About Mathematics in University Textbooks of Economics

ZERNER Martin
REHSEIS, Paris (France)

Abstract

In the specialization "economical and social studies" of the French baccalauréat (final high school degree) the mathematics curriculum has been replaced by "mathematics applied to economics and the social sciences". Problems which pretend to include economic applications will be analyzed.

In a text on economics, we will discuss how the use of mathematics masks the introduction of new economic hypotheses through the combined effects of the importance and the weakness of the mathematical reasoning.

There will be no prerequisit in economics and participants can choose between the two topics. This program may be slightly modified if I find new documents inbetween.

<u>Note</u>

No knowledge in economics is necessary to understand what follows.

Our project

As is well known, mainstream economics relies heavily on mathematics. However, students in economics are usually not so good at math and that is a problem for teaching. The usual way out is a heavy use of graphical representations. We want to analyze how it works in a widely used textbook, Barro's *Macroeconomics* (BARRO, 1984). This book has a short section with the title "A note on mathematics and economic reasoning" which runs thus :

> This book does not use any advanced mathematics. Rather, it relies on graphical methods and occasional algebraic derivations. Although calculus[1] would speed up the presentation in some places, it is unnecessary for the main economic arguments. Students should therefore not find the book difficult on technical grounds.
>
> What will be demanding from time to time is the economic reasoning. It is this aspect of economics that is the most difficult - as well as the most rewarding. Unfortunately, not all of this difficulty can be avoided if we wish to understand the economic events which occur in the real world. The feature that should help students to master the material is the use of a single, consistent model, which is then successively refined and applied to a variety of macroeconomic problems. Anyone who invests enough effort to understand the basic model will eventually see the simplicity of the approach, as well as its applicability to a wide variety of real world issues. Conversely, anyone who fails to master the basic model will be in serious trouble later on. (p.25-26)[2]

Our questions are how the mathematics are modified in this process and whether this modification has consequences for the economics. The first of these questions has been dealt with by Michèle Artaud (ARTAUD 1993, 1995) on the basis of another case study, namely a textbook of financial theory (COPELAND & WESTON, 1988). On the whole, her conclusions are consistent with ours. Our method will be to translate as closely as possible into rigorous mathematical terms.

It may be useful to say a word about research papers in economics. Of course, they use algebra and calculus. But they also do use graphical methods. An example is DORNBUSCH (1983) who studies the problem of the Third World debt[3]. This is a very real problem, but Dornbusch's developing country is very unreal. The article appeared shortly before the first edition of Barro's textbook and contains the very same kind of reasoning and figures as those which you can find in it.

Introducing Barro's *Macroeconomics*

This book is a success. It has had four editions (more if I am not up to date) and a French translation[4]. I have checked that it is actually in use in French universities. It is very carefully written and has many assets in its presentation (summaries, a glossary, etc). The fourth edition on which the present text is based has substantial improvements with respect to the preceding ones. The author is professor at Harvard and has been an advisor to several government and international agencies.

[1]Rappel pour les lecteurs francophones: *calculus* signifie calcul différentiel et intégral.

[2]References, when not otherwise stated, are to the 4th (1993) edition of BARRO (1984).

[3]I have explained that article in a summer school on didactics of mathematics (ZERNER, 1996).

[4]Armand Colin, Paris, 1987. La traduction a été faite sur une édition antérieure (très probablement la première) et moins bonne que celle qui est utilisée ici. Il y a aussi une certaine adaptation au contexte français, pas toujours heureuse.

A introductory chapter explains what macroeconomics is about and gives indications on the main macroeconomics indices : gross national product (GNP)[5], unemployment rate, etc. Here are the titles of the following parts :

I Microeconomic foundations and the basic market-clearing model
II Inflation
III Business fluctuations, unemployment and economic growth
IV Government behavior
V The international economy
VI Interactions between the monetary sector and the real sector.

Part I is of course devoted to the basic model (except for a final chapter on the labour[6] market). As we want to start from scratch, we'll concentrate on the beginning of the first chapter of that part : "Work effort, production, and consumption - the economics of Robinson Crusoe".

Basic principles

Mainstream economics is called neoclassical. Here we will pretend to believe in its starting postulates, whatever we think of them. It starts from a few lines from the classical economist Adam Smith (1723-1790) who stated that if every individual pursues his own profit, the invisible hand of the market will see to it that the outcome be the best possible for everyone. "The invisible hand of the market" is a phrase which you hear very often in discussions among economists. One of the conclusions they draw from it is that economic theory has to start from the study of the economic behaviour of individuals (methodological individualism).

That is why they are interested in Robinson Crusoe. To put it in BARRO's words:

> In any[7] economic analysis, the determination of work effort, production, and consumption depends on opportunities for production and on preferences about working and consuming. This basic interaction between opportunities and preferences shows up even in the simplest possible economy, which consists of isolated individuals, each of whom resembles Robinson Crusoe. (p.41)

As Robinson Crusoe has to grant for the basic model of every subfield of economics, each economist has his own Robinson. To the Robinson of Copeland and Weston, it is very important to be able to choose how much of his crop he will save for sowing. Not so with Barro whose Robinson cannot save anything and must consume immediately all that which he produces (and of course no more). The only choice left to him is how much to work. In economic terms he is a "basic economic unit which we think of as a combination of a household and a firm. [. . .] For most purposes this abstraction will be satisfactory because some households ultimately own the public businesses." (p. 41-42) This basic unit is called a household in the book. In the usual economic terminology, to be distinguished from everyday language, a household is any unit of consumption.

Another simplification is to suppose an economy with only one product. When it comes to comparison with statistical data, the amount of this product is identified with the GNP.

[5]En français: produit national brut (PNB).

[6]Except for mistakes, my own spelling is British.

[7]Incidentally, this implies that there is no economic analysis in Marx's works!

Robinson Crusoe at work

The more Robinson works, the more product he gets.

> Formally, the quantity of a household's commodity output per period, denoted by y, is a function of the quantity of labor input l. We write this relation as :
>
> $$y_t = f(l_t) \qquad (2.1)$$
>
> where f is the household's production function[8], which specifies the relation between the amount of work and the quantity of goods produced. (p.42)

Obviously, this function must be increasing. Moreover:

> The extra output produced by one more unit of work is called the **marginal (physical) product of labor,** henceforth designated **MPL**. We assume **diminishing marginal productivity,** which means that each successive unit of work generates progressively smaller, but still positive, responses of output. (p. 43)

In mathematical terms, this means that f, the production function, is concave. Notice that there is no justification of diminishing marginal productivity, an essential and most questionable assumption of neoclassical economics. It may be assumed that the students have had a course on microeconomics in which it has been discussed.

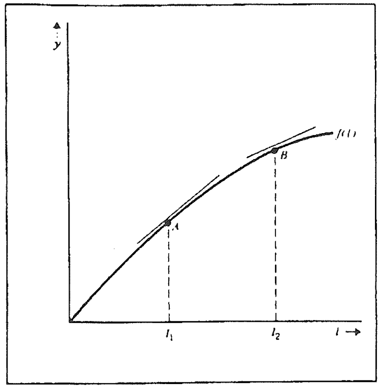

FIGURE 2.1 *Graph of Production Function*
The curve shows the level of output as a function of the quantity of labor input. At point A, the slope of the tangent straight line equals the marginal product of labor when $l = l_1$. The same is true for point B when $l = l_2$.

FIGURE 1 (from BARRO, 1984, p. 42)

> Figure 2.1, which is the graphical representation of equation (2.1), shows the relation of output to the quantity of labor input. [...] The positive slope of the curve (that is, of a straight line tangent to the curve) at any point indicates the additional output that results from extra labor input, which is the marginal product of labor. [...]
>
> The shape of the production function in Figure 2.1 implies that the slope becomes less steep as work effort increases. This property reflects the diminishing marginal productivity of labor. (p. 43)

[8] Actually all households are supposed to have the same production function throughout the book.

Through the use of the slope, which will be systematic, a new analytic property has been imposed on the production function: it must be differentiable. This is again questionable. Suppose the household is a French firm which still abides by labour regulations[9]. It has to pay 25% more for every hour of work in excess of 39 per worker and per week, which creates an angular point in its production function. A (weak) counter-argument might be that various angular points from different economic units average out. As a very important parenthesis, note that this example shows how utterly irrelevant the basic model is when it comes to real situations in which workers and bosses are implied. Which Robinson chooses the level of work effort? But here we pretend to believe in the basic principles of neoclassical economics.

Utility

It has already been said that Robinson consumes exactly what he produces. As a consequence :

> Then in the world of Robinson Crusoe we have:
>
> $$c_t = y_t = f(l_t) \qquad (2.2)$$
>
> where c_t is the amount of consumption in physical units. (p. 44-45)

Here a key notion steps in :

> Consumption in each period is a source of happiness or **utility** for households. (Henceforth, we use *utility*, the economist's standard jargon.) (p. 45)

Utility has now to be formalized and made graphic.

> Households have a fixed amount of time in each period, which they can divide between work and leisure. [...] We assume that leisure is intrinsically more enjoyable than time at work. In other words, leisure is a source of utility for households.
>
> Suppose that we can define a function to measure the amount of utility that derives each period from consumption and leisure. The form of this **utility function** is
>
> $$u_t = \underset{(+)(-)}{u(c_t\,,\,l_t)} \qquad (2.3)$$
>
> where u_t is the amount of utility (in units of happiness, which are sometimes called *utils*[10]) that someone obtains for period t. We assume that the form of the utility function, u, is the same for all periods. The positive sign under the quantity of consumption, c_t, indicates that utility rises with consumption. The negative sign under work effort, l_t, signifies the negative effect on utility of more work (that is, of less leisure).[11] (p. 46)

These properties are not enough for what is to follow. Above all, utility has to be made graphic. A slightly more sophisticated presentation going back to Pareto (1848-1923) gets rid of the utility function entirely, using only the indifference curves to be defined presently.

[9] As they are in November 1999.

[10] The util is never defined and never used.

[11] Though this is not a standard mathematical notation, it is legitimate, convenient, and often used by economists.

A basic assumption is that the utility gained from an extra unit of leisure, relative to that from an extra unit of consumption, diminishes as the ratio[12] of leisure to consumption rises. In other words, if someone has a lot of leisure but relatively little consumption, he or she is more concerned with adding to consumption rather than to leisure. Consider the amount of extra consumption needed to comprensate for the loss of a unit of leisure time. If a person starts with little consumption and a lot of leisure, then it is important to add to consumption. Therefore, he or she is willing to work a lot more to get additional consumption. If the person is already working quite a bit and has a high level of consumption, then leisure becomes more significant. Therefore, he or she is lesswilling to work more and give up leisure to obtain extra consumption.

The curve in Figure 2.5 summarizes this discussion. At zero work effort, $l = 0$, the curve specifies a level of consumption, c^0, on the vertical axis. This amount of consumption, together with full time leisure ($l = 0$), determines some level of utility from equation (2.3). Denote this level of utility by u^1. The curve shown in the figure connects this initial point to all other possible combinations of work and consumption to provide the same level of utility u^1.

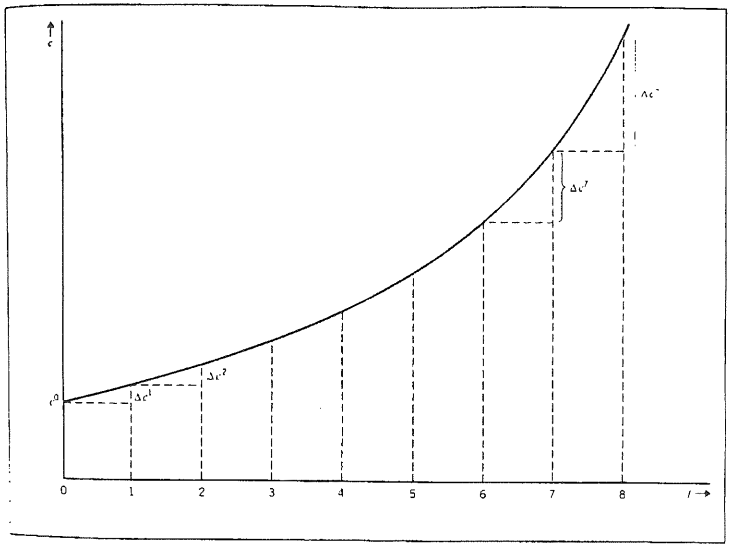

FIGURE 2.5 *An Indifference Curve for Work and Consumption*
All points (ℓ,c) on the curve yield the same level of utility, u¹. Hence, the household is indifferent among these pairs of work effort and consumption.

FIGURE 2 (from BARRO, 1984, p. 47)

Suppose the person works a positive amount, so that leisure becomes less than a full time activity. Assume that work is one hour per day, represented by $l = 1$ in Figure 2.5. By itself, this reduction

[12]The word "ratio" is used here in a loose and non technical sense. This is not important. M.Z.

in leisure lowers utility. But we want to know how much additional consumption would restore the original level of utility. Denote by Δc^1 the required amount of extra consumption. Then the new combination of work and consumption, where $l = 1$ and $c = c^0 + \Delta c^1$, yields the same utility as the initial pair, where $l = 0$ and $c = c^0$. Hence, the person is indifferent between the two pairs of work and consumption. We show that these two points yield the same level of utility by connecting them by the curve shown in the figure.

If the person works another hour, that is, $l = 2$, then some additional consumption is again needed to maintain the level of utility. Figure 2.5 assumes that the required extra consumption is the amount δc^2. Therefore, the point where $l = 2$ and $c = c^0 + \Delta c^1 + \Delta c^2$ again provides the same utility as the initial pair, where $l = 0$ and $c = c^0$.

We can continue this exercise as the amount of work rises. The result is the curve in Figure 2.5, which shows all pairs (l, c) that yield the same level of utility. Since people are indifferent among these pairs of work and consumption, the curve is called an **indifference curve**.

The previous discussion tells us something about the shape of an indifference curve. As someone works more, each additional unit of work requires a greater amount of consumption to maintain utility. Therefore, the size of each addition to consumption, Δc, is larger the higher the associated number of work hours. Note in particular that $\Delta c^1 < \Delta c^2 < \ldots < \Delta c^7 < \Delta c^8$ in Figure 2.5.

At any point along the indifference curve, the slope of a tangent straight line indicates the increment in consumption that a person requires to make up for the loss of a unit of leisure. Each of the additions to consumption, Δc, that appears in Figure 2.5 approximates this slope in the vicinity of the corresponding level of work. For example, the amount Δc^2 is a good measure of the slope when the level of work lies between one and two hours per day. The previous results imply that the slope of the indifference curve rises as the amount of work, l, increases.

This long quotation shows us the utter care which BARRO brings to his economico-mathematico-graphical explanations (with apologies for the long word). We extract from it the analytic properties of an indifference curve: it is the graph of an increasing, differentiable, and convex function of l.

These are the properties of one indifference curve. BARRO goes on to explain briefly what happens when one goes from one indifference curve to another one. Obviously, these curves do not intersect, and the higher the curve, the higher the constant level of utility on it. The fact that there is one such curve through each point of the positive quarter of the plane is implicit.

Robinson's optimization

We are now in a position to decide with Robinson how much he will work. We have skipped above several mentions of how this is done, namely by finding the balance between work and consumption which yields the highest possible level of utility. Beware that Robinson is a very wise man : he knows his production function and his utility function, so that in principle he is left with a purely mathematical problem. Here is the way BARRO explains the solution.

Suppose that a household begins from a particuliar combination of work and consumption (l, c). We can consult Figure 2.6 to find the indifference curve to which the point corresponds. The slope of the indifference curve at this point indicates how much extra consumption, Δc, someone insists on to work an additional unit of time. To determine how much one actually works, we combine the indifference curves with a description of people's opprtunities for raising consumption when work effort rises. In the model, these opportunities come from the production function, which appears in Figure 2.1. The marginal product of labor, MPL, is the amount of extra output generated by an extra unit of work. Further, we know from equation (2.2) that each addition to output corresponds to an equal addition to consumption.

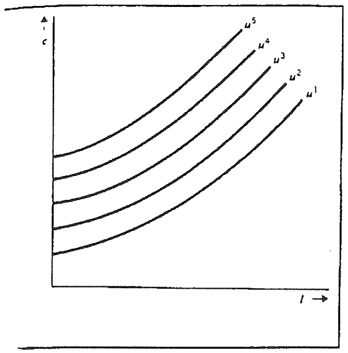

FIGURE 2.6 *A Family of Indifference Curves for Work and Consumption*
The level of utility rises as the household moves from the curve labeled u^1 to that labeled u^2, and so on.

FIGURE 3 (from BARRO, 1984, p. 49)

The MPL is the addition to production, and therefore to consumption, that results from an extra unit of work. The slope of the indifference curve is the amount of extra consumption that a person needs to make up for less leisure. Therefore, if the MPL exceeds the slope of the indifference curve, then the person will be better off if he or she works more and uses the added output to expand consumption. However,as work rises, the MPL declines, and the slope of the indifference curve rises. Therefore, the increase in work lessens the initial excess of the MPL over the slope of the indifference curve. When the gap vanishes, that is, when the marginal product equals the slope of the indifference curve, it no longer pays to work more.

[. . .]

To summarize, each household chooses the combination of work and consumption that maximizes utility. Therefore, the household selects the pair (l^*, c^*) at which the production function is tangent to an indifference curve. (p. 49-50)

The part which we have skipped describes the optimization process in more graphical terms on the basis of the figure reproduced below.

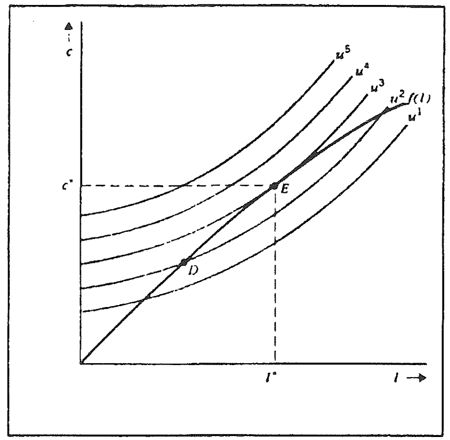

FIGURE 2.7 *Combining the Indifference Curves with the Production Function*
The household moves along the production function : to reach the highest possible indifference curve. This occurs at point E. where the production function is tangent to indifference curve u^3.

FIGURE 4 (from BARRO, 1984, p. 50)

We need some notations to analyze the mathematics involved. The partial derivatives of a function g of l and c will be denoted by g_l and g_c. The key mathematical object is the slope $p(l, c)$ of the indifference curve at the point (l, c). From it, we can recover the indifference curves by Cauchy's theorem on differential equations. Let us assume it is differentiable as well as the utility function.

By the definition of the indifference curves, we have :

$$u_l + pu_c = 0 \qquad (a)$$

The convexity of the indifference curve can be translated into the inequality :

$$p_l + pp_c > 0 \qquad (b)$$

Let us first examine BARRO's assertion that, as long as as the slope of the production function exceeds the slope of the indifference curve, the slope of the indifference curve increases with increasing work time along the graph of the production function. This would imply that the derivative along the graph, namely $p_l + f'p_c$, be non negative. Nothing of the above grants us that. Indeed, look at the following figure.

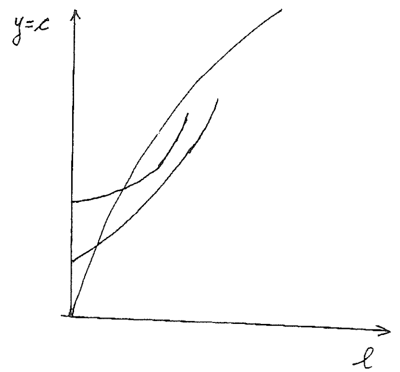

FIGURE 5

It is left as an exercise for the reader to find functions that fit it. (Hint: it is enough to find p). However, this objection vanishes if the slope of the indifference curve is an increasing function of c when l is kept constant[13]. Anyone who accepts the indifference curves as a legitimate economic concept and thinks that they are convex will probably be willing to grant this additional assumption.

So we can be convinced that, starting from a point were the MPL is lower than the slope of the indifference curve and increasing work the gap will diminish. Can it be proved, with all the preceding hypotheses, that it will vanish at some point? The answer is a square no. If you doubt it, just try and when you tired of it, check on the following counter-example.

$$f(l) = \sqrt{2l + l^2}.$$

We need not to know explicitly u, it is enough to have the indifference curves. Take for their equation :

$$c = r + \frac{1}{\pi}(\frac{\pi}{2} + \text{Arc tg } r)\sqrt{a^2 + l^2}$$

where r is a real parameter.
(Notice that $f' > 1$ and $p < 1$).

In economic terms, there are situations in which Robinson, in order to maximize his utility, has to work as much as he can[14].

[13] To show that $p(l_1, f(l_1)) > p(l_0, f(l_0))$ when $l_1 > l_0$ and $f' > p$ on $[l_0, l_1]$, use the point with abscissa l_0 on the indifference curve through $(l_1, f(l_1))$.

[14] How much is that? In early nineteenth century England, 16 hours a day must have been quite frequent. In the 1820s, the parliament passed a bill forbidding children to work more than 8 hours per day. As machines staying idle for 16 hours each day is not tolerable for profit, work was organized in two turns of 8 hours for children and one of 16 for grown up workers.

Now what are they going to do with that?

We are not through with the economics of Robinson Crusoe, not to speak of the basic model. The next step is to study shocks. By this word, economists mean a great variety of phenomena, ranging from technical improvements to bad harvests. All these are supposed to be formally representable by a change in the production function. This is handled graphically on the basis of the tangency of the graphs of the old and new production functions and the indifference curves (fixed for eternity). The first example is at the end of this same chapter where the long run evolution of working time is supposed to be explained by technical progress.

In the next chapter, Robinson is brought to a world equipped with markets and, most important, money and credit so that he has new indifference curves relative to his choice between consuming now or in the next period. But the rule that everything is consumed as soon as it is produced remains valid. Indeed, it is quite essential to make the basic model complete and operational, as is seen in chapter 5 "The basic market-clearing model".

The basic model is then applied to a large variety of real world issues indeed, including increase or decrease of taxes, level of unemployment and a lot of others. And all this relies on the reasoning we just found faulty.

References

ARTAUD, M. (1993) *La Mathématisation en économie comme problème didactique. Une étude exploratoire.* Thèse à l'Université de Provence.

ARTAUD, M. (1995) La graphisation en économie: étape de mathématisation et stratégie de démathématisation. *Actes du séminaire de didactique et de l'EIAO de l'IRMA de Rennes.*

BARRO, R. (1984) *Macroeconomics* Wiley, New York.

COPELAND, T. & WESTON, J. (1988) *Financial theory and corporate policy* (3rd edition) Addison-Wesley, New York.

DORNBUSCH, R. (1983) Real interest rates, home goods and optimal external borrowing *Journal of Political Economy* **91**, 141-153.

ZERNER, M. (1996) Explication d'un modèle économique. *Actes de la VIIIème école d'été de didactique des mathématiques IREM de Clermont-Ferrand.*

ZERNER, M. (1999) Les sujets de mathématiques du baccalauréat ES: le jour du BAC tu fais le con. *Repères IREM* **36**, 59-71.