The study of Nicomedes' Conchoid and Descartes' Folium, according to the Portuguese mathematician Francisco Gomes Teixeira in his *Traité des Courbes Spéciales Remarquables Planes et Gauches*

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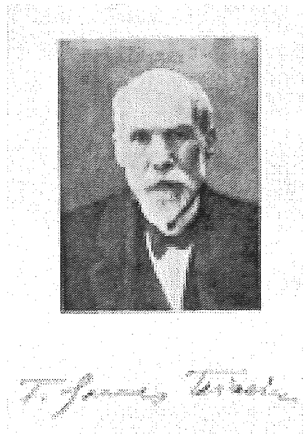
Abstract

One of the most interesting works of the Portuguese mathematician FRANCISCO GOMES TEIXEIRA (1851-1933) is the *Traité des Courbes Spéciales Planes et Gauches*.

This author analyses a great number of curves and gives, for each of them a detailed mathematical treatment that includes historical data with reference to original sources, as well as his personal contributions. As examples, we shall present the study of Nicomedes' conchoid and of Descartes' folium. For each one of these curves, we shall give:

- a brief historical summary;
- copies of texts by GOMES TEIXEIRA and by other authors whom he quotes;
- a work sheet inviting participants to collaborate, which will include some of the problems and challenges historically related with the curve.

Finally, the resolution of those problems will be presented in *data-show*.



1 The Portuguese mathematician FRANCISCO GOMES TEIXEIRA

1851-1933

1.1 Brief biographical details

FRANCISCO GOMES TEIXEIRA graduated from the University of Coimbra in 1875, and submitted a thesis on *Integração das equações às derivadas parciais da 2ª ordem* [Integration of Second Order Partial Differential Equations] for his doctorate.

While still a student in his third year, he submitted and published his first work, entitled *Desenvolvimento das funções em fracção contínua* [Expansion of Functions into Fractions Continuous], and in his fourth year he published the article *Aplicação das fracções contínuas à determinação das raizes das equações* [Application of Continuous Fractions to the Determination of Roots of Equations], which he submitted to the Academia das Ciências de Lisboa [Lisbon Academy of Science].

His dissertation for his application for a post at the Faculty was entitled *Sôbre o emprego dos eixos coordenados na mecânica analítica* [On the Use of Co-ordinate Axes in Analytical Mechanics], showing that

> ... Poinsot se enganara ao afirmar que, contrariamente à suposição de Lagrange, tais coordenadas são impróprias para a resolução dos problemas da Mecânica por meio do princípio do trabalho virtual. (Beires, 1950-1951, p. 11)
>
> [... Poinsot was mistaken in stating that, contrary to Lagrange's supposition, such coordinates are unsuitable for resolving problems of mechanics by means of the principle of virtual work].

That same year he was appointed substitute Professor of the Faculty of Science of the University of Coimbra, and he remained a Professor of that institution until he was appointed director of the Escola Politécnica do Porto [Porto Polytechnic] in 1883. There he took over the course of Differential and Integral Calculus, which he continued to teach throughout his time as a teacher.

In 1911 the University of Porto was founded, and GOMES TEIXEIRA was elected and designated as its first Rector. In 1918 he was proposed as, and became, honorary Rector. Although, in 1921, he had reached the legal age limit for the exercise of a public office, the Academia do Porto arranged a tribute at the University to bring him back to control of the course he had always taught.

GOMES TEIXEIRA was professor *honoris causa* of the Universidad Central de Madrid (1922) and the Université de Toulouse (1923). He took part in various international conferences: Boston, Caen, Besançon, Toronto, Zurich, Salamanca, Porto, Cambridge, Seville, Bilbao, etc.

His prestige was internationally recognised as he was a member of many institutions, among which, for their geographical diversity, let us quote: Madrid Royal Academy of Sciences, Lisbon Royal Academy of Sciences, Circolo Matematico da Palermo, Prague Academy of Sciences, Barcelona Royal Academy of Sciences and Arts, Liège Royal Society of Sciences, Krakow Mathematical Society, Lima Faculty of Sciences, Moscow Mathematical Society.

He belonged to the *Commission Internationale de l'Enseignement mathématique*, the *Comité Permanent International du Répertoire Bibliographique des Sciences Mathématiques* [Permanent International Committee for the Bibliographical Repertoire of Mathematical Sciences], the International Catalogue of Scientific Literature, and, from its foundation, the *Comité de

Patronage de l'Enseignement Mathématique* [Committee for the Patronage of the Teaching of Mathematics.

At the invitation of the organisers he was part of several international committees as a tribute to Lobatschewski, Lavoisier, Hermite, Mittag-Leffler, Descartes, Battaglini, etc.

Owing to his prestige as a man connected with teaching, GOMES TEIXEIRA held government positions connected to education: he was a member of the *Conselho Superior de Instrução Pública* [Higher Council for Public Instruction] and a Member of the *Junta Orientadora dos Estudos* [Studies Orientation Group].

1.2 The work of GOMES TEIXEIRA

Due to historical and social reasons it was during a period of weak scientific development, hardly favourable to research work, that GOMES TEIXEIRA began his career as a teacher and as a mathematician.

As a consequence of so little scientific production, the teaching of pure Mathematics in Portugal was carried out with the aid of compendia which, apart from rare exceptions, were translations of foreign books, essentially French ones, sometimes expanded and annotated.

Aware of the problems that this situation involved, GOMES TEIXEIRA published two books on Analysis in the Portuguese language, *Curso de Analyse Infinitesimal – Cálculo Diferencial* [Course in Infinitesimal Analysis – Differential Calculus] (1887) and *Curso de Analyse Infinitesimal – Cálculo Integral* [Course in Infinitesimal Analysis – Integral Calculus] (1889), whose contents dealt with the most up to date work of the time. These works were innovative and groundbreaking, as they contributed greatly to informing Portuguese mathematicians of the new currents in the Mathematical Analysis of the time. They also had the merit of introducing the precision of principle, the strictness of deduction and the amplitude of results already obtained in Europe into the teaching in Portuguese higher education institutions.

Study of his correspondence –more than 2 000 letters received from scientists in Europe, South America, Canada, Japan and the United States– shows that the work in question was not only noted in Portugal, as many foreign mathematicians gave him great praise, either personally or in scientific journals.

GOMES TEIXEIRA received two prizes for publications on Analysis: in 1888 in Portugal, he won the D. Luiz I prize for the book, *Curso de Analyse Infinitesimal – Calculo Diferencial*; in 1895, the memoir *Sobre o desenvolvimento das funções em série* [On the Expansion of Functions in Series] was submitted to the open competition organised by the Academia Real de sciencias exactas, physicas e anturaes de Madrid, winning a prize in 1897, and being published later. According to Vilhena

> O prémio era pecuniário e com diploma e medalha (1936, p. 125)
> [The prize was money, with a diploma and a medal]

but as it had been written in Portuguese, and as it was a condition that the languages adopted were Castilian or Latin, he was not awarded the medal.

In 1896 the Madrid Royal Academy of Sciences set the competition:

> Catálogo ordenado de todas las curvas de cualquier clase que han recibido nombre especial, acompañhado de una idea sucinta de la forma, ecuaciones y propriedades generales de cada una, con noticia, de los libros ó autores que primeramente las han dado à conocer. (TEIXEIRA, 1905, p. V)

[An ordered catalogue of all the curves of any class which have received a special name, accompanied by a succinct idea of the shape, equations and general properties of each, with notification of the books or authors who first made them known]

GOMES TEIXEIRA then produced a noticeable work entitled *Tratado de las curvas notables* [Treatment of the Noticeable Curves], written in Castilian, which was awarded a prize by the aforementioned academy, *ex aequo* with the Italian Gino Loria. The treatment was revised, enlarged and translated into French by the author, with the title *Traité des courbes spéciales remarquables planes et gauches*, for which he was awarded the Binoux prize of the Paris Academy of Sciences in 1917. In the report on GOMES TEIXEIRA's work for this prize, Appell wrote:

> . . . Sans doute des monographies de courbes de certaines espèces ou même des travaux plus complets ont été publiés à diverses époques. . . Mais il manquait un Ouvrage systématique et complet formant un catalogue ordonné de toutes les courbes remarquables, indiquant leurs équations et leurs propriétés essentielles, avec une Notice bibliographique des auteurs qui les ont étudiées. C'est cet Ouvrage qu'a composé le professeur F. Gomes Teixeira,. . . L'œuvre de M. Teixeira constitue également une histoire des Mathématiques envisagée sous un point de vue spécial. On trouve en effet, en étudiant les diverses courbes qui se sont introduites en Géométrie, l'illustration des progrès de la Géométrie pure, de la Géométrie analytique, de l'Analyse infinitésimale, de l'Algèbre et de la théorie des invariants et covariants, de la théorie moderne des fonctions, de la Mécanique, de la Physique et de l'Astronomie. . . En dressant un catalogue raisonné de ces courbes, en donnant leur histoire dans un important ouvrage, M. F. Gomes Teixeira a rendu à la Science un grand service, que la Commission propose de reconnaître en lui décernant le prix Binoux. (1918, pp. 126-128)
>
> [Without a doubt, at various times monographs on curves of certain kinds or even more complete works have been published. . . But what was missing was a systematic and complete Work, forming an ordered catalogue of all the noticeable curves, showing their equations and essential properties, with a bibliographical note on the authors who studied them. This is the Work that Professor F. Gomes Teixeira has composed,. . . The work of M. Teixeira is also a history of Mathematics, considered from a special point of view. As a matter of fact in studying the various curves which are introduced in Geometry, one finds the illustration of the progressions in Pure Geometry, Analytical Geometry, Infinitesimal Analysis, Algebra and in the theory of invariants and covariants, modern theory of functions, Mechanics, Physics and Astronomy. . . In establishing a rational catalogue of these curves, in supplying their history in an important work, M. F. Gomes Teixeira has done a great service to Science, which the Committee proposes to recognise by awarding him the "Binoux" prize.]

The "*Traité* . . ." was re-published in French in the United States in 1971 by the publisher Chelsea, and the French publisher Editions Jacques Gabay made a new edition in 1995, after a favourable opinion from many mathematicians and with an extremely positive review published in the French journal *Bulletin de l'APMEP*, n° 318, April-May 1995.

In 1902 the Portuguese government decided to publish all of GOMES TEIXEIRA's works, which were compiled in seven volumes with the title *Obras sobre Mathematica* [Works on Mathematics]. To illustrate how these works were appreciated, I quote the French mathematician M. Appell, who, in a report on GOMES TEIXEIRA at the time of the Bidoux prize already mentioned, among many other eulogies said:

> . . . Il nous est impossible de donner une analyse de la substance si riche des sept volumes. . . . (1918, pp. 126-128)
>
> [It is impossible for us to provide an analysis of the so-rich substance of its seven volumes. . .].

GOMES TEIXEIRA was also a historian; besides the many accurate historical quotes that he included in his works, and several articles that he wrote on Portuguese mathematics, at the end of his career, compiling and broadening his studies, he wrote the book *História das Matemáticas em Portugal* [History of Mathematics in Portugal]. However, he died in 1933 without completing it. It was revised and published after his death by Scipião de Carvalho in 1934. This work contains the lessons he gave from 12 to 19 April 1932, less than one year before his death.

The scientific writings of GOMES TEIXEIRA have a didactic charm inasmuch as they are accurate but clear, even when dealing with topics of great scientific difficulty. Reflecting concern about the teaching of Mathematics, in 1926 he published the book *Manual de Cálculo Diferencial* [Manual of Differential Calculus] in which, in "Advertência" [Notice] one can read:

> Èste volume contem a doutrina essencial para um primeiro estudo do Cálculo Diferencial a fazer nas cadeiras de Análise das faculdades de sciências [. . .] A êste volume seguir-se há por ventura outro do mesmo género, contendo os elementos do Cálculo integral. (TEIXEIRA, 1926)
>
> [This volume contains the essential doctrine for a first study of differential calculus to be done in the courses of Analysis at the faculties of sciences [. . .] This volume will be followed by another of the same kind, containing the elements of integral calculus].

Unfortunately, this prediction of the author was never realised, as only the first volume was published, and was not widely distributed.

But GOMES TEIXEIRA had a polyvalent side, and was not only interested in the study and teaching of pure mathematics. He was a traveller and a mountain climber, and in 1926 he published the book *Santuários de Montanha* [Mountain Sanctuaries] in which he told of his trips. In this work GOMES TEIXEIRA gives his opinion on the education of young people, arguing that

> . . . uma sociedade bem organizada deve dar à juventude perfeita educação moral, uma boa educação física e uma esmarada educação intelectual [. . .] Não basta preparar homens que saibam trabalhar bem; é necessário dar-lhes o vigor necessário para o poderem fazer. . . . (TEIXEIRA, 1926, pp. 23-24)
>
> [. . . a well organised society should give its youth a perfect moral education, a good physical education and an ideal intellectual education [. . .] It is not enough to prepare men who know how to work well; it is necessary to give them the strength needed for them to be able to do it. . .].

In the last years of his life, in parallel with the publication of historical and scientific material, he also dedicated himself to writing on religious themes: *Apoteose de S. Francisco de Assis* [Apotheosis of St. Francis of Assisi] (1928) *Uma santa e uma sábia* [A saint and a sage] (1930) and *Santo António de Lisboa* [St. Anthony of Lisbon] (1931).

1.3 Opening to the international scientific community: the contribution of GOMES TEIXEIRA

In the historic eulogy he made to the Portuguese mathematician Daniel da Silva, GOMES TEIXEIRA states:

> Nada há de mais prejudicial para a sciência de um povo do que o seu isolomento no meio da sciência dos outros. Este isolamento foi quási completo em Portugal na maior parte do século XIX, e o

motivo principal estava no desconhecimento da nossa língua nos meios scientíficos estranjeiros [. . .] As nossas revistas eram pouco lidas lá fora e os nossos sábios não recorriam às revistas mais vulgarizadas dos grandes países para apresentar os resultados das suas investigações. (TEIXEIRA, 1920, p. 6)

[There is nothing more harmful to the science of a people than their isolation in the midst of the science of others. This isolation was almost complete in Portugal in the greater part of the 19th century, and the main reason is the lack of knowledge of our language in foreign scientific environments [. . .] Our journals are little read outside this country and our scholars do not turn to the most popular journals of the large countries to present the results of their research].

These words of GOMES TEIXEIRA show how aware he was of the difficulties of communication between the Portuguese scientific community and the rest of the world owing to the language problem. So, he published his articles in various international mathematics journals, writing in French, Italian, Castilian and English.

GOMES TEIXEIRA was also the founder of the first Portuguese journal of mathematics, the *Jornal de Ciências matemáticas e astronómicas* [Journal of Mathematical and Astronomical Science], which was published from 1877 to 1904. While it was initially a journal aimed at a public covering those who were in some way connected with mathematics or its teaching, it quickly became a journal dedicated to mathematical research, with a view to making known in Portugal the most recent advances in the world. Mathematicians of international reputation, including Hermite, Bellavitis, Birger Hansted, Maurice d'Ocagne, Cesàro, Lerch, Gino Loria, Loriga, and Pirondini, therefore worked with the magazine, contributing research articles. To continue this policy, in 1905, GOMES TEIXEIRA started publishing the journal *Annaes Scientíficos da Academia Polytechnica do Porto* [Scientific Annals of the Porto Polytechnic Academy].

With his view of openness to the world, many articles, which related deep, precise and long research, were published in the most prestigious journals of the time, such as, *Journal für die reine und angewandte Mathematik, gegründet von Crelle; Bulletin des Sciences mathématiques; Journal de Mathématiques pures et appliquées de Liouville; Bulletin de la Société Mathématique de France; Mathesis; Intermédiaire des Mathématiciens; Reale Accademia dei Lincei; Nouvelles Annales de Mathématiques; Acta Mathematica; Jornal de sciencias mathematicas e astronomicas; Annales scientifiques de l'École Normale Supérieure de Paris; L'Enseignement mathématique; Comptes Rendus de l'Académie des Sciences de Paris.*

2 Traité des Courbes Remarquables Planes et Gauches

GOMES TEIXEIRA's work *Traité des Courbes Spéciales Remarquables Planes et Gauches* is included in volumes IV, V and VII of the Complete Works of GOMES TEIXEIRA. In 1907, in volume IV, in the preface to the first two volumes mentioned, the author wrote:

Le présent ouvrage est la traduction, avec plusieurs additions, d'un travail intitulé: *Tratado de las curvas especiales notables*, que l'Académie des Sciences de Madrid nous a fait l'honneur de couronner en 1899 et de publier en langue espagnole plus tard dans le tome XXII de ses *Mémoires*. Ce travail, qui répondait en même temps au programme proposé par cette Académie et à un programme proposé par M. Haton de La Goupillière dans le tome I de l'*Intermédiaire des mathématiciens*, a mérité l'approbation de cet éminent savant, et nous en publions, vivement encouragés par lui-même, cette traduction française. (1908, p. VII)

[This work is the translation, with some additions, of a work entitled: *Tratado de las curvas especiales notables*, which the Madrid Academy of Science did us the honour of awarding a prize in 1899 and later publishing in the Spanish language, in Volume XXII of its *Mémoires*. This work, which at the same time responded to the programme proposed by that Academy and to a programme proposed by M. Haton de la Goupillière in Volume I of L'*Intermédiaire des mathématiciens*, earned the approval of this eminent scholar, and we publish this French translation enthusiastically encouraged by him].

Still in the same preface, GOMES TEIXEIRA explains the structure of volumes IV and V:

. . . nous étudions la forme, la construction, la rectification et la quadrature, les propriétés et l'histoire de chaque courbe; nous considérons les relations de chaque courbe avec les autres; nous indiquons les problèmes où les courbes étudiées apparaissent; etc. Les auteurs de chaque question considérée sont mentionnés, quand cela nous a été possible [. . .] Cet ouvrage sera composé de deux volumes. Le premier est consacré aux courbes algébriques planes, à degré déterminé, de plus grand interêt. Dans l'autre seront envisagées de nombreuses courbes trancendantes planes, quelques classes de courbes planes et quelques courbes gauches plus remarquables. Dans la partie du deuxième volume consacré à l'étude de diverses classes de courbes planes, nous serons naturellement menés à envisager encore quelques courbes algébriques planes à degré déterminé qui en sont des cas particuliers. (1908, p. VII)

[. . . we study the shape, the construction, the rectification and the quadrature, the properties and the history of each curve; we consider the relationships of each curve to the others; we show the problems where the curves studied appear; the authors of each question are mentioned whenever this is possible [. . .] This work will be composed of two volumes. The first is dedicated to the algebraic plane curves, with a determined degree, of most interest. The other will consider the numerous plane transcendent curves, some classes of plane curves and some of the most noticeable skew curves. In the part of the second volume devoted to the various classes of plane curves, we will naturally be led to consider again some algebraic plane curves, with a determined degree, which constitute a particular case].

In this work GOMES TEIXEIRA. studies more than 150 curves, some of which, dealt with in the first volume, are taken up again in the second. Some examples are the Astroid, Cappa's Curve, Talbot's Curve, Watt's Curve, Circular Cubics, Planar Cubics, the Piriform Quartic, Agnesi's witch, etc.

Although the two volumes cited are, alone and together, a noticeable work of scientific and historical gathering, with important personal contributions, the author, in his constant and rigorous research and anxiety for knowledge, went still further and wrote Volume VII, which contains a supplement to Volumes IV and V.

In the preface to Volume VII, the author defines the goals of the work:

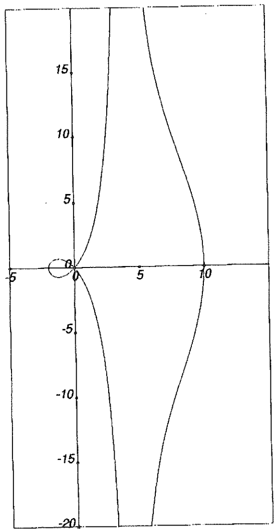
Nous revenons sur quelques-unes des courbes considérées dans cet ouvrage, pour ajouter de nouveaux développements à leur théorie, histoire et bibliographie, et nous exposons les théories de bien d'autres courbes remarquables qui n'ont pas été envisagés dans les volumes précédents. Parmi les pièces qui forment ce volume, on trouve les reproductions de quelques travaux sur les courbes spéciales que nous avons insérés en divers recueils scientifiques après la publication de l'ouvrage mentioné. (1915, p. VII)

[We return to some of the curves considered in this work, to add new developments to their theory, history and bibliography, and we explain the theories of many other noticeable curves which were not considered in preceding volumes. Among the pieces which make up this work are the reproductions of some works on spatial curves which we inserted in various compilations after publication of the work referred to].

There follows a list of works that show that GOMES TEIXEIRA constantly kept up to date with what he was writing about at the time.

But the author went even further, and added an Appendix dedicated to the famous problems of Geometry that cannot be solved with ruler and compass, whose solutions are found through various curves, focusing on the history of those problems which is closely linked to these curves. The author deals with the problems of duplication of the cube, trisecting the angle, and quadrature of the circle, giving a clear and complete exposition of the various solutions and methods related to each problem.

3 Nicomedes' Conchoid



3.1 Gomes Teixeira's approach to the conchoid

GOMES TEIXEIRA starts his study of Nicomedes' conchoid from the polar equation of the curve,

$$\rho = \frac{a}{\cos\theta} + h,$$

and deduces from it the cartesian equation

$$y^2 = \frac{x^2(h+a-x)(h-a+x)}{(x-a)^2}.$$

This, however, might not be adequate to pupils not yet familiar with polar coordinates, and so we present a direct deduction of the cartesian equation, involving only the proportionality of the

sides of similar triangles. In order to simplify some computations, we have introduced a line segment z such that $h^2 = z^2 + (x-a)^2$. In doing so, we were inspired by Isaac Newton's determination of the tangents to the conchoid in his *Treatise of the Method of Fluxions and Infinite Series*, which GOMES TEIXEIRA does not cite (NEWTON 1966, 51). We have interchanged the position of the coordinate axes of Newton's text, though, in order to obtain the equation in a form closer to the one in GOMES TEIXEIRA's treatise.

After establishing the polar and cartesian equations, GOMES TEIXEIRA proceeds with a detailed study of the conchoid, covering topics such as the points at infinity, normals, curvature and the points of inflexion, quadrature and volumes of solids of revolution generated by the curve. He then turns to the history of the conchoid, giving references that go from Antiquity up to Johann Bernoulli and Roger Cotes. Finally, he gives a detailed account of the use of the conchoid for the *neusis* construction of the solutions of the famous problems of the *angle trisection* and of the *cube duplication*.

3.2 The methodology for the workshop

In this workshop we rely extensively on texts that are not refered to by GOMES TEIXEIRA, but all the classroom material presented in the workshop is the result of the study of his *Traité des Courbes Spéciales Remarquables Planes et Gauches*. It was the reading of this treatise that inspired us not only to follow his clues and check his references, but also to look for other historical sources which he does not mention.

In this workshop, an analysis of the problem always precedes the corresponding synthesis. The former is sometimes provided (much in the way it could be done on the blackboard by the teacher) and at other times is asked for in a work-sheet; the latter is either included in a historical text of mathematics (the reading of which may be proposed to the class), or takes the form of a work-sheet, or both. By means of these few combinations we hope to call the teachers' attention to the great variety of possible strategies for the use of historical problems from Antiquity in the teaching of mathematics.

Although our source of inspiration for this part of the workshop was the reading of GOMES TEIXEIRA's treatment of the conchoid, our option was not to maintain the lettering of his diagrams. It was important to unify the notation in the related documents, and we have tried to keep closer to the original historical texts.

We deal with the trisection of the angle first, for the related constructions are much simpler than those of the cube duplication. This leads to the concept of *neusis* constructions and to the manipulation of a model of Nicomedes's trammel for the drawing of conchoids. We then take a look at the duplication of the cube.

3.3 The trisection of the angle

We start with a construction for the angle trisector presented by Pappus of Alexandria as proposition 32 of the fourth book of the *Mathematical Collection*. We introduce the construction by an analysis meant to prepare for the synthetic proof that follows; it is essentially the analysis in Heath (1981).

The synthesis that corresponds to this analysis is presented in two ways. The first one consists of the reading of an excerpt of Pappus' *Mathematical Collection*, where the synthetic treatment is provided.[1] An alternative to the reading of Pappus' text is the presentation of a work-sheet with the same synthesis.

3.4 Neusis constructions and the conchoid

The construction given by Pappus presuposes the insertion of a line segment of given length between two lines, in such a way that a given point lay on the segment produced. This is called *neusis* constructions.

An interesting question is whether such a construction is always reducible to a construction with ruler and compass, or not. The issue is not the object of any classroom material presented in this workshop, but the following short comment may be appropriate. The first known historical instance of a neusis construction dates from the fifth century B.C. (the construction of the third lunule of Hippocrates of Chios) and may indeed be carried out with ruler and compass alone, for it is the equivalent of applying a rectangle with a given area to a given line segment and exceeding by a square[2], which is a particular case of Euclid's.[3] In general, however, neusis constructions are not reduceable to ruler and compass constructions. Pappus proves that they may always be carried out through the intersection of a circle and a hyperbola (proposition 31 of the *Mathematical Collection* **IV**).

The extremities of the sought line segment are not always necessarily on two straight lines; they may be required to lie on two given curves. A neusis between a line and a circle is used by Archimedes to trisect the angle (proposition 8 of the *Book of Lemmas*). Besides, there are still other uses of the word *neusis* to describe more complex geometric constructions, in the ancient mathematical literature: usually the determination of a line segment, with a given point on its extension, and with a certain geometrical property other than having a given length. A famous instance may be found in Archimedes's construction of the regular heptagon.[4] Another one occurs in the Heronian construction of the two mean proportionals, mentioned below.

The simpler sort of neusis constructions (as defined at the beginning of this section) is suggestive of how the conchoid may have been discovered by ancient geometers and also of how it may be conceived by nowadays learners. Several ancient commentators relate Nicomedes (third and second centuries B.C.) to the invention of the conchoid. The most important of them are Pappus of Alexandria, Proclus of Lycia and Eutocius of Ascalon.

In book IV of the *Mathematical Collection*, Pappus gives the definition of the conchoid and states, without proof, some of its properties established by Nicomedes. Next, he explains how the conchoid may be used to effect a neusis construction (of the sort in which the line nearest to the given point is a *straight* line). Finally, Pappus presents Nicomedes' solution for the problem of the cube duplication.[5]

In his comment to Euclid's proposition concerning the bisection of a given rectilinear angle[6], Proclus links the discovery of the conchoid to the problem of the angle trisection.[7]

[1]VER EECKE 1982, **1**, 212-214.
[2]HEATH 1981, 1, 195-196
[3]*Elements* VI, 29.
[4]DIJKSTERHUIS 1987, 414-416.
[5]VER EECKE 1982, **1**, 185-190.
[6]*Elements* I, 9
[7]VER EECKE 1948, 233-234.

Eutocius collected an impressive amount of solutions of the Delian problem (the duplication of the cube, or its equivalent form of the finding of two mean proportionals to a line segment and its double) in his *Commentaries to Archimedes' treatise On the Sphere and the Cylinder*. Concerning the conchoid, he conveys information much on the line of Pappus, but often in greater detail. He tells about a book of Nicomedes called *On conchoid Lines*, in which the author defined the curve and described a trammel to draw it; Eutocius gives Nicomedes' deduction of two properties of the conchoid (those mentioned without proof by Pappus) and, like Pappus, presents the neusis construction by means of which Nicomedes duplicated the cube.[8]

Nicomedes' work *On conchoid Lines* is now lost, but both book IV of Pappus' *Mathematical Collection* and Eutocius' *Commentaries to Archimedes* may provide very interesting excerpts to be used in class, with the definition of the conchoid. We also use a model of the trammel described by Eutocius.[9] Exercises may be proposed to the effect of understanding how Nicomedes' curve is instrumental in obtaining the neusis construction implied in the angle trisection studied before. A word of caution may be appropriate, though: each angle amplitude requires a different conchoid. One may choose either the distance from the pole to the basis or the fixed length inserted between the basis and the curve according to the particular trammel at one's disposal, but then the other one will turn out too long or too short, except for one particular amplitude of the angle to be trisected. In other words, the given problem may easily adjust to one of the two parameters of a conchoid, but not to both.

3.5 The duplication of the cube

Before seeing how Nicomedes is reported by Pappus and by Eutocius to have used the conchoid in order to solve the problem of cube duplication, it is perhaps advisable to make two easy technical remarks. The first one concerns the equivalence of this problem to the one of finding two mean proportionals between a given line segment and its double, a fact accounted for in most history books on ancient mathematics, for example Heath (1981).

The second remark concerns proposition II, 6 of Euclid's *Elements*, which is used a few times in the rest of the workshop and should be reminded as a preliminary result.[10]

W.R. Knorr has shown that Nicomedes' procedure for the cube duplication transmitted by Pappus and Eutocius is an elaboration (and a simplification) of the solution given in the *Mechanics* and the *Belopoeica* of Heron of Alexandria; an analysis based on Heron's solution leads in a straightforward way to the synthesis attributed to Nicomedes by Pappus and Eutocius.[11] The classroom material presented in the workshop is directly inspired by Knorr's historical reconstruction.

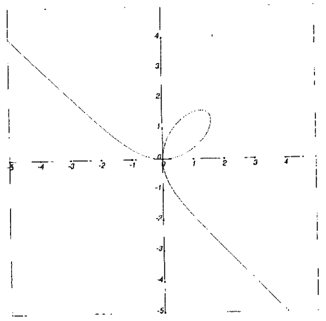
We start with an analysis for Heron's duplication of the cube. Once the pupils have been guided through such an analysis, we expect them to be able to reverse the argument and thus provide the synthesis. As before, however, we propose an alternative which we believe to be instructive: the reading of a historical excerpt that contains the synthesis, the interpretation of which being made simpler by the previous activity. This text is based on Eutocius' transcription of Heron's *Mechanics* and *Belopoeica*, in his commentaries to Archimedes[12]; however, we have reversed

[8]VER EECKE 1960, 2, 615-620.
[9]VER EECKE 1960, **2**, 615.
[10]HEATH 1956, **1**, 385.
[11]KNORR 1986, 225-226.
[12]VER EECKE 1960, **2**, 590-592

the diagram and changed the lettering of Eutocius' original, so that it conforms to the discussion above and to the next task on Nicomedes' method of cube duplication.

The construction involved in the Heronean solution is in itself a non-trivial problem. The best approach is by means of an analysis suggestive of the Nicomedean solution for the problem of the duplication of the cube. Once again, we provide the synthesis by means of an excerpt of Eutocius' *Commentaries* to Archimedes.[13]

4 Descartes' Folium



4.1 Forward

In this concise presentation of Descartes' folium, I must say that our main source is the text of GOMES TEIXEIRA on this curve, taken from the *Traité des Courbes Spéciales Remarquables Planes et Gauches*.[14]

All his references are given with accuracy as we could check, and some other references we give are related with the GOMES TEIXEIRA text.

Besides, we do not refer to all the points studied by GOMES TEIXEIRA, but only to some of them.

4.2 Definition and graphic; analytic equation

In 1637, the same year when *La Géometrie* of Descartes was firstly published, Fermat wrote a manuscript *Methodus ad Disquirendam Maximam et Minimam* (Method of finding Maxima and Minima). (Méthodes pour trouver les Maxima et les Minima) (FERMAT, Œuvres, 3).

It was as an appendix to that method that he gave his own method of finding tangents. Explicitly he wrote: "La théorie des tangentes est une suite de la méthode [. . .]"
(The theory of tangents is a continuation of the method [. . .]).

Apparently, Descartes did not interpret it well and he thought that it could not be universally applied, as Fermat claimed. So, Descartes, as a challenge to Fermat, in a letter to Mersenne of January 1638[15], wrote that Fermat's method could not be applied to the curve that he defined

[13] VER EECKE 1960, **2**, 618-620.
[14] TEIXEIRA, Obras, IV, 85-91.
[15] DESCARTES, Œuvres, I, 490

as:

> La ligne courbe BDN que je suppose être telle, qu'en quelque lieu de sa circonférence qu'on prenne le point B, ayant tiré la perpendiculaire BC, les deux cubes des deux lignes BC & CD soient ensemble esgaux au parallelepipede des deux mêsmes lignes BC&CD& de la ligne donnée P. [. . .]
> Car elle [la méthode de Fermat] ne se peut appliquer, ny a cet exemple, ny aux autres qui sont plus difficiles [. . .]
>
> (The curve line BDN that I assume such that, taking any point B and taking the perpendicular BC, the sum of the two cubes of the lines BC&CD equals the parallelipiped on BC&CD& other given line P. Because it is not possible to apply Fermat's method not only to this but also to others still more difficult).

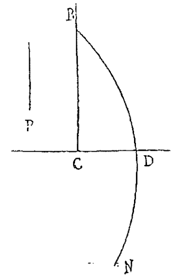


Fig.III.1

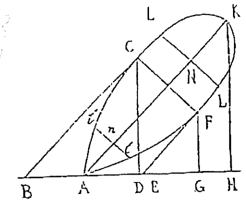


Fig.III.2

The drawing annexed by Descartes to his definition is, obsviously, not correct (Fig. III.1).

The first to consider the form of the curve was Roberval, that gave it the name of "galand" or "nœud de ruban", ("ribbon knot"). He also called it "fleur de jasmin", ("jasmine flower"). However in another letter to Mersenne, dated 23rd August 1638,[16] Descartes gives the correct drawing of the part of the curve situated in the first quadrant (Fig.III.2). We may wonder the reasons for this incompleted graphic. Descartes also writes the analytic equation of the folium, assuming that n is the length of the given line $P : x^3 + y^3 = xyn$.

According to GOMES TEIXEIRA, the first to give the correct drawing of the folium was Huygens[17], in 1692, (Fig.III.3), which we may compare with the modern graphic that can be easily drawn.

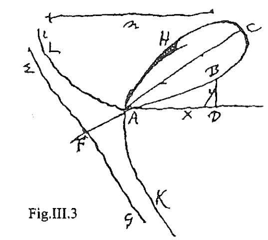


Fig.III.3

[16] DESCARTES, Œuvres, II, 313
[17] Œuvres, X, 351-352

4.3 The tangents

According to GOMES TEIXEIRA, the tangents to the folium, with an inclination of 45° on the x-axis were firstly determined by Descartes himself, in the same letter of 23$^{\text{rd}}$ August 1638. Afterwards, they were also determined by Fermat in a letter to Mersenne of 22$^{\text{nd}}$ October 1638[18], Sluse in a letter to Huygens in 1662[19] and by Barrow in his *Lectiones geometricae*, lesson X.

All the references given so accurately by GOMES TEIXEIRA challenge us to follow them, reading the original texts and learning how to find the tangents to the folium, specially according to the methods of Descartes and Fermat. We can also check the results, using the tools of the Infinitesimal Calculus and try to comment the statement of Morris Kline: Descartes' method was purely algebraic and did not involve any concept of limit, whereas Fermat's did, if rigorously formulated.[20]

4.4 The Quadrature

In his historical notes on the folium, GOMES TEIXEIRA states that the first to determine the areas of the folium was C. Huygens in a letter to L'Hospital, in 1692.[21] L'Hospital himself also solved the same problem in two letters to Huygens in 1693. Finally, the problem was also solved by Jean Bernoulli, who also determined the complete graphic of the folium in his *Lecciones mathematicæ*.[22]

GOMES TEIXEIRA determined these areas, through the known formulas of the Infinitesimal Calculus, as we are now going to show. He assumes the equation of the folium to be

$$x^3 - 3axy + y^3 = 0. \qquad (1)$$

He starts to remark the symmetry of the curve, in relation to the bisector of the first quadrant. Therefore, GOMES TEIXEIRA takes this axis, as the new x-axis, which corresponds to a rotation of 45° of the plane around the origin. He gives the formulas to do the rotation

$$x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}};\ y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$$

and obtains the new equation of the folium

$$Y^2 = \frac{X^2(3a - \sqrt{2}X)}{3(a + \sqrt{2}X)}. \qquad (2)$$

GOMES TEIXEIRA believes that Descartes was the first to have done this kind of transformation in the same letter of the 23$^{\text{rd}}$ August 1638. It is worthwhile following his references and making an incursion in Descartes' text. He was refering to the pleasure of Roberval in the consideration of the curve and added: [the segments mentioned in the text are the ones shown in Fig.III.2].

> [. . .] ie luy en veux ici donner une autre qui ne merite pas moins que celle-la les mesmes noms, & qui est beaucoup plus aisée a descrire, en ce que l'invention de tous ses poins ne depend d'aucune equation cubique. Celle cy donc est telle, qu'ayant pris AK pour l'aissieu de l'une de ses feuilles, & en AK le point N a discretion, il faut seulement faire que le carré de l'ordonnée LN soit au carré du segment AN comme l'autre segment NK est a l'aggregat de la toute AK& du triple d'NA& ainsy on aura le point L, c'est a dire tous ceux de la courbe, puisque le point N se prend a discretion.
>
> ([. . .] I want to give him another curve, that does not deserve less the same names, but that is much easier to describe, since its points do not depend on a cubic equation. This curve is such that, taking AK as an axis of one of the leafs and on AK any point N, it is enough to state that the square of the ordinate LN is to the square of the segment AN as the other segment NK is to AK added with the triple of AN and in this way we will obtain the point L, that is, all the points of the curve, since N is any point on AK.

[18]FERMAT, Œuvres, II, 169

[19]HUYGENS, Œuvres, IV, 246

[20]KLINE, 345.

[21]Œuvres, x, 351 and 374.

[22]Opera, III, 403.

We can translate this rethoric description in the proportion:

$$\frac{LN^2}{AN^2} = \frac{NK}{AK + 3AN}$$

that corresponds exactly to the equation (2), found by GOMES TEIXEIRA. Besides, this little extract of Descartes' letter makes us think about the meaning of some of his expressions, namely: "does not deserve less the same names", "one of the leafs", "its points do not depend on a cubic equation".

Afterwards, GOMES TEIXEIRA obtains the corresponding graphic, after the rotation (Fig.III.4).

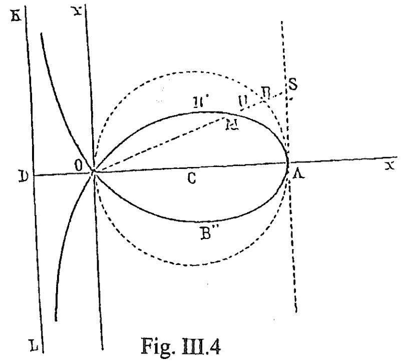


Fig. III.4

This allows him to obtain easily the equation of the asymptote $X = -\frac{1}{2}a\sqrt{2}$, the coordinates of the vertex $A\left(\frac{3}{2}a\sqrt{2}, 0\right)$, the tangents parallel to OA, passing through the points B and B', the area limited by the "leaf" and also the area limited by the curve and its asymptote.

GOMES TEIXEIRA applies the usual method of the Integral Calculus, as was already said. In order to determine the area of the "leaf" he uses the formula:

$$A = 2\int_{x_0}^{x_1} X\sqrt{\frac{3a - \sqrt{2}X}{a + \sqrt{2}X}}\,dX$$

where $X_0 = 0$ and $X_1 = \frac{3}{2}a\sqrt{2}$.

Next, GOMES TEIXEIRA makes the change of variables $\frac{3a-\sqrt{2}X}{a+\sqrt{2}X} = z^2$ obtaining, by integration

$$A = -\frac{8a^2}{\sqrt{3}}\left[\frac{z_1^3}{(1+z_1^2)^2} - \frac{z_0^3}{(1+z_0^2)^2}\right],$$

where $z_0 = \sqrt{3}$ and $z_1 = 0$. Finally he obtains $A = \frac{3}{2}a^2$.

He also obtains the same value for the area between the folium and its asymptote. So, GOMES TEIXEIRA concludes that the folium is a quarrable cubic, in the algebraic sense.

Again, his results challenge us to draw the square with the same area of the "leaf", using only a staight-edge and a compass.

Afterwards, GOMES TEIXEIRA clearly says that the fact of the folium being a quarrable cubic suggested him to look for all the cubics satisfying this condition. He referred to the results already obtained by Maximilien Marie, the first to consider the problem and claimed that those results were incompleted.

He writes that his basis for that work was the book of Newton *Enumeratio Linearum tertii ordinis* and he claims that his own research was successfully completed.

4.5 Parametric Equations

According to the definition of M. Kline[23], a curve is said unicursal when it has the maximum number of double points, which for a curve of degree n is $\frac{(n-1)(n-2)}{2}$. As Kline says,this definition was firstly given by Colin MacLaurin in his *Geometria Organica*, in 1720.

It is assuming that the folium is an unicursal curve that GOMES TEIXEIRA determines his parametric equations. He makes $y = tx$, which represents a line through the double point. In fact, that line cuts the curve in just another more point and so, it is possible to obtain x and y as rational functions of t. Consequently, he obtains:

$$x = \frac{3at}{1+t^3} \quad y = \frac{3at^2}{1+t^3}.$$

Although this not mentioned by GOMES TEIXEIRA, the parametric equations of the folium, allow us to determine the area of the "leaf" by an easier calculation.

In fact, through the formulas of the Infinitesimal Calculus, the area of leaf is given by

$$A = \frac{1}{2}\int_{t_0}^{t_1} x^2 dt.$$

In this case, we easily obtain:

$$A = \frac{1}{2}\int_{0}^{+\infty} \frac{9a^2t^2}{(1+t^3)^2} dt = \frac{3}{2}a^2.$$

This result leads us once more to formulate some questions and to do the effective calculations.

We can conclude that the GOMES TEIXEIRA text is as rich as stimulating and that, by its accuracy, it constitutes an excellent second source for the authors referred to. In addition, it is the essencial primary source for the GOMES TEIXEIRA's innovations and for his work of a very talented historian of Mathematics.

[23] KLINE, 2, p. 552

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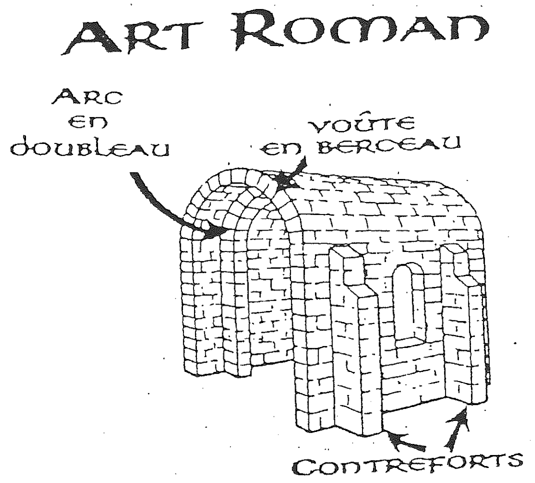

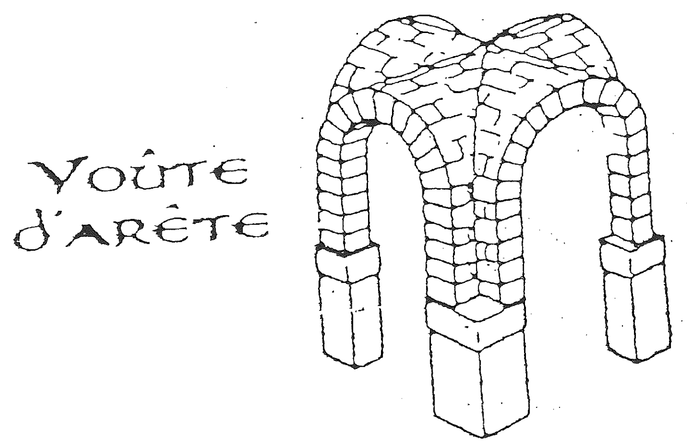

De la Math. Sup. à la classe de 5ème,
en passant par la lecture de textes d'Archimède....

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Abstract

L'idée de cet atelier est née un lundi matin, lors d'une de nos discussions dans le groupe M : A.T.H. à l'I.R.E.M. de PARIS VII. Je parlais d'un problème posé par Archimède à son ami Eratosthène et qui est aussi un exercice d'oral de l'Ecole des Arts et Métiers : il concerne le calcul du volume de l'intersection de deux cylindres de même rayon, d'axes orthogonaux et concourants. Maryvonne a été très intéressée car elle rentrait d'une visite - entre autres - du château de Falaise avec ses élèves de 5ème; c'est là qu'ils avaient appris que les premières croisées de transept ou voûtes d'arête, étaient des intersections de voûtes en berceau orthogonales, donc de demi-cylindres orthogonaux; elle voulait leur en proposer une construction.