathematics: the system

$$\begin{aligned} x+y+z &= 100 \\ 5x+y+0.25z &= 100 \end{aligned}$$

reduces to one equation in x and z. This equation, $16x = 3z$ must have integer solutions, and here the non-school mathematics entered. Since 3 and 16 have no common factors, the smallest solution in positive integers is $x = 3$ and $z = 16$, which gives $y = 81$. The other solutions are $(6, 62, 32)$, $(9, 43, 48)$, $(12, 24, 64)$ and $(15, 5, 80)$.

The problem, always referred to as the problem of the 100 birds, has travelled through the ages and through the world[1]. It appears already in China in the work of Chang Ch'iu-chien, about 485 A.D., with birds indeed (cocks, hens and chickens) and with different prices (5, 3, $\frac{1}{3}$). Chang Ch'iu-chien already recognises that for these prices there are three triples of integers that solve the problem. The problem is subsequently found in India (a.o. with Mahāvīra, 9th century, and Bhāskara II, 11th century), in arab texts (a.o. Abū Kāmil, 9th century), and in Byzantium. The oldest latin text, the *Problems to sharpen the young*[2], which is attributed to Alcuin (c. 732–804), is quite early. The oldest manuscript in which 'Alcuin' is preserved dates from the 9th century. 'Alcuin' has several variants of the problem (see [FOLKERTS 1978, p. 37]), but his Problem 39 is clearly rather close to the version I cited above. It stages a merchant who wants to buy in the east 100 assorted animals for 100 *solidi*. Acceptable prices are: 5 *solidi* for a camel, 1 *solidus*

[1] The following survey is based on [TROPFKE, 1980] and [FOLKERTS, 1978]

[2] Available in english translation in [HADLEY & SINGMASTER, 1992] and in latin with german commentary in [FOLKERTS, 1978]

for an ass and 20 sheep for 1 *solidus*. In this case there is a unique solution $(19, 1, 80)$, which 'Alcuin' checks without deriving it; in almost the same manner I presented my solutions to my father. Singmaster remarks that the same prices feature in the work of Abū Kāmil, but there for ducks, hens and sparrows. [HADLEY & SINGMASTER, 1992, p. 120]

Apparently the problem went around extensively. After 'Alcuin' many of the great medieval and renaissance mathematicians present it in one form or another: Fibonacci, Regiomontanus, Pacioli, Rudolff, Apianus, Ries, Cardano, Tartaglia and many later ones. Euler takes the hundred birds in his explanation of Diophantine equations (*Von der unbestimmten Analytic*, in his 'Complete Introduction to Algebra', Part 2, Section 2, Chapter 2, Question 4 [EULER, 1770, 1911, pp. 342–344]):

> Someone buys 100 pieces of cattle for 100 Rthl. 1 oxen for 10 Rthl. 1 cow for 5 Rthl. 1 calf for 2 Rthl. 1 sheep for $\frac{1}{2}$ Rthl. How many oxens, cows, calves and sheep were they?

From the equations

$$\begin{aligned} p+q+r+s &= 100 \\ 10p+5q+2r+\frac{1}{2}s &= 100. \end{aligned}$$

Euler eliminates s, and reworks the remaining equation to

$$r = 33 - 6p - 3q + \frac{1-p}{3}.$$

Since r is an integer, $p-1$ should be a multiple of 3, so Euler puts

$$\begin{aligned} p &= 3t+1 \\ q &= q \\ r &= 27-19t-3q \\ s &= 72+2q+16t. \end{aligned}$$

Since r cannot be negative, $19t+3q$ should be less than or equal to 27, which is only possible if $t=0$ or $t=1$. The first case ($t=0$) gives $p=1$ and since $r=27-3q \geq 0$, q can range from 0 to 9. The second case ($t=1$) leads to $p=4$, and via $r=8-3q \geq 0$ to three possible values for q, viz. $0, 1$ and 2. Euler finds a total of 13 solutions. When there should be at least one animal of each sort, 10 solutions remain valid.

4 Extending the repertoire

In later years I added several problems and stories to my collection. Lessons and lectures were occasions where I caught them. Obviously, we encounter a definition problem, since if I would consider everything I heard during lectures as 'telling mathematics', the length of this article would go beyond any reasonable limit. So, what I propose to include here is the unexpected sidestep, the bit about number theory in a calculus lecture, the unofficial part of the lecture.

Conjectures are good material for sidepaths. At the university Fermat's last theorem and the Goldbach conjecture were told in this manner. I shall concentrate here on Goldbach, which was told to us as a long standing problem: "Solve it and you'll be famous." One of the attractive aspects of the conjecture is that it is so easy to remember:

> Every even number from 4 onwards can be written as the sum of two prime numbers.

The conjecture was stated in 1742 by Goldbach in a letter to Euler, deliberately in order to solicit new mathematical theory. As a school teacher I often cited the Goldbach conjecture —not twice or more in the same class, I hope— as an example of a well understandable mathematical claim that still goes without proof. Not every claim in mathematics is so easily found true or false; and fame is reserved for the person who finds the clue, think about the case of Fermat's last theorem, for which a sum of money was awarded.

Another of these stories, that was told as a sidestep in a lecture (I do not remember what and whose lecture it was), was about the Hilbert hotel. The hotel has an infinite number of rooms, numbered 1, 2, 3 and so on. When the hotel is complete someone comes in who is looking for a room. No problem, for the management moves the persons in room n to room $n+1$ ($n \in \{1, 2, 3, \ldots\}$) or, if they are clever, the persons in room 10^n to room 10^{n+1}, since that will give less complaints.

A bus arrives. No rooms reserved, but yet forty customers to satisfy. No problem to house them. Either the moves from room n to room $n+40$, or the moves from room p^n to room p^{n+1} (where p runs through a set of 40 prime numbers), will vacate enough rooms. And again the Hilbert hotel is complete.
What then if a Hilbert-bus, with infinitely many passengers, stops at the parking lot? Here the hotel procedures prescribe that the occupants of room n "are kindly requested" to move to room $2n$, and the newcomers may occupy the odd-numbered rooms. And so the story went on. It ended with the case when the hotel was complete and infinitely many Hilbert-buses announced themselves. Here the diagonal procedure determines the order in which the newcomers may leave their bus and go to their room. In the secret language of mathematics: a countable union of countable sets is countable, and closely related to this: the rational numbers are countable.

I used the Hilbert-hotel is my school in a prefinal class (pupils aged 16,17), where I used to start the lesson with a brief —3 minutes long, say— mathematical story outside the regular topic of the lesson. The Hilbert-hotel featured for weeks (we had 4 hours a week), the pupils each time taking home the new problem that the hotel management had to meet.

As to the historical dimension: I do not know who has designed this pleasant and profitable transformation of the theory of countable sets into the Hilbert-hotel, but the theory itself is rather recent. The notion of countability was introduced by Cantor in the 1870's. In my class I ended the Hilbert-hotel series with some minutes about this history, and the increasing rigour and axiomatisation at the end of the 19th century. As a separate brief story this would have had little impact, but after the ground had been paved this historical bit went quite well.

Another acquisition in the field of 'telling mathematics' stems from a lecture given by a colleague. Since I had to lead a workshop about what this colleague would discuss, I went to his lecture. It was intended for a general audience of prospective science teachers, biologists as well as chemists, physicists and mathematicians. So it could not be too mathematical, chemical etc. The lecturer had decided that he wanted to give an example of a mathematical proof, in order to contrast the deductive method with the inductive method of the other, non-mathematical science. The amazing thing was that he presented a proof of the irrationality of $\sqrt{2}$ that I had never seen before. Later on I discovered that it originated in an article by H.-J. WASCHKIES (1971), but originally it came to me by the way of 'telling mathematics'. Although the proof is rather recent, its style is very classical and fits well to the pebble-arithmetic of the Greeks and to techniques of Nikomachos.

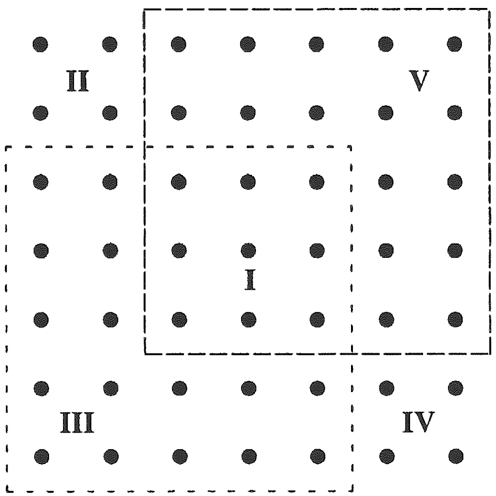

FIGURE 1 : 'Proof with pebbles' that $\sqrt{2}$ is irrational

The proof is by contradiction (also see figure 1). Suppose that $\sqrt{2}$ is rational, then it can be written as $\frac{p}{q}$, in most simplified form, so we suppose that p and q are the smallest possible numbers which satisfy $\sqrt{2} = \frac{p}{q}$.
Now make a square array of $p \times p = p^2$ pebbles. Since $p^2 = 2q^2$, one can replace the large square of p^2 pebbles by two smaller squares of q^2 pebbles each. If one draws these squares within the $p \times p$ array, one in the left lower corner, the other in the right upper corner, then for reasons of symmetry the two $q \times q$ squares (I+III and I+V in figure 1) overlap in a square (No. I). Since the two $q \times q$ squares contain all pebbles of the $p \times p$ array, the number of pebbles in (I+III) plus the number of pebbles in (I+V) is equal to the number of pebbles in (I+II+III+IV+V), and therefore the number of pebbles in I is equal to the number of pebbles in II+IV. That, however, is impossible, since the figures I, II and IV are squares, and II and IV are congruent; this would imply that there is a solution in smaller numbers to $x^2 = 2y^2$ than (p, q), whereas we supposed that (p, q) was the smallest. We conclude that $x^2 = 2y^2$ does not have a solution in integer numbers, in other words that $\sqrt{2}$ is irrational.

5 The educational value

I shall introduce the discussion of the educational value of 'telling mathematics' with yet another case.

Imagine a class of 16-year-olds during the last lesson before a holiday, a lesson that must be given even if you have done the whole syllabus for the period. Pupils proposed that we should do problems from outside the book, problems that they and I knew by heart. I agreed with pleasure, since this seemed to me a bright type of activity for such a lesson.

I remember one pupil and one problem in particular. The class decided that they would set a problem for me, and Liesbeth said that she had a good problem. Mathematics was not Liesbeth's

favourite discipine. But this time she was well-determined and in good spirits, bringing the following problem to the fore:

> You walk along a road when you arrive at a fork, where you have to choose between two ways to continue. One of the two leads to happiness, the other to eternal sorrow. Two brothers are guarding the fork, they both know the destination of the roads. One of them always speaks the truth, the other one always lies. You may ask one question, only to one of them. Whom are you going to ask what question?

In Cartesian style I prefer not to deprive the reader from the pleasure of searching and finding the solution. So let me now try to draw conclusions from the cases presented.

5.1 Challenge

What strikes me in the first place is the aspect of challenge. In the 26-factor problem and in the problem of the two brothers the pupils were curious whether their teacher would be able to solve the problem, just as I was curious which pupils would have a right answer to the next question in the series about the Hilbert-hotel.

Unexpected evidence for this aspect of challenge is provided in by a newspaper advertisement, where problems from the oral tradition are used to challenge the reader and to depict the advertising firm as a challenging environment to work in. The Eiffel company, which presents itself as a consultant in Financial-Economical and Economical-Juridical matters, and which works in three divisions (Commerce & Industry, Banking & Insurances, Public Affairs & Non-Profit), searched with an add in *Carp* 17 (27/04/1999) for "Legal personel with a drive". "If you m/f too think that the nice thing of a problem is not the problem, but just on the contrary the solution, you could very well fit precisely with Eiffel." Etc., etc.

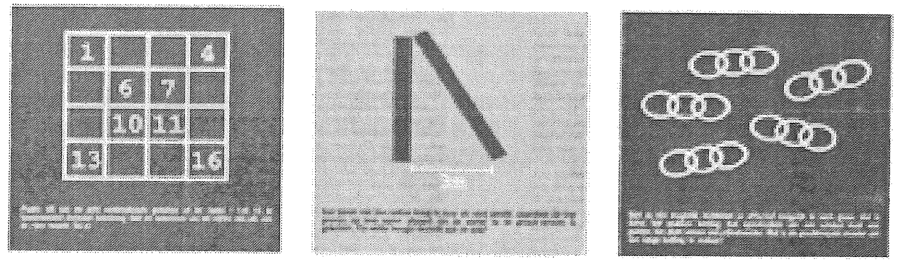

FIGURE 2 : Pictures from a personel advertisement

Towering texts, illustrated with six pictures which each state a more or less mathematical problem. In figure 2 three of these are presented: a 4×4 magical square of which 8 numbers that lack should be completed (this reminds strongly of the square in Dürer's *Melencholia* of 1514). In the text it is stated that the sum in each row and column should be 34. The second problem is about a tree of 10 meters high, which is broken and touches the ground 3 meters from its root. One is asked how high above the ground it is broken. The numerical values are still the same as in the *Chiu Chang Suan Shu* of the early Han-period (i.e. between 202 BC and 9 AD), only the Chinese bamboo has been replaced by a tree. But that was already the case in Renaissance Italy (with Calandri, in 1491, for example). The third problem makes an allusion to the work of the

Company, since the reader is asked to make the greatest possible effect with the least possible means. One is asked to make one chain from five parts, which each have three links. Breaking a link costs one guilder, closing it again costs 2 guilders and 50 cents. What is the cheapest way to do this.

Challenge is omnipresent, the work of the Company is depicted as challenging, but primarily the six problems are an overt challenge to the reader. For the company searches for "high potential starters" (this did not need translation; English is usual in the terminology of Dutch personel advertisements of this type); apparently you should not have the feeling that you are a high potential person if you cannot solve these mathematical riddles.

5.2 The usual mixed with the unexpected

A second characteristic that I would like to point at is the mixture of usual and unexpected elements. Generally the problems are stated in very common language. But also, the solution often requires an uncommon step, something that you tend to overlook. Well-known is this respect is the problem about the man who had to take a wolf, a goat and a bunch of cabbages across a river ('Alcuin's' problem 18, [HADLEY & SINGMASTER, 1992, p. 112]. The solution depends on the insight that it might be necessary for one or more of the passengers to cross the river more than once. Being on the other side does not imply that you should stay there. Not recognising this possibility can be compared with the monkey-trap. A coconut is emptied via a hole in the shell through which a monkey's hand can pass. The coconut is fixed somewhere, and some rice is put in the shell. The monkey puts its hand in the shell, and takes the rice in his hand. This increases the diameter of the hand so that it cannot pass through the hole any more. Freedom is close (the monkey only needs to open its hand), but what is close is easily overlooked.

Similar phenomena play a role in the solution of problems from the oral tradition. A look into the literature will present more evidence, see for example the problem of the division of 8 litres of wine in two portions of 4 liter, using cans of 3, 5 and 8 liters. A mathematics teacher told me that this problem was once posed to him on an air flight when his neighbour, a retired actor, had found out that he was a mathematician. The uncommon step is also required in the problem of drawing —without lifting your pencil— four straight lines through nine dots which are in a square array (three rows of three dots).

5.3 Beyond school mathematics

The tradition of 'telling mathematics' does not stop at the borders of school mathematics. Liesbeth's two brothers belong to the domain of logic, the 100 birds require some number theory, and the Hilbert-hotel is about the cardinality of sets. This aspect of crossing borders gives these problems a special flavour. They deal with something different (whereas school mathematics is again and again about the same subjects), and that also implies that they are open to everyone. Thinking properly should be sufficient.

5.4 Of all ages

Not every story or problem that comes through the air is centuries old, but quite a few are. If that is the case, teachers can use it to add to the flavour of the problem. The 'young', at least some of them, puzzled already about the wolf, goat and cabbage before the year thousand, and

the broken tree problem is even twice as old. In this respect mathematics is a very special discipline. Its knowledge is not broken by time, nor hindered by distance. Why not tell our students?

One interesting question pops up here: what qualifies a new story or problem to be added to the existing collection. For me the three squares that prove the irrationality of $\sqrt{2}$ is such a story. Point one is that I heard it, unexpectedly, and point two is that I like to tell the story. Why? I think that the proof is appealing in that it has great effect with simple means.

5.5 Fun

If the listener is well-tuned and she or he recognises these positive aspects of the problems that are in the air, pleasure is granted. And that is the best way to learn and teach mathematics.
With pleasure!

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