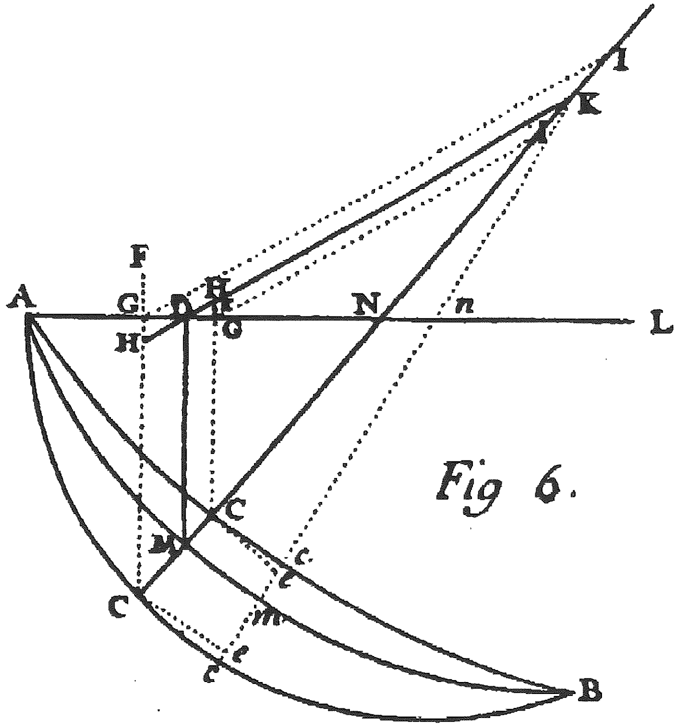

Fig 6.

“Mathematical Physics” and “Physical Mathematics”[1] : A historical approach to didactical aspects of their relation

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Abstract

A. The issue of the relevance of the history of mathematics to mathematics education is addressed and it is suggested that there are three possible ways to integrate historical aspects in the presentation of mathematics:

- By providing direct historical information, the emphasis being on learning about history.
- By following a teaching approach inspired by history, the emphasis being on learning mathematical topics.
- By presenting social and cultural aspects of Mathematics in a historical perspective, the emphasis being on mathematical awareness.

These possibilities are not restricted to mathematics only, but can be realized in the presentation of physics as well.

B. On the other hand, the historically continuous, close relation between mathematics and physics suggests that:

- mathematics and physics, as general attitudes towards the description and understanding of empirically and mentally conceived objects, are so closely interwoven, that any distinction between them, is related more to the point of view adopted while studying particular aspects of an object, than to the object itself. A historically inspired approach, though not necessary, is well suited to illustrate this point.
- The interwining referred to above, is expressed by both, the use of mathematical methods in physics (mathematical physics), and the use of physical concepts, thinking and arguments in mathematics (“Physical Mathematics”).

According to the above points (supported by many historically important examples), it is legitimate to consider mathematics and physics as different, but complementary, views of the world. This can be fruitful in teaching and understanding both disciplines.

C. The issues **A** and **B** raised above, are illustrated in some details by means of an example at the high-school level, namely, geometrical optics and differential calculus. This example admits considerable generalization, hence it is virtually important at the university level as well.

[1]The term “Mathematical Physics” is familiar, but the term “Physical Mathematics” is not; it has been taken from POLYA 1954, ch.IX.

1 Introduction

This paper is divided into two parts, both of which are related to two out of the twelve questions raised in the "Discussion Document" (DD) (FAUVEL et al. 1997), which motivated the writing of an ICMI Study Volume on *The role of the history of mathematics in the teaching and learning of mathematics* (FAUVEL et al., 2000). The first and smaller part[2] is related to question n°8 : "*What are the relations between the ... roles we attribute to history and the ways of introducing or using it in education?*". Its analysis suggests that "The [answer] ... involves a **listing of ways of introducing or incorporating a historical dimension**" (FAUVEL et al. 1997, p. 257, our emphasis). The first point to be made here is that it will become clear that this listing could be the **same** in both mathematics and physics.

The second and longer part is related to question n°5 of the DD : "Should different parts of the curriculum involve history of Mathematics [HM] in a different way?" Its analysis suggests that "Bearing in mind that history extends into the future, ... [this] could lead to suggestions for **new topics to be taught**" (FAUVEL et al. 1997, p. 256, our emphasis). The second point to be made here is that indeed, **history strongly suggests** the existence of a **close relation** between mathematics and physics, which should not be ignored in teaching and learning either of these disciplines.

Therefore, this paper is organized as follows: In Section 2 we provide a list of the possible reasons for introducing a historical dimension in mathematics education (ME), that have been or could have been put forward. This list clearly suggests, on one hand the possible general ways of introducing a historical dimension and on the other hand that they are equally valid in physics education (PE) as well. This is not accidental, but in our opinion it is related to two epistemological and historical theses concerning the relation between mathematics and physics. Their formulation and clarification is the subject of section 3. In sections 4 and 5 we present an outline of some historically important examples that illustrate these theses, at the same time providing evidence for their correctness. Finally, in section 6, the possibility to implement in the teaching process the historically and epistemologically suggested close relation between mathematics and physics described in sections 3 to 5, is illustrated by means of an example.

2 Arguments for integrating history of mathematics in mathematics education[3]

Integrating HM in ME may support, enrich and improve:

1. *The learning of mathematics* by (a) contrasting the historical development of mathematical knowledge vs. its final form presented as a deductive structure; (b) using history as a resource of relevant questions, problems and ideas that may motivate, interest and engage the learner; using history as a bridge between different mathematical domains or between mathematics and other disciplines.

2. *The development of views on the nature of mathematics and mathematical activity*. This concerns the appreciation of the fact that both the **form** of mathematics (notation, terminology, computational methods) and its **content** have an evolutionary nature that underlines the relative —with respect to time—character of fundamental metaconcepts like rigor, proof, evidence, error etc.

3. *The didactical background of teachers and their pedagogical repertoire*, by helping them (a) to identify the **motivations** for the introduction of (new) mathematical knowledge, (b) to become aware of the **difficulties** that appeared in the past and **may** reappear in the classroom, (c) to get involved into the **creative process** of "doing mathematics", e.g. in the context of historically inspired projects, (d) to enrich **their didactical repertoire** of questions, problems, teaching sequences etc, (e) to become more sensitive and tolerant towards **nonconventional ways of doing mathematics**.

4. *The affective predisposition towards mathematics*, by helping both students and teachers (a) to see mathematics as an **evolving human endeavour** requiring intellectual effort, (b) to appreciate the creative nature of failure, mistakes, misunderstanding etc.

5. *The appreciation of mathematics as a cultural-human endeavour* by letting both students and teachers appreciate (a) the fact that mathematics evolves under the influence of **both social and cultural factors** and by **intrinsic ones** like aesthetics, curiosity, challenge, recreational purposes etc and (b) mathematics as part of the cultural heritage of particular civilizations and societies, and the role it played in this context.

A closer inspection and analysis of the above arguments suggests that there are three **different** but **complementary general directions** and **emphases** for introducing the historical dimension in ME:

(a) To **learn history** by providing **direct historical information**.

(b) To **learn mathematical topics** by following a **teaching approach inspired by history**.

(c) To develop what may be called **mathematical awareness** (i) by learning **about mathematics** and (ii) by highlighting **social and cultural aspects of mathematics in a historical perspective**.

These general ways for introducing the historical dimension in ME can be implemented in practice in a variety of ways, from employing original sources, worksheets, research projects etc, to using theater plays, movies, the Internet etc (e.g. see section 6 for an outline of such an implementation of (b)). However, we are not going to describe them here. Instead, we would like to point out that both the arguments presented in this section and (a)-(c) above, can equally well be valid in PE. This is not accidental but in our opinion it is related to two epistemological / historical theses, which form the subject of the next section.

3 The relation between mathematics and physics

The following two theses form the central core of the present paper and partly explain why the historical dimension plays a similar role and has a similar nature in ME and in PE.

<u>*Thesis* A</u>: Mathematics and physics have always been closely interwoven, in the sense of a "two-ways process":

[2]It summarizes part of the work done by the group, which was responsible for writing chapter 4.1 of the above mentioned ICMI Study Volume on *An analytical survey of the possible ways of integrating History of Mathematics in the classroom*; see acknowledgements here.

[3]This is a summary of the detailed analysis provided in Ch. 4.1, Sections 2 and 3 in FAUVEL et al. (to appear); see footnote 2 and acknowledgements here.

- Mathematical methods **are used** in physics

- Physical concepts, arguments and modes of thinking **are used** in mathematics (for this fact seen in a somewhat different perspective see TZANAKIS 1996, TZANAKIS 2000).

Apparently, this thesis seems more easily acceptable than thesis **B** below. Nevertheless, the term "use" will be clarified and deepened after stating thesis **B**, in a way that makes thesis **A** to appear less naive and readily acceptable.
Thesis **B**: Any distinction between mathematics and physics, seen as general attitudes towards the description and understanding of an object[4], is related **more** to the point of view adopted while studying particular aspects of this object, than to the object itself.

In sections 4 and 5 we will provide some evidence for theses **A** and **B**, by commenting on some historical examples. However, if these theses are accepted, then the following conclusions can be drawn:

- Any treatment of the HM independent of the history of physics (HP) is necessarily incomplete (and vice versa).

- By accepting the importance of the historical dimension in education (for the reasons given in section 2), the relation between mathematics and physics should not be ignored in teaching these disciplines.

Before illustrating theses **A** and **B** by means of examples, we will elaborate more on thesis A. Conventionally thesis **A** is interpreted as follows:

(1) *From mathematics to physics*: Mathematics is simply the **language** of physics.

(2) *From physics to mathematics*: (i) Physics is an **exterior to mathematics, huge reservoir of problems** to be solved mathematically; (ii) Physics is simply a **domain of application** of already **existing** mathematical **tools**.

Though both (1) and (2) are true, they do not exhaust the multifarious interconnection of the two disciplines and they need refinement in the following sense:
For (1): Mathematics is not only the "language"of physics (i.e. the tool for expressing, handling and developing logically physical concepts and theories), but also, it often **determines** to a large extent the **content** and **meaning** of physical concepts and theories themselves. It is in this broader sense that the term "**mathematical physics**" is used in this paper.
For (2): Physics provides, not only problems "ready-to-be-solved"mathematically, but also **ideas**, **methods** and **concepts** that are crucial for the **creation** and **development** of **new** mathematical concepts, methods, theories, or even whole mathematical domains. It is in this broader sense that the term "**physical mathematics**" is used in this paper.

In the next two sections we provide some evidence for the above by means of two groups of examples, one illustrating mathematical physics and the other one physical mathematics. At the same time, some of them provide evidence for thesis **B** as well. However, it should be

[4]By this term we mean not only concrete, empirically conceived objects, but also mental objects like concepts, questions, problems etc.

emphasized that lack of space makes our presentation sketchy, hence incomplete, and more details can be found in the litterature.

4 Examples : Mathematical Physics

The first two examples illustrate the way in which some strictly mathematical development can lead to the introduction and the specification of meaning of an important physical concept.

1. *The concept of antimatter*: After the invention of quantum mechanics, Dirac tried to develop a relativistic theory of the electron. Based partly on mathematical criteria of symmetry, he arrived in 1928 at the relativistic equation now bearing his name, by an essentially mathematical approach. However, the trouble with this equation, was that it admits solutions with negative energy for the electron, a physically unacceptable result. Instead of rejecting his equation on the basis of this physically absurd result, Dirac proposed in 1931 that these negative energy solutions should be retained as describing, not electrons, but "antielectrons" (or positrons as they are now called), a different kind of particles with energy and charge opposite to those of ordinary electrons, with which they are mutually annihilated when they interact. Originally, Dirac believed that these negative energy solutions would correspond to protons, but finally he changed his mind on the basis of objections mainly of a mathematical nature that had been raised by Weyl (KRAGH 1990 pp. 102-103). In this way the bold new concept of antimatter was introduced into physics by interpreting the result of a strictly mathematical deduction. Apparently this could not have been done otherwise if one wanted to avoid the rejection of Dirac's theory (for a detailed historical account see SCHWEBER 1994 section 1.6, KRAGH 1990 pp. 57-59, 87-103).

2. *The wave nature of matter*: In 1900, Planck introduced his quantum hypothesis for the energy E of light as a function of its frequency $\nu, E = h\nu$ (h being Planck's constant)[5]. In 1908, three years after the original formulation of special relativity (SR), it became clear through Minkowski's work that the momentum p and the energy E of a particle are **different coordinates** of the **same** four dimensional (4D) vector in space time (p, E) and the same is true respectively for the wave number k and the frequency ν of any wave (k, ν) (PAULI 1981 sections 29, 37, PAIS 1982 section III.7(c), PIERSEAUX 1999, section 2.IV.4-1)[6]. In 1924, de Broglie observed that given the analogy between geometrical optics and classical mechanics (see example 5 below), if one wants to accept both Planck's relation and the above consequences of SR, then one is **mathematically** led to the relation $p = hk$ which clearly suggests that not only the light, but also any kind of particles has (is associated to) a wave nature (wave phenomenon) (DE BROGLIE 1925, ch.II section V). This idea, on the one hand stimulated Schrödinger's effort towards the formulation of wave mechanics and on the other hand was confirmed experimentally a few years later (SCHRÖDINGER 1982 p. 20, JAMMER 1965 pp. 257-258, KRAGH 1982 pp. 155-157).

The next two examples illustrate how the introduction of a new mathematical concept may accelerate the development of a physical theory, or, conversely its absence may prevent its development. Both examples provide evidence for thesis **B** as well.

[5]Strictly speaking, originally Planck's relation concerned the exchange of energy between matter and light. It was Einstein who, in 1905, extended this relation to the light itself

[6]This is already implicity contained in Einstein's original paper (reprinted in SOMMERFELD 1952 paper III, p. 56) and can be inferred as long as the geometric formulation of SR based on the concept of spacetime is used. This is clear in de Broglie's work (DE BROGLIE 1925 ch.II section IV).

3. *The concept of spacetime and the theory of relativity*: It is known that SR is based on the so-called "Lorentz coordinate transformations" (LT) between inertial coordinate systems moving relative to each other at constant velocity. Originally they have been derived in a **physically oriented way** (*thesis* **B**) by Lorentz (1904) as those transformations leaving invariant Maxwell's electrodynamic equations (SOMMERFELD 1952, paper II). Einstein's 1905 derivation was also of this nature, but it was based on an epistemological analysis of the intuitive concept of simultaneity (SOMMERFELD 1952, paper III). However, in 1908, Minkowski followed a more **mathematically oriented approach** (*thesis* **B**). He introduced the crucial **geometric** concept of spacetime as a 4D manifold with a particular type of (pseudo)distance for any two of its points. He then derived the LT as those transformations that leave invariant this (pseudo)distance[7] (SOMMERFELD 1952, paper V; for a didactically appropriate reconstruction with more historical comments and references to the original litterature, see TZANAKIS 1999). This was a crucial step which accelerated the development of SR, by unfolding the geometrical ideas hidden in Einstein's original paper. Nevertheless, one could imagine the development of SR **without** the concept of spacetime. However, and this is the second point to be made here, it is completely impossible to imagine the development of the general theory of relativity (GR) without this concept. The reason is simple: without it, riemannian geometry and tensor calculus could not have been used as the absolutely indispensable tool for the formulation of Einstein's physical ideas about gravitation, which led to GR. This last point is even more clearly illustrated by the next example.

4. *The concept of a singularity in spacetime*: We all have heard about the existence of a singularity in the original "big-bang" of the universe, or inside blackholes. Originally, a singularity was conceived in a rather **intuitive physical way** (*thesis* **B**), as a point (or region) of spacetime in which (some) geometrical and physical quantities become infinite (see e.g. WHEELER 1964 p. 317, HARRISON et al. 1965, ch.11, p. 141). This idea does not permit to understand whether the already (theoretically) known existence of singularities in particular cases, is an accidental fact, or is an intrinsic feature of GR (HAWKING et al 1973 pp. 261-262, JOSHI 1993 pp. 157-163).

On the other hand, in the 1960's a different **mathematical approach** (*thesis* **B**) was initiated by Penrose and developped by him, Hawking and Geroch. It was based on the **quite familiar idea of singularity in riemannian geometry** as a limit point of a curve that does not belong to the manifold (see e.g. CLARKE 1993 section 1.2. for a rigorous definition)[8]. This simple, but radically different idea was the necessary crucial step for the formulation and proof of the famous "singularity theorems" in GR by Penrose, Hawking and Geroch (1965-1970), which showed that the existence of singularities is essentially an intrinsic characteristic of GR (HAWKING et al. 1973 ch.8). At the same time, we have here a fruitful **feedback** to Mathematics, namely the development of (pseudo)riemannian geometry as an independent mathematical discipline with prototype example the geometry of spacetime, which remains however, always closely connected to its physical applications (good examples of such monographs are those by O'NEILL 1983 and BEEM et al. 1981).

[7]The prefix "pseudo" stems from the fact that this spacetime distance may be positive, zero or negative for noncoinciding points.

[8]E.g. a spherical surface in which a point has been removed has a singularity in this sense; that is, it has curves which are **incomplete** in the way described above. In riemannian geometry, the opposite concept of a **complete** curve (i.e. loosely speaking, a curve containing its limits points) was quite familiar (see e.g. the classical treatises by HICKS 1971 and HELGASON 1962 and original references therein).

5. *The invention of wave mechanics*: Schrödinger in 1926 laid the foundations of his wave mechanics, based on the analogy between classical mechanics and geometrical optics (SCHRÖDINGER 1982, paper II and lecture I). This analogy was known long before. In the period 1833-1835, based on this analogy, Hamilton developped a unified mathematical approach to the description of these two theories, in which they appear as different but isomorphic structures. Schrödinger's crucial argument was based on the remark that we know that geometrical optics is only an approximation to the exact wave optics. Therefore, we may look for a new (wave) mechanics, such that classical mechanics is an approximation to it, in such a way that the above mentioned isomorphism is preserved. This was sufficient for arriving mathematically at the formulation the partial differential equation (PDE) now bearing his name and which is the cornerstone of wave mechanics (for a reconstruction of Schrödinger's approach stressing the role of analogy as a pattern of discovery, see TZANAKIS 1998, see also TZANAKIS et al. 1988 and references to the original litterature therein).

This example also illustrates thesis **B** at the teaching level: In a **mathematically** oriented treatment, Hamilton's approach may be considered as a method for solving a first order PDE, the so-called Jacobi method, or its equivalent, the solution of such an equation by using its system of characteristic ordinary differential equations. In a **physically** oriented treatment it can be seen as a starting point for developing analytical mechanics and more precisely, the Hamilton-Jacobi theory of solving a mechanical problem, or its equivalent, solving it with the aid of the corresponding system of Hamilton's canonical equations[9] (TZANAKIS, 2000, section 3.3).

5 Examples : Physical Mathematics

The first two examples illustrate the fact that a **physical** concept or idea may act as a (partial) motivation for the emergence of important **mathematical** concepts.

1. *Velocity and the derivative concept*: In its modern form, the velocity concept was the product of a long and complicated evolution over almost three centuries and it was partially formulated by Galileo, who discussed only uniformly accelerated motion (BOYER 1959 pp. 72-73, 82-83, 113-114, DUGAS 1988 pp. 57, 59-61, 66-67, WHITROW 1980 pp. 181, 183-184). It is known that this fact influenced the emergence of the concept of instantaneous velocity (BOYER 1959 pp. 177, 180), which in turn acted as a basic motivation for the formulation of the derivative concept (HALL 1983 pp. 288-289). This fact can be illustrated by a short extract from Newton's "Principia". After Newton introduces his conception of the derivative as "an ultimate ratio of evanescent quantities", he tries to refute possible objections to it by writing:

> Perhaps it may be objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none. But **by the same argument** it may be alleged that a body arriving at a certain place, and there stopping, has no ultimate velocity; because the velocity, before the body comes to a place, is not the ultimate velocity; when it has arrived, is none. But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives... **And in like manner**, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish. (NEWTON 1934, p. 39, our emphasis)

It is clear that Newton tried to legitimize the concept of the derivative on the basis of the physical

[9]Actually, the latter is the system of characteristic ordinary differential equations for the Hamilton-Jacobi equation which is the basis of the Hamilton-Jacobi theory.

concept of instantaneous velocity which he considered intuitively more clear (and apparently he never defined it exactly; BOYER 1959 pp. 193-194), which thus appears as the prototype example of a derivative.

2. *Dirac's δ - function and generalized functions*: A similar, more recent example is provided by Dirac's δ-function in quantum mechanics introduced in 1927 (KRAGH 1990 pp. 40-41). In his famous book on the foundations of quantum mechanics, first published in 1930 (DIRAC 1958), he introduces the δ-function

$$\delta(x) = 0 \text{ for } x \neq 0, \int \delta(x)dx = 1.$$

It is easily seen that strictly speaking, this object has no mathematical meaning; actually only an approximate physical picture of it as a highly peaked graph can be given. Dirac was well aware of this fact when he wrote that

> . . . $\delta(x)$ **is not a function** of x according to the usual mathematical definition of a function, which requires a function to have a definite value for each point in its domain, but **is something more general**, which may be called an 'improper function'. . . Thus $\delta(x)$ is not a quantity which can be generally used . . . as an ordinary function, but **its use must be confined** to certain simple types of expression for which it is obvious that **no inconsistency can arise**. . . although an improper function does not itself have a well-defined value, **when it occurs as a factor in an integrand the integral has a well-defined value**. (DIRAC 1958, p. 59, our emphasis)

This physical picture, the operational effectiveness of $\delta(x)$ in quantum mechanical calculations and its mathematically self-contradictory nature, acted as a partial motivation for the emergence of the concept of generalized function introduced originally by Sobolev in 1936 and more systematically by L. Schwarz from 1945 onwards, as a functional (i.e. a function) acting on an appropriate space of functions (KRAGH 1990 p. 41, BOYER 1968 p. 671, DIEUDONNÉ 1981 pp. 225-226). Actually, the last sentence of the above quotation implicitly expresses the idea of the δ-function as a functional.

Finally, this example provides evidence for thesis **B**: In the above quotation, Dirac adopts a **physical** attitude by admitting that although this concept is mathematically unacceptable, still he uses it since it is operationally effective[10]. On the other hand, a **mathematical** approach to it, was to accept that such a concept may be useful in applications and possibly in pure Mathematics, therefore one should try to make it a logically consistent concept.

3. *Brownian motion and stochastic differential equations*: In a similar way, the study of Brownian motion was the main motivation for the development of a whole mathematical domain, namely the theory of stochastic differential equations.

Brownian motion is the irregular motion done by a heavy particle suspended in a fluid, due to its random collisions with the (much lighter) molecules of the fluid. Though it was first observed by Brown in 1828, it was not until 1908 that the first **mechanical** model was proposed by Langevin (LANGEVIN 1908), who wrote for the velocity v of the Brownian particle as a function of time t, the equation

$$\frac{dv}{dt} = -\beta v + F(t), \beta = \text{constant}$$

[10]For more details on this attitude of Dirac towards Mathematics, see KRAGH 1990 pp. 280-281.

The novel feature here is the nature of the force $F(t)$ due to the collisions of the particle with the molecules of the fluid. Since the motions of the latter are random, only the **average** properties of F could be postulated upon physical considerations. This was an important equation since it was the first **mechanical** model which allowed for the experimental verification of the molecular structure of matter[11]. On the other hand, Wiener (WIENER 1923) proved that, although the velocity $v(t)$ of the Brownian particle seen as a stochastic process (due to the random nature of $F(t)$), is continuous (and of unbounded variation) with probability one, it is **nowhere** differentiable with probability one. Hence, Langevin's equation has no meaning as an ordinary differential equation. This contradiction, together with the average properties of $F(t)$, postulated on physical grounds, were the main input for the emergence of the concept of a **stochastic integral** introduced by Itô in 1951 (see ARNOLD 1974, Introduction and references therein). This is the cornerstone for the development of the theory of stochastic integration and of stochastic differential equations. In this context, Langevin's equation is no longer an unacceptable object, but acquires a meaning as an equation of this kind.

4. *The development of vector analysis in the 19th century*: Often an intuitive physical method for tackling some problems quantitatively, may lead to the development of new mathematical methods and theories. This is the case of Bernoulli's "brachistochrone problem" as a main motivation for the development of the calculus of variations. This example will be discussed from a somewhat different perspective in the next section. Below, we will consider vector analysis as another such example, in which the new mathematical concepts and methods were developped in their final form mainly by physicists. Here we have a very complicated interaction between mathematics and physics, hence the discussion that follows is necessarily sketchy, mainly confined on Maxwell's contribution.

By the mid 19th century there were important physical investigations containing deep mathematical insights that form part of the foundations of modern vector analysis; Green's essay on "The applications of Mathematical Analysis to the theories of electricity and magnetism" (1828) containing his and Gauss' theorems, W. Thompson's early work on analogies between electric phenomena with heat conduction and elasticity (1846-47; WHITTAKER 1951, pp. 241-242) and Stokes' Smith Prize Essay of 1854 in which the theorem bearing now his name is contained (for more details on the history of these theorems, see CROWE 1985, note 29 pp. 146-147). The general significance of these theorems, as well as the importance of vector methods, were explicitly acknowledged by Maxwell in his classical "Treatise on Electricity and Magnetism" published in 1873 (MAXWELL 1954, vol. I, sections 16, 21, 24, 95b, see also below). Actually, Maxwell was well aware of this fact when in 1871 he wrote in a more general context:

> . . . when the student has become acquainted with several different sciences [i.e. domains or theories in physics], he finds that the **mathematical processes and trains of reasoning on one science resemble those in another** so much that his **knowledge of the one science may be made a most useful help in the study of the other**.
>
> When he examines the reason of this, he finds that in the two sciences he has been dealing with systems of quantities, in which the **mathematical form of the relations** between the quantities **are**

[11]Einstein's paper of 1905 (EINSTEIN 1956, paper I) was the first theoretical work on which conclusive experiments on the existence of molecules could be based, like those of Perrin in 1908 (PERRIN 1991, ch.IV). However, Einstein's theory was not a genuine mechanical model. Such a model was provided by Langevin (see the equation above) who rederived Einstein's basic result. His model was further elaborated by others and especially by Ornstein and Uhlenbeck (UHLENBECK et al. 1930) and played an important role in the development of the theory of stochastic processes and of stochastic differential equations (for a brief historical survey see BLANCHARD et al. 1987 section I.1a; cf. NELSON 1967 sections 3, 4, 9).

the same in both systems, though the **physical nature** of the quantities may be **utterly different**. (quoted in CROWE 1985, p. 130, our emphasis)

In this quotation it is evident that Maxwell stresses the importance for both mathematics and physics of the determination of isomorphic mathematical structures, helpful for developing mathematical methods on the basis of which problems in different domains can be tackled in the same way. He becomes more explicit in 1872 when he stressed the importance in this context of the "calculus of quaternions" developped mainly by Hamilton and Tait:

> A most important distinction was drawn by Hamilton when he divided the quantities with which he had to do into Scalar quantities... and Vectors... The invention of the calculus of Quaternions is a step towards the knowledge of quantities related to space which can only be compared for its importance, with the invention of triple coordinates by Descartes. **The ideas** of this calculus, as **distinguished from its operations and symbols**, are fitted to be of the greatest use in all parts of science. (quoted in CROWE 1985, p. 131, our emphasis)

These qualitative remarks were transformed into exact mathematics in his "Treatise", in which on the basis of the calculus of quaternions he stresses the importance of the basic operators *grad, div, curl* of modern vector analysis (originally introduced by Tait; MAXWELL 1954, section 25) and revealed the **general significance** of its basic theorems, already known in special cases (theorems of Stokes, Gauss and Green). In fact, according to Maxwell "... the doctrine of Vectors... is a method of **thinking** and not a method for saving thought..." (quoted in CROWE 1985, p. 133, our emphasis). Thus in his view, "... by means of the vectorial approach, the physicist attains to a direct mathematical representation of physical entities and is thus aided in seeing the physics involved into the mathematics" (CROWE 1985, p. 134). In fact, in the preface to his "Treatise", he expresses very clearly the role of physical insight for appreciating the significance of mathematical results:

> I also found that several of the most fertile methods of research discovered by mathematicians could be expressed **much better** in terms of ideas derived from Faraday than in their original form... Hence many mathematical discoveries of Laplace, Poisson, Green and Gauss **find their proper place** in this treatise and their **appropriate expressions in terms of conceptions mainly derived from Faraday**. (MAXWELL 1954, pp. ix-x, our emphasis)

It is through such deep insights into the mathematical structure of physical theories, together with an outline of vector methods and concepts contained in his "Treatise" that modern vector analysis emerged and was established in the hands of physicists like Gibbs (1881-1884) and Heaviside (from 1883 onwards) (CROWE 1985 pp. 138-139).

5. *Quantum mechanics and functional analysis*: As the last example we mention the role of quantum mechanics (QM) in stimulating the development of functional analysis. The subject is vast and here we only mention the role of physics in the emergence of the abstract concept of a Hilbert space.

In 1925, originally Heisenberg and later Heisenberg, Born and Jordan developped matrix mechanics as a new theory of atomic phenomena, based on matrix algebra, a subject unfamiliar to physicists at that time (VAN DER WAERDEN 1967, papers 12, 13, 15). In 1926 Schrödinger founded wave mechanics (cf. section 4.5) based on the familiar theory of PDE. The main mathematical problem of the two theories was respectively, the diagonalization of a certain (often infinite dimensional) matrix, the hamiltonian matrix and the solution of Schrödinger's equation (see e.g. HEISENBERG 1949, Appendix). The strange thing was that these two **conceptually**

totally **different** theories, gave **identical** results, compatible with experiments. Hence the question of finding their relation naturally arose. It was tackled by both Schrödinger (1926) and von Neumann (1927-1932) in a different way that provides some support for thesis **B**:

Schrödinger provided a **formal** proof that, by choosing a basis for the wave functions he was using, solving his PDE becomes a matrix eigenvalue problem identical to that of matrix mechanics and vice versa (SCHRÖDINGER 1982, paper 4).

Von Neumann's approach was of a rather different character (VON NEUMANN 1947, ch.I, particularly section 4 and p. 19). He tried to identify the basic properties of the objects with which the two theories were dealing of and in this way he was led to define axiomatically what became known as a separable Hilbert space (VON NEUMANN 1947, ch.II, STONE 1932, p. 2)[12]. Then he proved that all such spaces are isomorphic, thus giving a definite answer to the question above: the two conceptually different theories were just different representations of the same abstract mathematical structur that forms the mathematical substratum of the formalism of QM (VON NEUMANN 1947, ch.II, theorem 9 and pp. 41-42). For an outline of a possible didactical sequence, see TZANAKIS 2000, section 3.4).

6 Implementing the relation between mathematics and physics in teaching : an example

The examples presented in sections 4 and 5 provide enough evidence for the deep interplay between mathematics and physics. Therefore, in this section we will describe how their close relation could be implemented in practice, by analyzing an example on the basis of an approach inspired by history (see (b) in section 2). To this end, the following general scheme will be employed (TZANAKIS, 2000, section 1, TZANAKIS 1996 section 1 and in more detail, FAUVEL et al., 2000, ch.7, section 3.2 - cf. acknowledgements here):

(a) The teacher has a **basic** knowledge of the historical evolution of the subject.

(b) On the basis of this knowledge he identifies the **crucial steps** of this evolution (key ideas, questions and problems that stimulated it, difficulties and errors that have been faced and possibly overcome etc).

(c) These crucial steps, are **reconstructed**, probably using modern terminology and notation, so that they become didactically appropriate.

(d) To keep the presentation to a reasonable size, many **details** in (c) can be given as sequences of historically motivated exercises of an increasing level of difficulty, such that each one presupposes (some of) the preceding ones.

As an example we outline below a **possible** teaching sequence for Bernoulli's "brachistochrone problem" mentioned in section 5, as a subject for making practice in differential calculus at the high school level or early undergraduate level (TZANAKIS et al. 2000; for a similar approach to this subject see CHABERT 1993). Following the above mentioned general scheme, we have:

The basic historical steps (for (a) and (b))

[12] The term "Hilbert space" was used earlier to denote only the space ℓ^2 of complex sequences of numbers having a finite sum of the squares of their norms (DIEUDONNÉ 1981, p. 172, VON NEUMANN 1949, p. 23).

(i) Hero's proof of the law of reflection on the basis of the assumption that light moves between two points by following the shortest path. As a geometrical extremum problem the proof is elementary and well known, see figure 1 (THOMAS 1941, p. 496-499).

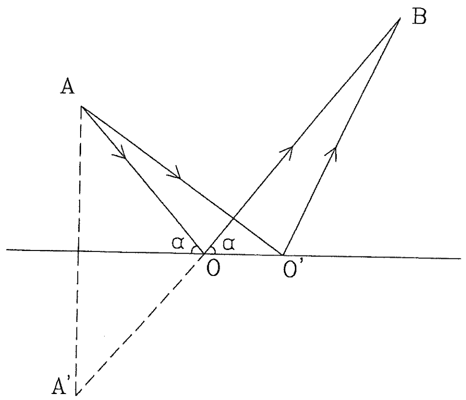

FIGURE 1

(ii) Fermat's derivation of the law of refraction (1662), already formulated empirically by Harriot (1601), Snel (1621) and Descartes (1637) (HALL 1983 p. 197), on the basis of his principle of Least Time "... Nature always acts in the shortest ways" which he interpreted in the present context as follows (see figure 2): Light goes from a point A in which it has speed v_1 to a point B in which it has speed v_2 by following the shortest path (DUGAS 1988, p. 254; here $v_1 > v_2$).

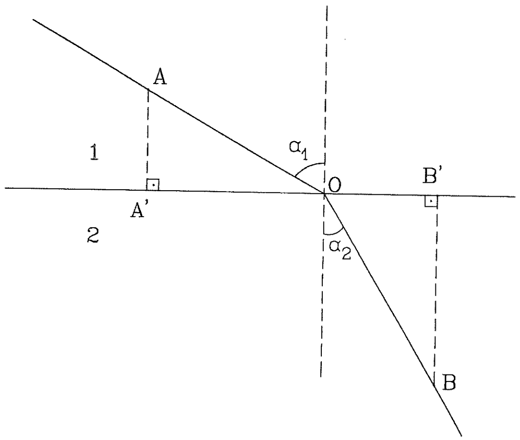

FIGURE 2

In this way he specified point O by deriving the relation (DUGAS 1988, part III, section V.1).

$$A'O/B'O = v_1/v_2 = \sin a_1/\sin a_2 \text{ if } AO = BO \tag{1}$$

(iii) In 1696, Johann Bernoulli formulated the "brachistochrone problem": "To find the trajectory of a point, which starting from a given point A and moving on a vertical plane under its weight only, arrives at a given point B in the least time" (HAIRER et al. 1996, pp. 136-137). His solution was given in **analogy** to Fermat's approach in (ii), by dividing the vertical distance between A and B into thin horizontal layers in which the velocity of the particle could be considered approximately constant. In this way an equation similar to (1) is valid in each layer. By passing to the limit of a vanishing width of the layers, he derived and solved an equation for the unknown curve which turns out to be the cycloid.

A teaching sequence (for (c) and (d))

(i) One may introduce coordinates in figure 2 and express AOB **analytically** as a function of one of them, e.g. of $OA'(AO \neq BO$ in general). Requiring the time to be a minimum, leads to the vanishing of the derivative of this function, which gives the second of equations (1), i.e. the law of refraction in its usual form $\sin a_1/v_1 = \sin a_2/v_2$.

(ii) One may give as an exercise (possibly in several steps) a derivation along these lines of the much simpler law of reflection.

(iii) The idea in (i) may be used to **reconstruct** Bernoulli's solution of the brachistochrone in the form of a differential equation for the unknown curve $y(x)$. Equation (1) and the law of the conservation of the energy of the particle, give (SIMMONS 1974, section 1.6)

$$y(1 + y'^2) = 2c = \text{constant} \tag{2}$$

(iv) That the cycloid, given in parametric form as

$$x = c(\theta - \sin\theta), y = c(1 - \cos\theta)$$

satisfies this equation is a simple exercise.

(v) A somewhat more advanced related subject is to use Newton's dynamic law and the differentiation rules (especially the chain rule), to derive the equation of motion of a point particle which is constraint to move along a cycloid under its own weight only. The result is (g = the acceleration of gravity)

$$\frac{d^2}{dt^2}(\sin\frac{\theta}{2}) = -\frac{g}{4c}c\sin\frac{\theta}{2} \tag{3}$$

This is the equation of motion of a **simple** pendulum ($\sin\theta/2$ denoting the amplitude of its oscillation; $\sin\theta/2$ is proportional to the arc length of the cycloid). Eq(3) is an important result since, on the one hand it describes a strictly isochronous oscillation (i,e, its period is independent of its amplitude) and on the other hand it is only an **approximation** for the **simple** pendulum, but an **exact** result for cycloidal motion. This was the basis of Huygens' construction (1673) of the first isochronous clock based on the motion along a cycloid (cycloidal pendulum; SOMMERFELD 1964 section 17).

(vi) At the university level, one may use (iii) above to express the time as an **integral** involving y and y'. In analogy with the differential of a function in the Calculus, one may introduce the concept of the variation of an integral (functional), which should vanish for an extremal curve. In this way equation (2) is obtained again and the problem is solved. Thus, one may appreciate clearly the generality of this approach, which supersedes the particular problem of the brachistochrone and indicates the path for a systematic introduction to the calculus of variations (see e.g. CHABERT 1993).

In this paper an effort was made to support the claim that there is a continuous in time, fruitful deep interplay between mathematics and physics, which should be conceived as different, but complementary views of the same world (mentally or empirically conceived). Therefore, this interplay and complementary character should not be neglected in teaching and learning these disciplines; on the contrary, both ME and PE can profit from it, possibly taking into account aspects of the historical evolution of this interplay.

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