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COORDINATING REPRESENTATION REGISTERS: LINEAR ALGEBRA
STUDENTS' UNDERSTANDING OF ORTHOGONAL LEGENDRE
POLYNOMIALS IN THE INNER PRODUCT SPACE \mathbb{P}_n IN A
TECHNOLOGY-ASSISTED LEARNING ENVIRONMENT

Abstract. The purpose of this research study was to understand how linear algebra students in a university in the United States make sense of the orthogonal Legendre polynomials as vectors of the inner product space \mathbb{P}_n in a DGS (Dynamic Geometry Software)-MATLAB facilitated learning environment. Math majors came up with a diversity of innovative and creative ways in which they coordinated semiotic registers (Duval, 1993, 2006) for understanding inner products of Legendre polynomials along with other notions inherent in the inner product space, such as Triangle Inequality, Pythagorean Theorem, Parallelogram Law, Orthogonality and Orthonormality, Coordinates Relative to an Orthonormal Basis. Research participants not only produced such creative inner product space visualizations of the Legendre polynomials with the associated integral inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$ on the DGS, but they also verified their findings both analytically and visually in coordination. The paper concludes by offering pedagogical implications along with implications for mathematics teaching profession and recommendations for future research.

Keywords. Semiotic registers, linear algebra, Legendre polynomials, inner product space, visualization, dynamic geometry software, orthogonality.

Résumé. Coordonner des registres de représentation : compréhension par les étudiants des polynômes orthogonaux de Legendre dans l'espace euclidien \mathbb{P}_n au sein d'un environnement technologique. Le but de cette étude était de comprendre comment les étudiants en algèbre linéaire d'une université américaine donnent un sens aux polynômes de Legendre orthogonaux en tant que vecteurs de l'espace euclidien \mathbb{P}_n dans un environnement d'apprentissage facilité par DGS (Dynamic Geometry Software)-MATLAB. Les étudiants en mathématiques ont trouvé une variété de moyens novateurs et créatifs dans lesquels ils ont coordonné des registres sémiotiques (Duval, 1993, 2006) pour comprendre les produits scalaires des polynômes de Legendre ainsi que d'autres notions inhérentes à l'espace euclidien, telles que l'inégalité triangulaire, le théorème de Pythagore, la loi des parallélogrammes, l'orthogonalité et l'orthonormalité, les coordonnées dans une base orthonormée. Les participants à la recherche ont non seulement produit de telles visualisations créatives de l'espace euclidien des polynômes de Legendre pour le produit scalaire intégral associé $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$ sur le DGS, mais ils ont également vérifié leurs résultats à la fois analytiquement et visuellement en coordination. L'article conclut en offrant des implications pédagogiques ainsi que des implications pour la profession d'enseignant en mathématiques et des recommandations pour de futures recherches.

Mots-clés. Registres sémiotiques, algèbre linéaire, polynômes orthogonaux de Legendre, espace euclidien, visualisation, logiciel de géométrie dynamique, orthogonalité.

1. Introduction

The Legendre polynomial P_n of order n satisfies the 2nd-order ODE (Legendre differential equation) $\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] + n(n+1)y = 0$. Named after the French mathematician Adrien-Marie Legendre (1752-1833), the set $\{P_0, P_1, P_2, \dots\}$ forms an orthogonal set of polynomials with the associated integral inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$. This article presents the results of a qualitative research that shows how mathematics majors in a university in the United States made sense of the integral inner product in the vector space $\mathbb{P}_n(\mathbb{R})$ of polynomials of degree $\leq n$ with real coefficients – the mapping $\mathbb{P}_n(\mathbb{R}) \times \mathbb{P}_n(\mathbb{R}) \rightarrow \mathbb{R}$ defined via $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$ for $f, g \in \mathbb{P}_n(\mathbb{R})$ – in a DGS/MATLAB-facilitated learning environment. The mathematics major research participants explored the defining properties (symmetry, linearity, positive definiteness) of the integral inner product along with other notions inherent in the inner product space, such as norm, distance, orthogonal projection, Cauchy-Schwarz Inequality, Triangle Inequality, Pythagorean Theorem, Parallelogram Law, orthogonality and orthonormality, orthonormal basis, and coordinates relative to an orthonormal basis. The article also demonstrates the diversity of original and innovative ways through which mathematics majors visualized and made sense of the Legendre polynomials as an orthogonal basis for the inner product space $\mathbb{P}_n(\mathbb{R})$ in a technology-assisted learning environment. Student thinking was analyzed within representation registers and interplay between registers theoretical perspective (Duval, 1993, 2006).

1.1. Teaching and Learning Linear Algebra

Researchers have investigated the teaching and learning of linear algebra from an educational perspective since the mid-eighties (Dorier, 1991, 1995, 2000; Dreyfus & Hillel, 1998; Harel, 1985, 1987; Pavlopoulou, 1993; Robert & Robinet, 1989; Rogalski, 1994; Sierpiska, 1995, 2000; Sierpiska, Dreyfus & Hillel, 1999). In 1990, educators across mathematics departments in the United States formed the linear algebra curriculum study group (LACSG) whose goal was to “initiate substantial and sustained national interest in improving the undergraduate linear algebra curriculum” (Carlson, Johnson, Lay & Porter, 1993, p. 41). The LACSG recommended that linear algebra curricula consider “the needs and interests of students as learners;” and “utilize technology in the first linear algebra course” (p. 45). LACSG further recommended that the core syllabus of a first course in linear algebra encompass matrix addition and multiplication, systems of linear equations,

determinants, properties of \mathbb{R}^n , eigenvalues and eigenvectors, orthogonality, and supplementary topics (pp. 43-44). In particular, eigenvalue-eigenvector topics would include characteristic polynomial, algebraic multiplicity, eigenspaces, geometric multiplicity, similarity and diagonalization, symmetric matrices, orthogonal diagonalization, and quadratic forms (p. 44).

Harel and Kaput (1991), and Harel (1985, 1990) formulated the concreteness principle, as a fundamental approach for the teaching and learning of linear algebra, originated from Piaget's (1977) idea of conceptual entities. According to this principle, "for students to abstract a mathematical structure from a given model of that structure the elements of that model must be conceptual entities in the student's eyes; that is to say the student has mental procedures that can take these objects as inputs" (Harel, 2000, p. 180). Concreteness principle requires that "students build their understanding of a concept in a context that is concrete to them" (p.182). He recommends MATLAB as a tool that would help students visualize vectors and matrices as concrete mathematical objects, in accordance with the concreteness principle. Harel (2000) proposed a first course in linear algebra as a "natural continuation of what students have learned in high-school," which would "build on rich concept images of linear algebra already possessed by students" (p. 188).

1.2. Vector Spaces in Linear Algebra

Although there are no prior research studies in linear algebra with primary focus on inner product spaces and orthogonal polynomials in particular, mathematics education researchers investigated university students' understanding of vector spaces (Dorier, 1995, 1988; Dorier, Robert, Robinet & Rogalski, 2000; Parraguez & Oktac, 2010) and subspaces of \mathbb{R}^n in linear algebra (Wawro, Sweeney & Rabin, 2011). It is also possible to highlight past research that has dealt with such neighboring topics as span, linear independence and basis (Dogan-Dunlap, 2010; Hannah, Stewart & Thomas, 2013; Wawro et al., 2012); eigenvalues and eigenvectors (Sinclair & Gol Tabaghi, 2010; Thomas & Stewart, 2011; Gol Tabaghi & Sinclair, 2013; Caglayan, 2015); use of geometry in linear algebra (Dorier & Sierpinska, 2001; Gueudet-Chartier, 2006); representations in \mathbb{R}^n and the multi-representational aspect of linear algebra (Andrews-Larson, Wawro & Zandieh, 2017; Dorier, Sierpinska, 2001; Hillel, 2000); diagonalization (Zandieh, Wawro & Rasmussen, 2017).

Dorier (2000) highlights epistemological causes for students' difficulties (formalism). Hillel (2000) argued that students' difficulties in linear algebra are primarily proof related: "not understanding the need for proofs nor the various proof techniques; not being able to deal with the often implicit quantifiers; confusing necessary and sufficient conditions; making hasty generalizations based on very shaky and sparse evidence" (p.191). Dorier and Sierpinska (2001) suggested that

“the use of geometrical representations or language is very likely to be a positive factor, but it has to be controlled and used in a context where the connection is made explicit” (267). These researchers also outlined several reasons for why students consider linear algebra as a cognitively and conceptually difficult area to master: (i) axiomatic approach and the level of sophistication in the abstraction (e.g., “the concept of vector space being an abstraction from a domain of already abstract objects like geometrical vectors, n -tuples, polynomials, series of functions” (p. 257); (ii) the language of linear algebra (e.g. “the geometric language of lines and planes, the algebraic language of linear equations, n -tuples and matrices, the abstract language of vector spaces and linear transformations”); (p. 270) (iii) the registers of the language of linear algebra (e.g., graphical, tabular, symbolic, etc.); (iv) variety of representation systems (e.g. Cartesian, parametric, etc.); (v) cognitive flexibility “in moving between the various languages, viewpoints and semiotic registers” (p. 270).

Research studies taking into account the diversity of representation registers (not only restricted to linear algebra) and interplay between registers that inform the present report are important (Ramírez-Sandoval, Romero-Félix & Oktaç, 2014). A general reference concerning general representation registers is Duval (1993, 2006). Among other things, the main representation registers are: algebraic, formal, geometric, graphical, numerical, symbolic, verbal, visual. Some other work also includes other semiotic resources like gestures and artifacts (Radford, 2002; Arzarello, 2006). The question of representation registers with respect to the notion of integral is central in the work by McGee and Martinez-Planell (2014). Finally, Oktaç and Vivier (2016) also highlight the importance of the interplay between different registers to favour understanding: in the particular case of analysis, they also conclude on the importance of the graphical register as a visual aid to favour understanding.

Dorier (2000) and Grenier-Boley (2014) further highlighted the following students' difficulties in relation with linear algebra: (i) difficulties with prerequisites in logic, set theory and geometry; (ii) difficulties in using the first notions of linear algebra; (iii) difficulties in seeing the link with familiar situations; and (iv) difficulties in flexibly converting between registers. It is also worth noting the interpretation of several difficult concepts by the students at the beginning of university as FUG (formalizing, unifying, generalizing) concepts (Dorier, 2000; Vandebrouck, 2013): when first introduced to students, such notions introduce greater generality by unifying earlier objects through a new formalism. Examples of such concepts are given by basic notions of linear algebra (vector space, vector subspace, vector subspace spanned by vectors) and basic notions of real analysis such as the formal definitions of limit (and maybe the integral). In the case of linear algebra, the formalism is intrinsically difficult as it is inherent to its generalization and

unification (Dorier, 2000). Recent research about the teaching of linear algebra takes into account an interpretation of some of its concepts as FUG concepts (see for example Grenier-Boley, 2014).

In a research study on students' conceptualization of subspaces of \mathbb{R}^n , Wawro et al. (2011) identified the three factors associated with students' concept imagery: "Geometric Object (e.g., subspace is a plane in a space), Part of a Whole (e.g., subspace is contained within a space), and Algebraic Object (e.g., subspace is a collection of vectors)" (p. 7). They also observed the "technically inaccurate "nested subspace" conception that \mathbb{R}^k is a subspace of \mathbb{R}^n for $k < n$ " (p. 1). Framed in APOS theory, Parraguez and Oktac (2010) focused on students' coordination between the vector addition and the scalar multiplication operation associated with the vector space structure in relation to other concepts such as spanning sets and linear independence. They developed a genetic composition of the vector space notion in an attempt to predict "how students might construct the vector space concept as a schema" (p. 2 123). The purpose of the present report is to understand how linear algebra students make sense of the orthogonal Legendre polynomials as vectors of the polynomial inner product space $\mathbb{P}_n(\mathbb{R})$ with the associated integral inner product. Of particular interest are innovative and creative ways in which mathematics majors coordinate visual and analytic approaches in understanding inner product properties of orthogonal Legendre polynomials and their connections to the other concepts of linear algebra.

1.3. Motivation for the Study

This article deals with students' understanding of the family of orthogonal Legendre polynomials and some of their properties associated with the inner product spaces \mathbb{P}_n or \mathbb{P} for a certain inner product defined by an integral. These properties are either algebraic or geometric and particularly related to the orthogonality or orthonormality of this family with respect to the inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$. This subject is at the crossroads of both bilinear algebra and functional analysis and is therefore crucial for the continuation of students' studies. From an algebraic point of view, this subject is related to the understanding of basic bilinear algebra (and not only linear algebra) when the considered bilinear form is symmetric positive definite on a vector space E over \mathbb{R} . In this case, the inner product considered endows E with a structure of euclidian space (if $\dim_{\mathbb{R}} E$ is finite) or inner product space (in general). In the article, $E = \mathbb{P}_n$ or \mathbb{P} is endowed with a structure of inner product space for a form defined via an integral. The properties studied in this article are therefore natural attributes related to this form, some of which having a geometric interpretation; it is also based upon basic linear algebra.

From an analytical point of view, the inner product considered in the article is in fact that of $L^2(I)$ where I is an open interval of \mathbb{R} and the inner product is defined by $\langle u, v \rangle = \int_I u(x)v(x)dx$. Then $L^2(I)$ endowed with this inner product is a Hilbert space. Moreover, the orthogonality of the family of Legendre polynomials for this inner product precedes the notion of Hilbert basis. In the article, the research is part of a project which focuses on connections between three branches of advanced mathematics by means of Computer Algebra Systems (CAS) or DGS: geometry, real analysis and linear/abstract algebra. As highlighted above, the study of this subject is clearly related to these three branches. In this aspect, and aligned with the LASCg recommendations (Carlson et al., 1993), the chosen subject (Legendre polynomials) in relation to the connections between these different branches certainly reveals itself as a crucial motivation for the study.

2. Theory and Method

2.1. Registers of Semiotic Representations

Representations are central to mathematics teaching and learning. “A representation is something that stands for something else” (Duval, 2006, p. 103). The guiding theoretical framework used to frame the present report is Duval’s theory of registers of semiotic representations (1993, 2006). According to Duval, the mathematical knowledge object is not to be confused with the representation of the object itself: “the distinction between an object and its representation is a strategic point for understanding mathematics” (Duval, 1993, p. 37). Object-representation duality is prone to lead to the notion of cognitive paradox of mathematical thought: “on the one hand, the apprehension of mathematical objects can only be a conceptual apprehension and, on the other hand, it is only by means of semiotic representations that an activity on mathematical objects is possible” (p. 38). Duval (2006) further posits the requirement that, from a cognitive point of view, the three important characteristics be considered in the analysis of a mathematical activity: (1) [The paramount importance of semiotic representations] It runs through a transformation of semiotic representations, which involves the use of some semiotic system; (2) [The large variety of semiotic representations used in mathematics] For carrying out this transformation, quite different registers of semiotic representations can be used; (3) [The cognitive paradox of access to knowledge objects] Mathematical objects must never be confused with the semiotic representations used, although there is no access to them other than using semiotic representation (p. 126).

Not all semiotic systems are registers; for a system of representation to be considered as a register of representation, the representation must permit three fundamental cognitive activities: (1) The *formation* of an identifiable representation as a representation in a given register; (2) The *processing* (treatment) of a representation (i.e., the internal transformation taking place in the same register where it was

formed); (3) The *conversion* of a representation (i.e., the transformation of this representation into a representation of another register while retaining all or part of the content of the representation) (Duval, 1993, pp. 41-42). It is worth noting that treatments and conversions stand as two types of transformations of semiotic representations that are radically different (Duval, 2006, p. 111). On the one hand, a treatment is a transformation of representations that happen within the same register: “The treatments, which can be carried out, depend mainly on the possibilities of semiotic transformation, which are specific to the register used” (p. 111). On the other hand, a conversion is a transformation of representation consisting of “changing a register without changing the objects being denoted” (e.g., passing from the definite integral (algebraic) notation of an inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$ to its graphic representation (as the area between the curve and the x -axis). In this vein, a conversion as a representation transformation is “more complex than treatment because any change of register first requires recognition of the same represented object between two representations whose contents have very often nothing in common” (p. 111).

What does it mean to understand in mathematics then? Although using signs and semiotic representations is the only way to access mathematical objects and to deal with them, “mathematical objects must never be confused with the semiotic representations that are used” (Duval, 2006, p. 106). Treatments and conversions, in that aspect, are fundamental transformations leading towards mathematical comprehension. As fundamental is the “simultaneous mobilization of at least two registers of representation, or the possibility of changing at any moment from one register to another... conceptual comprehension in mathematics involves a two-register synergy, and sometimes a three-register synergy” (p. 126). The focus of the present report is restricted largely to graphical (visual) and algebraic (analytical) semiotic registers of representations with an analysis in terms of coordination and conversion between different representation registers.

2.2. Data Collection and Analysis

Qualitative-descriptive interview data (Patton, 2002) were collected over three years in a university in the United States, as part of a research project which was designed for the purpose of enhancing mathematics and mathematics education majors’ content knowledge of advanced mathematics, with particular focus on geometry-calculus/analysis-linear/abstract algebra triad and connections within branches of advanced mathematics using technology. Qualitative interview sessions on advanced mathematical thinking were videotaped using one camera, with primary focus on students’ constructions and cursor-gestures on the MATLAB and GeoGebra DGS along with hand gestures and inscriptions.

The data for the analysis of inner product space concept came from the videotapes of interview sessions in a computer lab that included a total of sixteen mathematics majors who successfully completed a one-semester-long linear algebra course with course grades ranging from C to A (6A, 7B, 3C), who were interviewed by the author individually on separate days. The university linear algebra course students took had covered the first eight chapters of the textbook “Elementary Linear Algebra 7th Edition” by Ron Larson (2013). Students were familiar with the MATLAB and DGS as they had already used them during the preceding interview sessions on linear algebra topics; they were not provided any training or instructional help during the interviews. The qualitative interviews were based on a semi-structured interview model (Bernard, 1994; Kvale, 2007) in the course of which the interviewer – who is also the author of this article – followed-up with probes and questions on the interviewees’ responses. Students were asked to respond to a variety of interview questions with primary focus on orthogonal Legendre polynomials and inner product properties¹.

Students had already known the definition of an inner product [Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be vectors in a vector space V , and let c be any scalar. An inner product on V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors \mathbf{u} and \mathbf{v} and satisfies the axioms listed below. 1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ 2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ 3. $c\langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$ 4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (Larson, 2013)] in general, and applied this textbook definition in various contexts of inner product spaces, for instance, to determine whether a certain given function satisfies the four axioms of an inner product. They were also familiar with other notions of relevance such as vector norm (length), distance between vectors, angle between vectors, orthogonal vectors and orthogonal projections in inner product spaces, and geometric type inequalities and theorems. Prior to the exploration of orthogonal Legendre polynomials, apart from various inner products (e.g., the Euclidean inner product) defined on vector spaces \mathbb{R}^2 , \mathbb{R}^3 , \mathbb{R}^4 and \mathbb{R}^n ; students also had previously learned and explored the Frobenius inner product $\langle A, B \rangle = \sum_{i,j} A_{ij}B_{ij} = \text{tr}(B^T A)$ for real $m \times n$ matrices A and B in the vector space $\mathbb{R}^{m \times n}$; along with the polynomial inner product $\langle p, q \rangle = a_0b_0 + a_1b_1 + \dots + a_nb_n$ for polynomials $p(x) = a_0 + a_1x + \dots + a_nx^n$ and $q(x) = b_0 + b_1x + \dots + b_nx^n$ in $\mathbb{P}_n(\mathbb{R})$ in a similar DGS-MATLAB facilitated learning environment. In all these inner product spaces, students had also explored various inner product properties including orthogonality and orthonormal bases, along with other relevant topics of linear algebra such as span and spanning sets, linear independence and basis, coordinates relative to an orthogonal/orthonormal basis, change of basis, transition matrices.

¹ Most of the time, the students did not really investigate beyond \mathbb{P}_3 or \mathbb{P}_4 . Appendix lists the interview outline (with tasks and sample probing questions) students explored during the interviews.

During the interviews, students were asked to think aloud, to clearly indicate their problem solving procedure, and to explain their reasoning in detail. Students were also granted access to scratch paper along with MATLAB and DGS to facilitate and clarify their explanations. All interview tasks were designed in such a way that they could be explored via both algebraic (analytical) and graphical (visual) approaches, in accord with the guiding theoretical framework used in the study. Though it was optional, all students were very eager and passionate about using MATLAB and DGS, primarily for checking their work, visualizing analytic approaches, testing conjectures, or providing examples and counter-examples. Analysis of data, which consists of videotaped qualitative interviews along with MATLAB-DGS work and inscriptions, was carried out using thematic analysis (Boyatzis, 1998) along with the theory of registers of semiotic representations (Duval, 1993, 2006). These analytical frameworks were primarily used to describe how research participants came up with a diversity of innovative and creative ways in which they coordinated and converted among various representation registers for understanding orthogonal Legendre polynomials and the inner product properties. After transcribing all interview videos, the original transcripts were reviewed line by line to gain access to the strategies (e.g., algebraic, geometric, graphical/visual, numerical, symbolic/notational, verbal/natural language, gestural semiotic registers) embraced by students as they explored Legendre polynomials and its relationship to other core concepts of linear algebra, in accordance with the research objective. The last cycle of data analysis process consisted of a holistic review of the corpus of data multiple times, in accordance with constant comparative methodology (Glaser & Strauss, 1967).

2.3. A-Priori Analysis: A Study of Semiotic Registers of Representation: The Case of Orthogonal Legendre Polynomials

Aligned with the constructs of the theoretical framework, a DGS/MATLAB-based exploration of Legendre polynomials in the polynomial inner product space was crucial, primarily to gain access into students' thinking and multiple ways of representing this very important content, which is at the crossroads of linear algebra and analysis. This section outlines a diversity of representation registers and their relationships that could come to interplay when Legendre polynomials are considered as vectors of the inner product space $\mathbb{P}_n(\mathbb{R})$ with the associated definite integral inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$ for $f, g \in \mathbb{P}_n(\mathbb{R})$ in a technology-assisted learning environment.

2.3.1. Analytic/Algebraic Register

This seems to be the most widespread semiotic register in dealing with Legendre polynomials, which, in turn, is prone to a diversity of representations. To name a few, we have: (i) the implicit expression representation of the sequence of Legendre polynomials via the recurrence relation $(2n + 1)xP_n = (n + 1)P_{n+1} + nP_{n-1}$ with

$P_0 = 1$ and $P_1 = x$; (ii) the explicit expression representation of the sequence of Legendre polynomials via the Rodrigues' formula $P_n = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$ for all $n \geq 0$; (iii) When the Pythagorean Theorem is considered, for example, the equivalence of the algebraic expansion representation of $\|P_n + P_m\|^2$ in terms of inner products to the algebraic expansion representation of $\|P_n\|^2 + \|P_m\|^2$ within the algebraic register; (iv) When the coordinates representation $[\mathbf{p}]_B$ of an arbitrary polynomial \mathbf{p} relative to the orthonormal basis B of normalized Legendre Polynomials are considered, for instance, the equivalence of the squared norm representation $\|\mathbf{p}\|^2$ (with respect to the definite integral inner product) to the squared norm representation $\|[\mathbf{p}]_B\|^2$ (with respect to the ordinary Euclidean inner product) within the algebraic register.

2.3.2. Visual/Graphical Register

Possible representations are: (i) [in coordination with 2.3.1(i)] an equivalent implicit expression representation of the sequence of Legendre polynomials via the recurrence relation $P_{n+1} = \frac{-nP_{n-1} + (2n+1)xP_n}{n+1}$ with $P_0 = 1$ and $P_1 = x$ which may come handy in the graphical approach, for example, when graphing using the input bar in *Graphics View* or using the cells in *Spreadsheets View*; (ii) The y -axis / origin symmetry representation of the product function $P_n P_m$ when $n + m = \text{odd} / \text{even}$ within the graphical register; (iii) The “always above the x -axis over the interval $[-1, 1]$ ” visualization of the product function $P_n P_m$ when $n = m$ within the graphical register; (iv) [in coordination with 2.3.2(iii)] When modelling the Parallelogram Law for $n \neq m$, for instance, the realization of the (vector sum) squared norm $\|P_n + P_m\|^2$ graphical representation and the (distance between vectors) squared norm $\|P_n - P_m\|^2$ graphical representation as “each other's y -axis symmetric graphical representations” within the graphical register.

2.3.3. Symbolic Register

Closely related to algebraic/analytic representation register, symbolic register could be thought of as encompassing signs or symbols in general (not necessarily solely pertaining to algebra). Examples of representations in this register include: (i) The squared norm representation $\|P_n\|^2 = \langle P_n, P_n \rangle$ in relation to the inner product notation in the symbolic register; (ii) The vector norm representation $\|P_n\| = \sqrt{\langle P_n, P_n \rangle}$ in relation to the square root of the inner product notation within the symbolic register; (iii) When the Triangle Inequality or the Pythagorean Theorem is considered, for example, an accurate labeling of the sides of the drawn triangle using the vector norm representation $\|P_n\|$ within the symbolic register; (iv) Using the chain of symbols $m \neq n \Rightarrow P_n \perp P_m \Rightarrow \langle P_n, P_m \rangle = 0$ in representing orthogonality of a pair of Legendre polynomials; (v) Use of various signs or symbols (not only relevant to algebra) in general.

2.3.4. Geometric Register

Possible representations are: (i) A right triangle drawn representation modelling the orthogonality of a pair of orthogonal Legendre polynomials within the geometric register; (ii) The “obvious cancelling areas resulting in zero net area between the curve and the x -axis over the interval $[-1,1]$ ” visualization of the product function $P_n P_m$ when $n \neq m \wedge n + m$ odd within the geometric register; (iii) [in coordination with 2.3.2(iii)] The “positive area between the curve and the x -axis over the interval $[-1,1]$ ” visualization of the product function $P_n P_m$ when $n = m$ within the geometric register; (iv) The “nontrivial zero net area between the curve and the x -axis over the interval $[-1,1]$ ” visualization of the product function $P_n P_m$ when $n \neq m \wedge n + m$ even within the geometric register; (v) [in coordination with 2.3.2(iv)] When modelling the Parallelogram Law for $n \neq m$, for instance, the equivalence of the “area between the curve $(P_n + P_m)^2$ and the x -axis over the interval $[-1,1]$ ” and the “area between the curve $(P_n - P_m)^2$ and the x -axis over the interval $[-1,1]$ ” as “equality in areas” geometric representation within the geometric register.

2.3.5. Numerical Register

Examples of representations in this register include: (i) When the case $m + n$ odd ($m \neq n$ sous-entendu) is considered, for instance, the equivalence of the inner products $\langle x^m, x^n \rangle$ and $\langle P_m, P_n \rangle$ in zero numerical value; (ii) When the case $m = n$ is considered the equivalence of the inner products $\langle x^m, x^n \rangle$ and $\langle P_m, P_n \rangle$ in numerical value (both having the $\frac{2}{2n+1}$ numerical value representation within the numerical register; (iii) [in coordination with 2.3.3(iii)] When the Triangle Inequality or the Pythagorean Theorem is considered, for example, an accurate labeling of the sides of the drawn triangle using the numerical value representation $\sqrt{\frac{2}{2n+1}}$ of the vector norm (e.g., for $n = 2$, the numerical value of $\|P_n\|$ is $\sqrt{2/5}$) within the symbolic register.

2.3.6. Verbal (Natural Language) Register

Plenty of representations in this category are possible: (i) P_n is an even / odd function for n even / odd (and contains only even / odd powers of x) accordingly its graph is symmetric with respect to the y -axis / origin; (ii) In the context of orthogonal Legendre polynomials, Cauchy-Schwarz Inequality $\langle P_n, P_m \rangle \langle P_n, P_m \rangle \leq \langle P_n, P_n \rangle \langle P_m, P_m \rangle$ can be verified by merely noting that the right-hand side is always positive whereas the left-hand side is either zero or equal to the right-hand side; (iii) In establishing Legendre polynomials as an orthogonal basis for \mathbb{P}_n , it is sufficient to use orthogonality to test for a basis.

2.3.7. Gestural Register

In coordination with graphical or geometric representation registers, for instance: (i) Finger gestures pointing to the lower and upper integration limits when the definite integral inner product is considered; (ii) [in coordination with 2.3.2(ii)] Finger gestures pointing to the pairs of cancelling areas in the graphical representation of the definite integral of the product function $P_n P_m$ when $n + m = \text{odd}$.

2.3.8. Artifactual Register

In coordination with geometric or symbolic representation registers, for instance: (i) Modelling a relationship of Legendre polynomials as a triangle or a parallelogram artifact; (ii) Modelling the inner product $\langle P_n, P_m \rangle = \int_{-1}^1 P_n P_m dx = \frac{2}{2n+1} \delta_{nm}$ as a diagonal matrix model, the Gramian matrix artifact, with generic diagonal entry of the form $2 \int_0^1 P_n P_n dx$; (iii) When the normalized Legendre polynomials are considered, modelling the completeness relation $\langle \mathbf{p}_i, \mathbf{p}_j \rangle = \int_{-1}^1 p_i p_j dx = \delta_{ij}$ as a special diagonal matrix, the Gramian matrix, with n^2 entries consisting of the n^2 inner products, within the artifactual register.

2.3.9. Interplay Among Registers

In a DGS/MATLAB-facilitated investigation of the family of Legendre polynomials, *conversions* of a representation can occur in a variety of ways. Some examples are: (i) Changing from the algebraic/analytic register to the graphical register in the exploration of the inner product of two Legendre polynomials: passing from the algebraic definite integral inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$ to the graphical definite integral inner product as the area between the curve and the x -axis over the interval $[-1, 1]$; (ii) Passing from the implicit algebraic representation of the sequence of Legendre polynomials via the recurrence relation $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$ with $P_0 = 1$ and $P_1 = x$ to the implicit graphical representation of the sequence of Legendre polynomials via the recurrence relation $P_{n+1} = \frac{-nP_{n-1} + (2n+1)xP_n}{n+1}$ with $P_0 = 1$ and $P_1 = x$.

Treatments, that is, representations happening within the same register can be observed in many ways: (i) The equivalence of the inner products $\langle x^m, x^n \rangle$ and $\langle P_m, P_n \rangle$ in numerical value when the case $m + n$ odd is explored, within the numerical register; (ii) The equivalence of the vector norms $\|P_n + P_m\|$ and $\|P_n - P_m\|$ in the investigation of Parallelogram Law within the algebraic register, or within the geometric register as either norm representing the length of either one of the congruent diagonals of a special parallelogram, that is, a rectangle.

3. Results

This section presents the findings on math majors' representations of Legendre polynomials and integral inner product properties in a MATLAB-DGS assisted learning environment. Overall, research participants provided a diversity of visual and analytic representations in their exploration of the orthogonal set $\{P_0, P_1, P_2, \dots\}$ of Legendre polynomials with the associated integral inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$. The first part of the analysis focuses on students' evaluation and interpreting of inner products $\langle P_m, P_n \rangle$ for $m = n$ and $m \neq n$, parity relationships and comparison of inner products $\langle x^n, x^m \rangle$ and $\langle P_m, P_n \rangle$ [3.1-3.3]. The section then concludes with students' establishing of the orthogonality via analytic and visual approaches, followed by the verification of Cauchy-Schwarz Inequality, Triangle Inequality, and Pythagorean Theorem [3.4-3.5].

3.1. Generating Legendre Polynomials $\{P_n\}$ Recursively or Explicitly

Students began their exploration of the Legendre polynomials $\{P_n\}$ by actually obtaining the sequence of Legendre polynomials $\{P_0 = 1, P_1 = x, P_2 = \frac{3x^2-1}{2}, P_3 = \frac{5x^3-3x}{2}, \dots\}$ either recursively via the recurrence relation $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$ with $P_0 = 1$ and $P_1 = x$ (e.g., fig. 1); or explicitly using the Rodrigues' formula $P_n = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$ for all $n \geq 0$ (e.g., fig. 2).

Handwritten calculations for Legendre polynomials using the recurrence relation:

$$\begin{aligned} n=1 \\ 2P_2(x) &= (3)P_1(x) - P_0(x) \\ &= \frac{3x^2-1}{2} \\ n=2 \\ 3P_3(x) &= 5xP_2(x) - 2P_1(x) \\ P_3(x) &= \frac{5x(3x^2-1)}{2} - 2x = \frac{5}{2}(3x^2-1)x - \frac{2}{3}x \\ n=3 \\ 4P_4(x) &= 7xP_3(x) - 3P_2(x) \\ P_4(x) &= \frac{1}{4} \left[7x \left(\frac{5}{2}(3x^2-1)x - \frac{2}{3}x \right) - 3 \left(\frac{3x^2-1}{2} \right) \right] \end{aligned}$$

Figure 1. Gaby's Recursive Formula

Handwritten derivation of Rodrigues' formula for Legendre polynomials. The formula is shown as $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$. The derivation shows the calculation of $P_2(x)$ using the formula, resulting in $P_2(x) = \frac{3x^2 - 1}{2}$.

Figure 2. Andy's Rodrigues' Formula

In either group, students made use of analytic and visual approaches in a coordinative manner. Whereas analytic approach proved easy and straightforward for the most part, the visual approach offered a diversity of creative and innovative visualizations of Legendre polynomials using the Geogebra Dynamic Geometry Software (DGS). Among those who visualized Legendre polynomials on the DGS, one group of students including Andy and Kyra graphed the Legendre polynomials by typing the formula of each Legendre polynomial that was obtained recursively algebraically (fig. 3 and 4). Another group of students used the spreadsheet feature of the DGS and obtained the expression for each Legendre polynomials for $n \geq 2$ by using the recurrence relation $P_{n+1} = \frac{-nP_{n-1} + (2n+1)xP_n}{n+1}$ with $P_0 = 1$ and $P_1 = x$. There were also those who used the Rodrigues' formula $P_n = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$ to obtain all Legendre polynomials $n \geq 0$ using spreadsheet functionalities.

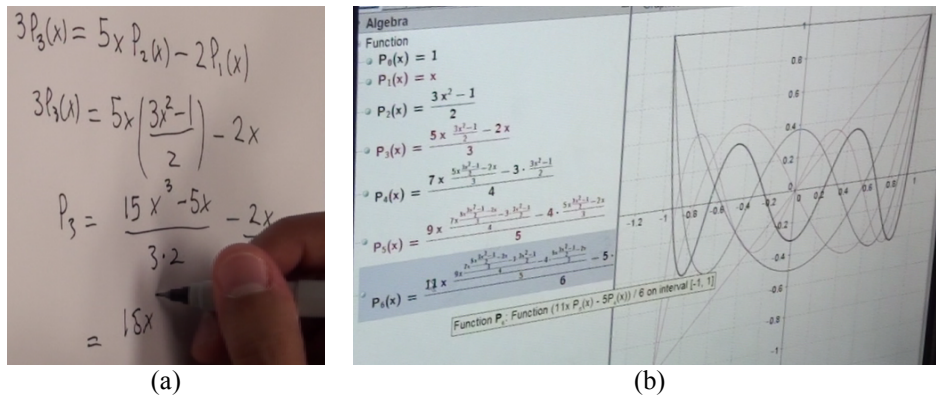
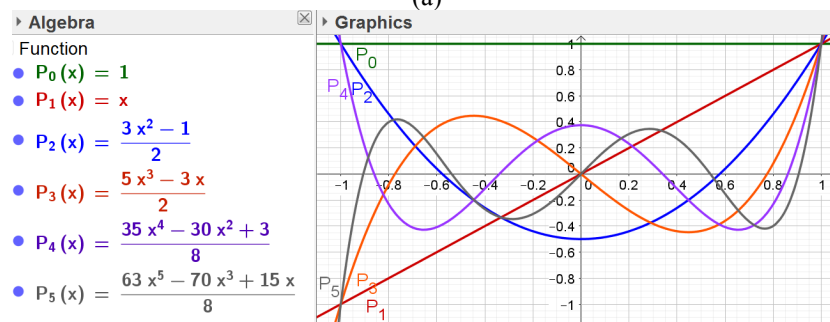


Figure 3. Andy's Analytic and Visual Approaches

$P_0(x) = 1$
 $P_1(x) = x$
 $(1-x)P_1(x) = 1(1+x) + P_1(x) - xP_0(x)$
 $2P_2(x) = 3x(x) - 1 \rightarrow P_2(x) = \frac{3x^2 - 1}{2}$
 $3P_3(x) = 5xP_2(x) - 2P_1(x)$
 $3P_3(x) = 5x\left(\frac{3x^2 - 1}{2}\right) - 2x \rightarrow P_3(x) = \frac{5x^3 - 3x}{2}$
 $4P_4(x) = 7(5x^2 - 1)P_3(x) - 3P_1(x)$
 $4P_4(x) = 7x\left[\frac{5x^3 - 3x}{2}\right] - 3\left[\frac{3x^2 - 1}{2}\right]$

(a)



(b)

Figure 4. Kyra's Analytic and Visual Approaches

Because the integral inner product for the Legendre polynomials is defined over the interval $[-1, 1]$, several students including Liz and Dana used the piecewise defined function syntax to graph these polynomials. To graph P_2 and P_3 over the interval $[-1, 1]$, for instance, Dana respectively typed **P_2=Function[(3x^2-1)/2,-1,1]** and **P_3=Function[(5x^3-3x)/2,-1,1]**. Dana then used the syntax **Integral[P_2*P_3,-1,1]** to verify the orthogonality of P_2 and P_3 (fig. 5a). To calculate the squared norms $\|P_2\|^2$ and $\|P_3\|^2$, she respectively typed **Integral[P_2*P_2,-1,1]** and **Integral[P_3*P_3,-1,1]** (fig. 5b-c).

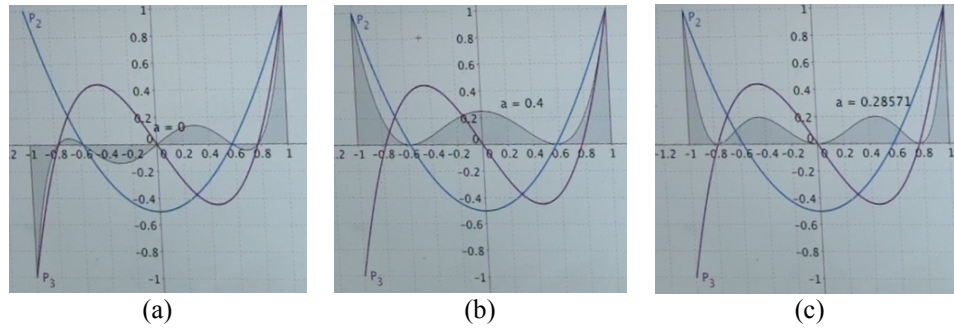


Figure 5. Dana's Piecewise Defined Function Syntax for Graphing Legendre Polynomials

Students in either group came up with various discoveries regarding the behavior of the Legendre polynomials along with graphing strategies associated with the DGS: (i) Every time a Legendre polynomial's expression is obtained in *Spreadsheet View*, the graph corresponding to the Legendre polynomial immediately appeared in *Graphics View*; (ii) The “**Simplify**” syntax proved crucial in obtaining the sought Legendre polynomials; (iii) Some other DGS related strategies such as “Zoom out” or “Scale Change” played an important role in the analysis of Legendre polynomials (e.g., in establishing the fact that the Legendre polynomial of order n has n roots, or the orthogonality of two Legendre polynomials via the cancelling areas argument), especially for $n \geq 4$; (iv) P_n is an odd function for n odd (and contains only odd powers of x) whence its graph is symmetric with respect to the origin; (v) P_n is an even function for n even (and contains only even powers of x) accordingly its graph is symmetric with respect to the y -axis; (vi) Some students came up with the parity equation $P_n(-x) = (-1)^n P_n(x)$ designating the odd/even case all at once; (vii) The discovery of **Derivative(<Function>, <Variable>, <Number>)** syntax took some time for some of the students who embraced the Rodrigues' Formula approach; (viii) For the case n odd, $P_n(0) = 0$; (ix) For the case n even, $P_n(0)$ is alternately positive and negative as in the sequence of numbers $1, -\frac{1}{2}, \frac{3}{8}, -\frac{5}{16}, \frac{35}{128}, -\frac{63}{256}, \dots$; (x) Each Legendre polynomial P_n has n distinct roots in the interval $[-1, 1]$.

3.2. Evaluating and Interpreting $\langle P_m, P_n \rangle$ for $m = n$ and $m \neq n$

Once Legendre polynomials were generated algebraically and visually, students moved to the task of evaluating inner products. At first, all students made use of basic calculus integration properties in evaluating inner products $\langle P_m, P_n \rangle = \int_{-1}^1 P_m P_n dx$ for $m = n$ and $m \neq n$ (e.g., fig. 6 and 7).

$$\begin{aligned}
 & \int_{-1}^1 \left(\frac{3x^2-1}{2} \right) \left(\frac{3x^2-1}{2} \right) dx \\
 & \frac{1}{4} \int_{-1}^1 9x^4 - 6x^2 + 1 dx = \frac{1}{4} \left[\frac{9}{5}x^5 - \frac{6}{3}x^3 + x \right]_{-1}^1 \quad \begin{array}{l} P_0 = 0 \\ P_1 = x \\ P_2 = \frac{3x^2-1}{2} \end{array} \\
 & \frac{1}{4} \left[\frac{9}{5} - 2 + 1 + \frac{9}{5} - 2 + 1 \right] = \frac{1}{4} \left[\frac{18}{5} - 2 \right] = \frac{1}{4} \left[\frac{18}{5} - \frac{10}{5} \right] \\
 & = \frac{1}{4} \left[\frac{8}{5} \right] = \boxed{\frac{2}{5}} \\
 \\
 & \int_{-1}^1 x \left(\frac{3x^2-1}{2} \right) dx = \frac{1}{2} \int_{-1}^1 3x^3 - x dx = \frac{1}{2} \left[\frac{3}{4}x^4 - \frac{1}{2}x^2 \right]_{-1}^1 \\
 & \frac{1}{2} \left[\frac{3}{4} - \frac{1}{2} - \frac{3}{4} + \frac{1}{2} \right] = \frac{1}{2}(0) = \boxed{0}
 \end{aligned}$$

Figure 6. Gaby's Integrations for the Inner Products $\langle P_m, P_n \rangle$

$$\begin{aligned}
 \int_{-1}^1 P_3 P_3 dx &= \int_{-1}^1 \frac{(5x^3-3x)^2}{4} dx = \frac{1}{4} \int_{-1}^1 25x^6 - 30x^4 + 9x^2 dx \\
 &= \frac{1}{4} \left[\frac{25x^7}{7} - 6x^5 + 3x^3 \right]_{-1}^1 = \frac{1}{4} \left[\frac{25}{7} - 3 - \left(-\frac{25}{7} + 3 \right) \right] \\
 &= \frac{1}{4} \left[\frac{50}{7} - 6 \right] = \frac{1}{4} \left(\frac{8}{7} \right) = \frac{2}{7} \\
 \\
 \int_{-1}^1 P_2 P_4 dx &= \int_{-1}^1 \frac{(3x^2-1)}{2} \cdot \frac{(35x^4-48x^2+9)}{8} dx = \int_{-1}^1 \frac{(3x^2-1)(35x^4-48x^2+9)}{16} dx \\
 &= \frac{1}{16} \int_{-1}^1 105x^6 - 144x^4 + 27x^2 - 35x^4 + 48x^2 + 9 dx = \frac{1}{16} \int_{-1}^1 105x^6 - 179x^4 + 75x^2 + 9 dx \\
 &= \frac{1}{16} \left[15x^7 - \frac{179x^5}{5} + 7x^3 + 9x \right]_{-1}^1 = \frac{1}{16} \left[15 - \frac{179}{5} + 7 + 9 - \left(-15 + \frac{179}{5} - 7 - 9 \right) \right] \\
 &= \frac{1}{16} \left[31 - \frac{179}{5} - \left(\frac{179}{5} - 31 \right) \right] = \frac{1}{16} \left[-62 - \frac{358}{5} \right] = \frac{1}{16} \left[-\frac{668}{5} \right] = -\frac{668}{80}
 \end{aligned}$$

Figure 7. Kyra's Integrations for the Inner Products $\langle P_m, P_n \rangle$

Because the powers of x repeatedly appeared in the integrand, many students came up with a formula which they thought would become handy in simplifying calculations. Among those who realized this fact were Val, Gaby, Liz, and Macy, who developed a formula for the inner product $\langle x^m, x^n \rangle$ for the two particular cases where $m + n$ is odd or even, respectively (fig. 8-11). Gaby, for instance, explained: “If $m + n$ is even then $m + n + 1$ is odd for which the inner product equals the fraction $\frac{2}{m+n+1}$; otherwise zero, meaning they are orthogonal.” Andy referred to the odd/even function properties in his explanation for why the inner product $\langle x^m, x^n \rangle$ would vanish for the case where the integrand is an odd function when the lower and upper limits of integration are -1 and 1 . Liz was among those who made use of the even integrand property to simplify $\int_{-1}^1 x^{m+n} dx$ as $2 \int_0^1 x^{m+n} dx$ before ultimately evaluating the definite integral (fig. 5c). In this aspect, Andy and Liz can be considered to have adopted a simplified definite integral representation within the symbolic register for why the inner product $\langle x^m, x^n \rangle$ should vanish, or should be simplified as $2 \int_0^1 x^{m+n} dx$, respectively, in terms of the representation registers theory (Duval, 1993, 2006). Overall, students seemed to have made use of a diversity of Calculus properties in their exploration of the defining properties of the integral inner product $\langle P_m, P_n \rangle = \int_{-1}^1 P_m P_n dx$ in general, and $\langle x^m, x^n \rangle = \int_{-1}^1 x^m x^n dx$ in particular.

$$\int_{-1}^1 x^{m+n} dx = \frac{x^{m+n+1}}{m+n+1} \Big|_{-1}^1$$

Consider $m+n = 2j+1$.

$$\frac{x^{2j+2}}{2j+2} \Big|_{-1}^1 = \frac{1^{2j+2}}{2j+2} - \frac{(-1)^{2j+2}}{2j+2}$$

$$= \frac{1^{2j+2}}{2j+2} - \frac{1^{2j+2}}{2j+2} = 0$$

Consider $m+n = 2j$

$$\frac{x^{2j+1}}{2j+1} \Big|_{-1}^1 = \frac{1^{2j+1}}{2j+1} - \frac{(-1)^{2j+1}}{2j+1}$$

$$= \frac{1^{2j+1}}{2j+1} + \frac{(-1)^{2j}}{2j+1} = \frac{1}{2j+1} + \frac{1}{2j+1}$$

$$= \frac{2}{m+n+1}$$

Figure 8. Val's Formulas for $\langle x^m, x^n \rangle$

$$\int_{-1}^1 x^{m+n} dx = \frac{1}{m+n+1} x^{m+n+1} \Big|_{-1}^1$$

$$= \frac{1}{m+n+1} (1)^{m+n+1} - \frac{1}{m+n+1} (-1)^{m+n+1}$$

if $m+n$ is even, then $m+n+1$ is odd

$$= \frac{1}{m+n+1} - \left(-\frac{1}{m+n+1} \right) = \frac{1+1}{m+n+1} = \frac{2}{m+n+1}$$

if $m+n$ is odd, then $m+n+1$ is even

$$= \frac{1}{m+n+1} - \frac{1}{m+n+1} = 0$$

3 $P_2(x) = \frac{1}{2}(3x^2 - 1)$

Figure 9. Gaby's Formulas for $\langle x^m, x^n \rangle$

$$\langle x^m, x^n \rangle = \int_{-1}^1 x^{m+n} dx = 0 \quad \text{where } m+n = \text{odd}$$

$$= 2 \int_0^1 x^{m+n} dx = \frac{2}{m+n+1} \quad \text{where } m+n = \text{even}$$

Figure 10. Liz's Formulas for $\langle x^m, x^n \rangle$

$$\frac{2}{m+n+1} = \int_{-1}^1 x^{m+n} dx = \frac{x^{m+n+1}}{m+n+1} \Big|_{-1}^1 = \frac{1}{m+n+1} - \frac{(-1)^{m+n+1}}{m+n+1}$$

if $m+n$ is even, then $m+n+1$ is odd

$$= \frac{1}{m+n+1} - \left(-\frac{1}{m+n+1} \right) = \frac{1+1}{m+n+1} = \frac{2}{m+n+1}$$

if $m+n$ is odd, then $m+n+1$ is even

$$= \frac{1}{m+n+1} - \frac{1}{m+n+1} = 0$$

Figure 11. Macy's Formulas for $\langle x^m, x^n \rangle$

Upon coming up with such formulas for the inner products $\langle x^m, x^n \rangle$, many students made use of defining properties of inner product (e.g., symmetry, linearity) in progression towards an understanding of the expansion of inner products $\langle P_m P_n \rangle$ in terms of the inner products $\langle x^m, x^n \rangle$. In this sense, all students appeared to progress organically toward generalizations by way of a formula they just discovered: Coming up with a formula for the inner products $\langle x^m, x^n \rangle$ was a significant one. The other remarkable achievement was students' ability to actually apply this formula in various inner products involving Legendre polynomials. Among those were Val, Jill, Gaby, Liz, Suzan, and Andy who primarily made use of the linearity and symmetry of the integral product $\langle P_m P_n \rangle = \int_{-1}^1 P_m P_n dx$, which is what enabled them to involve the formula for $\langle x^m, x^n \rangle$ throughout their calculations (fig. 12-16). In Duval's terms, this can be thought of as an expanded sum representation of $\langle P_m P_n \rangle$ in the symbolic register as a linear combination of inner products $\langle x^m, x^n \rangle$.

$\langle P_1, P_1 \rangle = \int_{-1}^1 (P_1(x))^2 dx = 0.4$
 $\langle P_1, P_2 \rangle = \int_{-1}^1 P_1(x) P_2(x) dx$
 $\langle \frac{3}{2}x^2 - \frac{1}{2}x^0, \frac{5}{2}x^3 - \frac{3}{2}x^1 \rangle = \int_{-1}^1 (\frac{3}{2}x^2 - \frac{1}{2}x^0)(\frac{5}{2}x^3 - \frac{3}{2}x^1) dx$
 $= \int_{-1}^1 (\frac{15}{4}x^5 - \frac{9}{4}x^3 - \frac{5}{4}x^3 + \frac{3}{4}x^1) dx$
 $= \frac{15}{4} \int_{-1}^1 x^5 dx - \frac{9}{4} \int_{-1}^1 x^3 dx - \frac{5}{4} \int_{-1}^1 x^3 dx + \frac{3}{4} \int_{-1}^1 x^1 dx$
 $= \frac{15}{4} \cdot \frac{2}{6} - \frac{9}{4} \cdot \frac{2}{4} - \frac{5}{4} \cdot \frac{2}{4} + \frac{3}{4} \cdot \frac{2}{2}$
 $= \frac{15}{12} - \frac{9}{8} - \frac{5}{8} + \frac{3}{4}$
 $= \frac{15}{12} - \frac{14}{8} + \frac{3}{4} = \frac{15}{12} - \frac{14}{12} + \frac{9}{12} = \frac{10}{12} = \frac{5}{6}$
 Because the inner products are from $\int_{-1}^1 x^a x^b dx$ where $a+b$ is odd.

Figure 12. Val's Written Work

$\langle P_2, P_2 \rangle = \langle \frac{3}{2}x^2 - \frac{1}{2}x^0, \frac{3}{2}x^2 - \frac{1}{2}x^0 \rangle$
 $= \frac{3}{2} \cdot \frac{3}{2} \langle x^2, x^2 \rangle - \frac{3}{2} \cdot \frac{1}{2} \langle x^2, x^0 \rangle - \frac{1}{2} \cdot \frac{3}{2} \langle x^0, x^2 \rangle + \frac{1}{2} \cdot \frac{1}{2} \langle x^0, x^0 \rangle$

Figure 13. Suzan's Written Work

$\langle \frac{5}{2}x^3 - \frac{3}{2}x^1, \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}x^0 \rangle$
 $\frac{5}{2} \cdot \frac{35}{8} \langle x^3, x^4 \rangle - \frac{30}{8} \cdot \frac{5}{2} \langle x^3, x^2 \rangle + \frac{3}{8} \cdot \frac{3}{2} \langle x^3, x^0 \rangle$
 $- \frac{3}{2} \cdot \frac{35}{8} \langle x^1, x^4 \rangle + \frac{3}{2} \cdot \frac{30}{8} \langle x^1, x^2 \rangle - \frac{3}{2} \cdot \frac{3}{8} \langle x^1, x^0 \rangle = 0$
 Annotations: $3+4=7$, $3+2=5$, $3+0=3$, $4+1=5$, $2+1=3$, $1+0=1$.

Figure 14. Gaby's Written Work

$\langle P_2, P_3 \rangle = \langle \frac{3}{2}x^2 - \frac{1}{2}x^0, \frac{5}{2}x^3 - \frac{3}{2}x^1 \rangle$
 $= \frac{3}{2} \cdot \frac{5}{2} \langle x^2, x^3 \rangle - \frac{3}{2} \cdot \frac{3}{2} \langle x^2, x^1 \rangle$
 $- \frac{1}{2} \cdot \frac{5}{2} \langle x^0, x^3 \rangle + \frac{1}{2} \cdot \frac{3}{2} \langle x^0, x^1 \rangle = 0$

Figure 15. Jill's Written Work

$P_2 = \frac{1}{2}(3x^2 - 1)$ $P_3 = \frac{1}{2}(5x^3 - 3x)$
 $\langle P_2, P_2 \rangle = \frac{1}{4} \langle 3x^2 - 1, 3x^2 - 1 \rangle = \frac{1}{4} (9 \langle x^2, x^2 \rangle - 6 \langle x^2, 1 \rangle + \langle 1, 1 \rangle)$
 $\langle P_2, P_2 \rangle = \frac{9}{4} \langle x^2, x^2 \rangle - \frac{3}{2} \langle x^2, x^0 \rangle - \frac{3}{2} \langle x^0, x^2 \rangle + \frac{1}{4} \langle x^0, x^0 \rangle$
 $= \frac{9}{4} \cdot \frac{2}{5} - \frac{3}{2} \cdot \frac{2}{3} - \frac{3}{2} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{2}{1}$
 $= \frac{9}{10} - 1 + \frac{1}{2}$
 $= \frac{9}{10} - \frac{1}{2} = \frac{4}{10} = \frac{2}{5}$
 $\langle P_2, P_4 \rangle = \frac{1}{4} \langle 3x^2 - 1, 5x^3 - 3x \rangle = \frac{1}{4} (15 \langle x^2, x^3 \rangle - 9 \langle x^2, x^1 \rangle - 5 \langle x^0, x^3 \rangle + 3 \langle x^0, x^1 \rangle)$
 $= \frac{15}{4} \langle x^2, x^3 \rangle - \frac{9}{4} \langle x^2, x^1 \rangle - \frac{5}{4} \langle x^0, x^3 \rangle + \frac{3}{4} \langle x^0, x^1 \rangle$
 $= \frac{15}{4} \cdot \frac{2}{7} - \frac{9}{4} \cdot \frac{2}{3} - \frac{5}{4} \cdot \frac{2}{5} + \frac{3}{4} \cdot \frac{2}{1}$
 $= \frac{15}{10} - \frac{9}{10} - \frac{5}{10} + \frac{3}{2}$
 $= \frac{15}{10} - \frac{9}{10} - \frac{5}{10} + \frac{15}{10} = \frac{16}{10} = \frac{8}{5}$
 $\frac{15}{8} - \frac{9}{4} = \frac{15}{8} - \frac{18}{8} = -\frac{3}{8}$
 $-\frac{7}{8} + \frac{5}{4} = -\frac{7}{8} + \frac{10}{8} = \frac{3}{8}$
 $-\frac{3}{8} + \frac{3}{8} = 0$

Figure 16. Andy's Written Work

3.3. Parity Relationships : Students' Comparison of $\langle x^n, x^m \rangle$ and $\langle P_m, P_n \rangle$

Upon generating Legendre polynomials recursively (both analytically and visually), Gaby first verified orthogonality of Legendre Polynomials visually. As an example, she used P_2 and P_3 ; she indicated that the inner product $\langle P_2, P_3 \rangle = \int_{-1}^1 P_2 P_3 dx$ would vanish, while emphasizing the integration limits as -1 and 1 with finger gesture (fig. 17a). She further stated that the inner product might not have vanished had she used different integration limits. Gaby also visually showed that the inner products $\langle P_2, P_4 \rangle$ and $\langle P_3, P_4 \rangle$ both vanish, respectively (fig. 17b-c). She also illustrated the nonvanishing case with the inner products $\langle P_2, P_2 \rangle$ and $\langle P_3, P_3 \rangle$, respectively (fig. 17d-e). Gaby's gestures accompanied with verbal explanations for the vanishing inner product recount the diversity of representation registers (symbolic, gestural, visual, verbal) that are available at this particular instantiation, and the interplay among these registers in an attempt to discover an important fact about Legendre polynomials family.

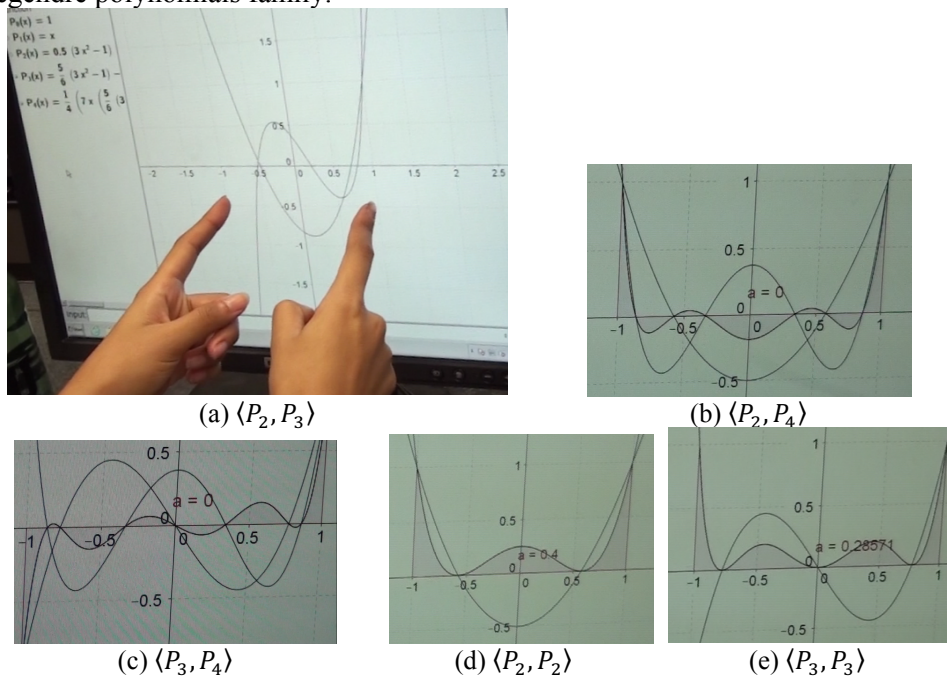


Figure 17. Interplay between Graphical and Gestural Registers

It is important to compare and contrast Gaby's visual arguments about $\langle P_2, P_4 \rangle$ and $\langle P_3, P_4 \rangle$, since these two vanishing inner products cannot be visualized in the same way. As was also articulated by many students including Gaby, the orthogonality of P_2 and P_3 can be visualized in a convincing way by observing that the algebraic product of the functions P_2 and P_3 itself is an odd function; so the orthogonality is a

consequence of the fact that the center of the interval $[-1,1]$ is zero². It is also worth noting the very different situation with P_2 and P_4 , for which there is no equivalent argument (since the orthogonality of these polynomials does not come from the oddity of the algebraic product anymore). In this case, the values -1 and 1 are of crucial importance, not the fact that they are opposite values. Taking opposite values for the bounds of the interval may still simplify the visualization (since P_2P_4 is an even function), but the main issue looks more like an educated guess of the value of a number b for which positive and negative areas defined by P_2P_4 in the interval $[-b,b]$ cancel out. Unfortunately, Gaby's finger gestures do not seem to indicate an understanding of this difficult case.

Several students indicated that the inner product $\langle x^m, x^n \rangle$ becoming zero for the case $m+n$ odd would be analogous to the inner product $\langle P_m, P_n \rangle$ becoming zero for the case $m+n$ odd as well. Suzan, for instance, stated that when $m+n$ is odd, then this would require that $m \neq n$ which implies $P_m \neq P_n$, therefore, the zero inner product $\langle P_m, P_n \rangle = 0$. As shown above, an important stage was students' realization that the inner product $\langle x^m, x^n \rangle$ could be easily calculated via the formula $\langle x^m, x^n \rangle = \begin{cases} \frac{2}{m+n+1}, m+n \text{ even} \\ 0, m+n \text{ odd} \end{cases}$. That said, the realization that the inner product $\langle P_n, P_m \rangle$ could

too be easily calculated by way of a similar formula was not immediate. Liz's first conjecture was that the inner product $\langle P_n, P_m \rangle$ would be nonzero for the case $m+n$ even, just as how it was for the inner product $\langle x^m, x^n \rangle$ for which she had discovered a formula earlier. She suggested the same formula she had established earlier for the inner product of Legendre polynomials as $\langle P_n, P_m \rangle = \frac{2}{m+n+1}$ and used it to calculate several inner products (fig. 18a). She was very happy to see that her visual approach was in agreement with her formula for the calculations of $\langle P_2, P_2 \rangle$ and $\langle P_3, P_3 \rangle$, both pertaining to the case where $m+n$ is even (fig. 18b-c). She then visualized more inner products as she graphed $\langle P_2, P_3 \rangle$ and noted that it vanished (fig. 18d). Liz further deduced that just because $\langle x^m, x^n \rangle$ does not vanish for the case $m+n$ even would not necessarily imply that $\langle P_m, P_n \rangle$ should not vanish for the case $m+n$ even as well. She offered the pair $\langle x^2, x^4 \rangle = \frac{2}{7}$ and $\langle P_2, P_4 \rangle = 0$ as an example to illustrate her thinking. She further stated that the only time the inner product $\langle P_m, P_n \rangle \neq 0$ occurs is when $m=n$. She concluded her explanation by referring to the

² Although she did not generalize her interpretation for any interval of the form $[-a, a]$ in her work with Legendre polynomials, in a follow-up study with the exploration of the orthogonal Hermite polynomials with the associated improper integral inner product $\langle f, g \rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x)g(x)dx$, Gaby often made use of such balls on the x -axis centered at zero as $[-5,5]$, $[-10,10]$, $[-100,100]$, etc. In that aspect, Gaby's finger gestures can be thought of as emphasizing this symmetry.

visualization that the nonvanishing inner products $\langle P_2, P_2 \rangle = 0.4 \neq 0$ and $\langle P_3, P_3 \rangle = 2/7 \neq 0$. Her final comment was that her formula would work only for the case where $m = n$, irrespective of the parity of $m + n$.

The more they explored inner products involving Legendre polynomials, all students eventually discovered that $\langle P_m, P_n \rangle$ would equal zero for $m \neq n$. For the case $m = n$, upon observing the emerging pattern $\langle P_1, P_1 \rangle = \frac{2}{1}, \langle P_2, P_2 \rangle = \frac{2}{3}, \langle P_3, P_3 \rangle = \frac{2}{5}, \langle P_4, P_4 \rangle = \frac{2}{7}, \langle P_5, P_5 \rangle = \frac{2}{9}, \dots$ within the numerical register, students suggested the formula $\langle P_n, P_n \rangle = \frac{2}{2n+1}$ in compact form and frequently used it in the rest of their explorations.

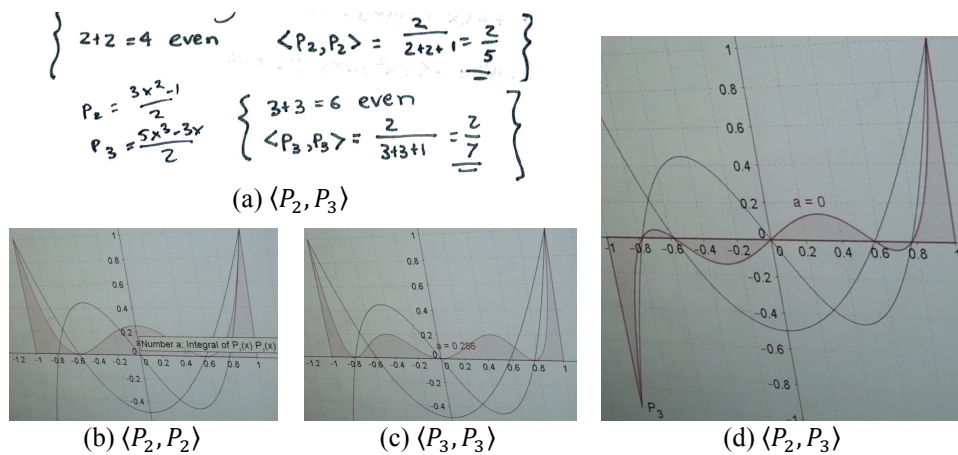


Figure 18. Liz's Conjecture $\langle P_n, P_m \rangle = \frac{2}{m+n+1}$ Tested

3.4. Establishing Orthogonality via Analytic and Visual Approaches

Students used a diversity of techniques in establishing orthogonality of Legendre polynomials both analytically and visually. Within the analytic approach, odd/even integrand Calculus properties were among the more frequently utilized ones. As an example, in his analytic approach, Andy conjectured that the integral $\langle P_1, P_2 \rangle$ would vanish because the integrand would be an odd function. He algebraically verified his conjecture, and even came up with an odd function generic representation of the graph of the integrand (fig. 19a). He further explained that all inner products of the form $\langle P_n, P_n \rangle$ would result in an even integrand, $(P_n)^2$, whose graph is always above the x -axis, producing a nonzero integral. As an example, he worked out the inner product $\langle P_3, P_3 \rangle$; he indicated that the integrand would be even so the integral would not vanish. He also drew an even function generic representation for the integrand as well (fig. 19b).

$$\begin{aligned}
 \int_{-1}^1 P_1(x)P_2(x)dx &= \int_{-1}^1 x \left(\frac{3x^2-1}{2} \right) dx \\
 &= \int_{-1}^1 \left(\frac{3x^3}{2} - \frac{x}{2} \right) dx \\
 &= \left[\frac{3x^4}{8} - \frac{x^2}{4} \right]_{-1}^1 \\
 &= \frac{3}{8} - \frac{1}{4} - \left(\frac{3}{8} - \frac{1}{4} \right) = 0
 \end{aligned}$$

(a)

$$\begin{aligned}
 \int_{-1}^1 P_3(x)P_3(x)dx &= \int_{-1}^1 \left(\frac{5x^3}{2} - \frac{3x}{2} \right) \left(\frac{5x^3}{2} - \frac{3x}{2} \right) dx \\
 &= \int_{-1}^1 \left(\frac{25x^6}{4} - \frac{15x^4}{4} - \frac{15x^4}{4} + \frac{9x^2}{4} \right) dx \\
 &= \int_{-1}^1 \left(\frac{25x^6}{4} - \frac{30x^4}{4} + \frac{9x^2}{4} \right) dx \\
 &= \left[\frac{25x^7}{28} - \frac{30x^5}{20} + \frac{9x^3}{12} \right]_{-1}^1 \\
 &= \frac{25}{28} - \frac{30}{20} + \frac{9}{12} - \left[-\frac{25}{28} + \frac{30}{20} - \frac{9}{12} \right] \\
 &\Rightarrow \frac{2 \cdot 25}{28} - \frac{2 \cdot 1}{6} + \frac{2 \cdot 1}{108} \\
 &= \frac{25}{14} - \frac{1}{3} + \frac{1}{54} \\
 &= \frac{25}{14} - \frac{30}{108} + \frac{9}{108} \left[\frac{25}{14} - 3 + \frac{3}{2} \right] \\
 &= \frac{25}{14} - \frac{3(14)}{14} + \frac{2 \cdot 1}{14} \\
 &= \frac{25}{14} - \frac{42}{14} + \frac{2 \cdot 1}{14} = \frac{4}{14} = \frac{2}{7}
 \end{aligned}$$

(b)

Figure 19. Andy's Reference to Odd/Even Function Properties

Andy's drawn representations are reminiscent of the notion of "content of a representation" in a treatment. Although the integrand $P_1P_2 = 1.5x^3 - 0.5x$ (the actual object) is a vertically stretched cubic polynomial with three x -intercepts, what Andy drew seems to be *his* generic representation of an odd function within the graphical register. Similarly, the integrand $P_3P_3 = (5x^3 - 3x)^2/4$ is a polynomial of degree 6, yet Andy's drawn representation communicates a different content – that of x^2 . The difference between the content of the representation and the actual object in either case (integrand P_1P_2 and integrand P_3P_3) does not seem to pose an obstacle for Andy's explanation for $\langle P_1, P_2 \rangle = 0$ and $\langle P_3, P_3 \rangle \neq 0$, respectively. It is also worth noting that Andy did not recognize the most difficult case (the case $\langle P_m, P_n \rangle (m \neq n)$ with m and n both even or both odd)³.

³ The case $m \neq n$ with even integrand did not catch Andy's attention. Only Liz and Val recognized this situation.

When the interviewer asked Liz her thoughts about the inner products she evaluated (fig. 18a-d), she further explored the three inner products $\langle P_2, P_3 \rangle$, $\langle P_2, P_2 \rangle$, $\langle P_3, P_3 \rangle$ respectively. Regarding the graph of $\langle P_2, P_3 \rangle$, she commented: “*Something seems wrong with it... Like I am not... It does not seem like what I was expecting to see. Like, the value [meaning the value of the inner product $\langle P_2, P_3 \rangle$] seems right but the way that it... I don't know I just... I did not expect that shape [respectively pointing to the areas below and above the x-axis that cancel for the inner product $\langle P_2, P_3 \rangle$ she just calculated (fig. 20a)] For some reason I was just trying to combine the shapes not really multiply them.*” Her reaction about how the shape she obtained was not what she was expecting to see made the interviewer probe further on her response. The interviewer asked her thoughts about the shapes of the inner products $\langle P_2, P_2 \rangle$ and $\langle P_3, P_3 \rangle$ as well. She redid $\langle P_2, P_2 \rangle$ and commented: “*You are multiplying that [meaning P_2] by itself so some type of... I don't think it will actually be orthogonal to itself... I don't expect the ending value to be zero.*” Upon visualizing the inner product $\langle P_2, P_2 \rangle$, she pointed to the area under curve always being above the x-axis as the reason for the nonzero inner product (fig. 20b). She provided a similar explanation for the nonvanishing inner product $\langle P_3, P_3 \rangle$ (fig. 20c).

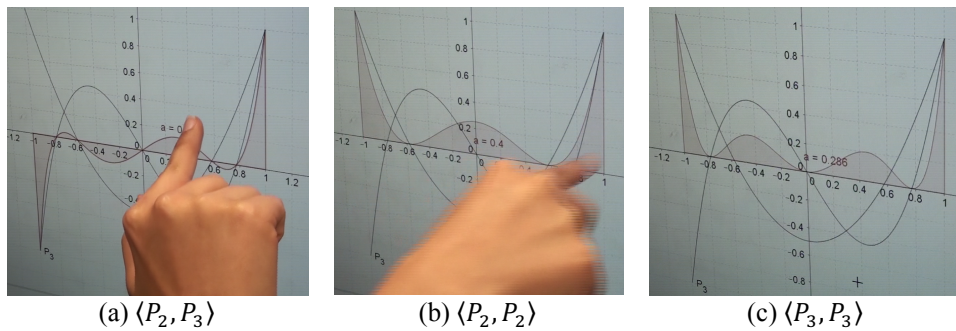


Figure 20. Liz's Visualization of the Inner Products $\langle P_2, P_3 \rangle$, $\langle P_2, P_2 \rangle$, $\langle P_3, P_3 \rangle$

Like Liz, Val did not seem to be bothered with the shape of nonvanishing inner products. However, for vanishing inner products, she was surprised just as how Liz was surprised regarding the shapes that emerged. Val thought aloud: “*So like... I was just thinking... What does it mean for two polynomials to be orthogonal to each other cuz that's like kind of weird because I guess having the property of being orthogonal is normally a geometric thing instead of an algebraic thing... but it's being defined as the inner product is equal to zero I guess... because I tried to zoom out and I tried to see like maybe they look orthogonal but they don't look orthogonal to me.*”

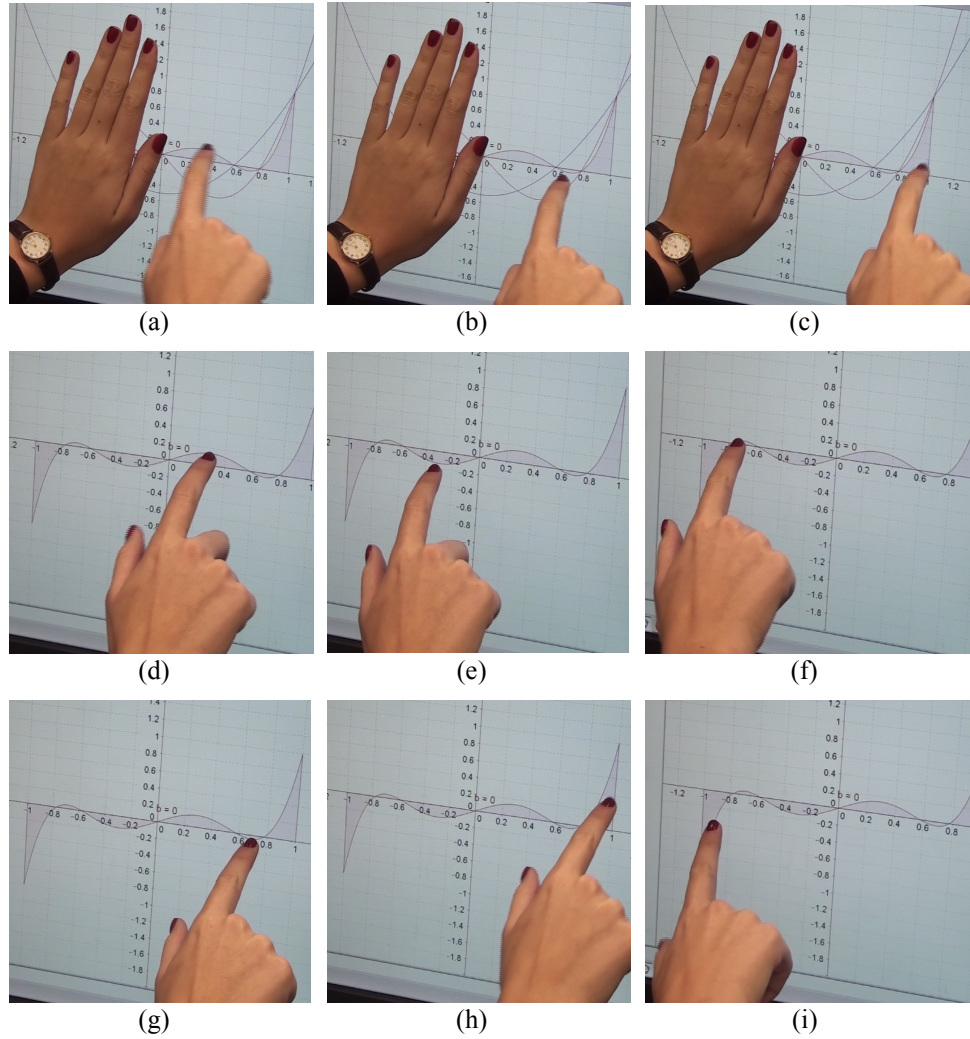


Figure 21. Val's Visualization of the Inner Product $\langle P_2, P_3 \rangle$

Upon completing her algebraic work showing the orthogonality of two Legendre polynomials, P_2 and P_3 (fig. 16a), Val then thought aloud again: “*and how is the inner product defined? The integral of the product of the two functions* [typed **Integral[P_2*P_3,-1,1]** and obtained the definite integral that showed $b=0$ value] *and how does this tell me that they are orthogonal... is it because it's symmetric cuz if you cover this side* [hand gesture covering the left part of the graph] *and cuz it's symmetric with respect to the origin that's why... no way!!!* [very excitedly] *okay yeah so if you cover this side then the area under the curve are these* [covering the

left part of the graph with left hand while respectively finger-pointing to the area pieces on the right (fig. 21a-c)] *and it's symmetric with respect to the origin so it's zero.*"

To clarify her thinking further, she hid the original function graphs and worked with the area pieces only. She emphasized that she focused on the area under the curve of the product of the two Legendre polynomials [pointing to P_2 and P_3 on the Algebra Sideview]: *"I think it mainly has to do with P_3 .. because P_3 is odd so it's symmetric with respect to the origin so I think this one called the shots [highlighting the curve of P_3] in the relationship because this one makes the integral also symmetric with respect to the origin and then because of that fact that means that it's equal to zero because these two cancel out and then these two sections cancel out and then these two cancel out [respectively pointing to the three pairs of congruent pieces (fig. 21d-i)]."* Val's visualization could be thought of as the manifestation of the well-known functional parity property which says that the product of an odd function by an even function is an odd function.

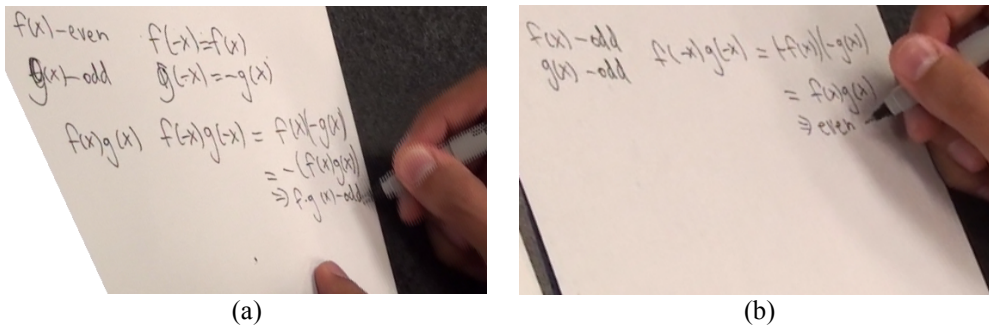


Figure 22. Andy's Analysis of the Parity of the Product Function

Val's visualization is closely related to Andy's analytic approach via which he established this fact. Upon referencing parity relationships in his analysis of inner products involving Legendre polynomials (fig. 22), Andy was certain that the product of a pair of even functions produces an even function, hence, the resulting integrand would be symmetric with respect to the y -axis; he did not feel the need to prove this. He disregarded the even/even case as he thought he already knew that the integrand would be an even function. Nevertheless, he was very eager to prove the other two cases where: (i) one function is even and the other is odd (fig. 22a); (ii) both functions are odd (fig. 22b).

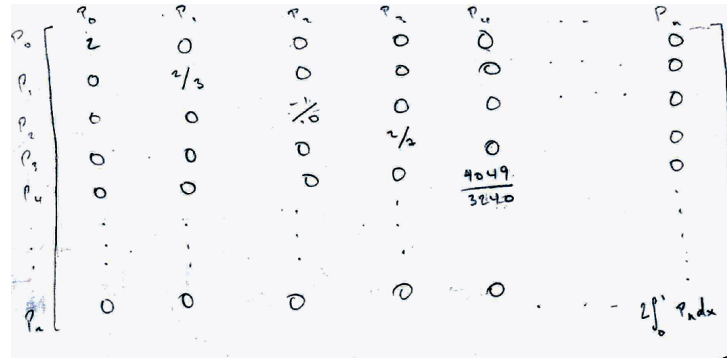
As shown above, graphing inner products involving $\{P_n\}$ was prone to further student observations regarding their connections to various Calculus properties. Val concluded her interpretation of orthogonality of Legendre polynomials with further insight, with particular attention to the inner product of two orthogonal Legendre polynomials with an integrand that is not symmetric with respect to the origin: "So

if the inner product between two functions is symmetric with respect to origin then that means that their inner product is equal to zero but probably if they are not symmetric with respect to origin like let's say.. there's probably some weird functions that are not symmetric with respect to origin but are orthogonal [offers the inner product of P_2 by P_4 as an example] so they are always orthogonal as long as it's not the inner product with respect to itself."

Why did Liz and Val seem to get surprised when they saw geometrically the graphs of the vanishing inner products? This is a very important and highly relevant question which can be addressed within the representation registers theoretical framework (Duval, 1993, 2006). On the one hand, there is the notion of geometric orthogonality – which students are already familiar with – within the geometric register. Having already explored geometric orthogonality in the context of Euclidean inner product spaces \mathbb{R}^2 and \mathbb{R}^3 , it can be postulated that perhaps these students were expecting some kind of geometric type objects (like line segments or triangles or polygons) in which perpendicularity could be observed geometrically. On the other hand, there is the notion of functional orthogonality which is principally defined as a definite integral inner product in the context of polynomial inner product spaces within a totally different semiotic register – a combination of graphical, visual, algebraic representation registers. A cognitive conflict, perhaps, can be thought of as arising as a result of the clash of registers, in particular that of geometric vs. analytic/graphical registers. Liz's confusion "*I did not expect that shape...For some reason I was just trying to combine the shapes not really multiply them*" and Val's confusion "*I tried to see like maybe they look orthogonal but they don't look orthogonal to me*" could be explained with this notion of registral conflict. Liz perhaps wanted to have a linear combination of some objects kind of like how she used to express a certain given inner product as an expansion of certain number of other inner products. As for Val, she perhaps was searching for some kind of perpendicular looking objects, as corroborated by her statement "*kind of weird because I guess having the property of being orthogonal is normally a geometric thing instead of an algebraic thing*" These two students' confusions seemed to be addressed and resolved at the end by their willingness to make sense of the situation via some sort of bridge-building strategies in which they formed a synergy of multiple representation registers that are meaningfully connected to one another.

To summarize, students' successful conversions between the analytic and graphical semiotic registers resulted in the discovery of the following three aspects of inner products $\langle P_m, P_n \rangle$ involving Legendre polynomials: (1) $m + n$ odd $\Rightarrow m \neq n$ so the inner products $\langle x^m, x^n \rangle$ and $\langle P_m, P_n \rangle$ should behave in a similar way, that is, they must both vanish – the result (possibly noticed only for the particular case of P_2 and P_3) that $\langle P_m, P_n \rangle = 0$ whenever m is odd and n even (or m even and n odd), proved by the fact that $P_m P_n$ is an odd function; (2) if $m + n$ is an even number, then there

are two possibilities, that is, $m = n$ or $m \neq n$; (2a) When $m + n$ is even and $m = n$ then the inner products $\langle x^m, x^n \rangle$ and $\langle P_m, P_n \rangle$ behave in a similar way, that is, they both equal $\frac{2}{n+m+1}$ in value – the fact that $\langle P_n, P_n \rangle$ is strictly positive for any n , visually proven by the fact that the curve $(P_n)^2$ always remain above the x -axis; (2b) When $m + n$ is even and $m \neq n$ (the case m and n both even or both odd (and different from each other), the most difficult one) then the inner products $\langle x^m, x^n \rangle$ and $\langle P_m, P_n \rangle$ differ in behavior, that is, while the inner product $\langle x^m, x^n \rangle$ equals $\frac{2}{n+m+1}$ in value, the inner product $\langle P_m, P_n \rangle = 0$.



(a)

```
>> syms x
>> P0=1;P1=x;P2=(3*x^2-1)/2;P3=(5*x^3-3*x)/2;
>> syms x
>> P0=1;P1=x;P2=(3*x^2-1)/2;P3=(5*x^3-3*x)/2; P4=(35*x^4-30*x^2+3)/8;
>> [int(P0*P0,x,-1,1) int(P0*P1,x,-1,1) int(P0*P2,x,-1,1) int(P0*P3,x,-1,1) int(P0*P4,x,-1,1);
int(P1*P0,x,-1,1) int(P1*P1,x,-1,1) int(P1*P2,x,-1,1) int(P1*P3,x,-1,1) int(P1*P4,x,-1,1);
int(P2*P0,x,-1,1) int(P2*P1,x,-1,1) int(P2*P2,x,-1,1) int(P2*P3,x,-1,1) int(P2*P4,x,-1,1);
int(P3*P0,x,-1,1) int(P3*P1,x,-1,1) int(P3*P2,x,-1,1) int(P3*P3,x,-1,1) int(P3*P4,x,-1,1);
int(P4*P0,x,-1,1) int(P4*P1,x,-1,1) int(P4*P2,x,-1,1) int(P4*P3,x,-1,1) int(P4*P4,x,-1,1)]

ans =

[ 2, 0, 0, 0, 0]
[ 0, 2/3, 0, 0, 0]
[ 0, 0, 2/5, 0, 0]
[ 0, 0, 0, 2/7, 0]
[ 0, 0, 0, 0, 2/9]
```

(b)

Figure 23. Kyra's Matrix Model for Orthogonality of Legendre Polynomials

This section concludes with Kyra's modeling of the orthogonality condition in coordination with her analytic and numerical/symbolic approaches in MATLAB. Upon establishing that the inner product $\langle P_n, P_m \rangle = \int_{-1}^1 P_n P_m dx = \frac{2}{2n+1} \delta_{nm}$ would be nonzero for $m = n$ only, she proposed a diagonal matrix model, the Gramian matrix, with generic diagonal entry of the form $2 \int_0^1 P_n P_n dx$. Like Liz, Kyra can be

said to embrace a simplified definite integral representation within the symbolic register, that is, the equivalence of $\int_{-1}^1 x^{m+n} dx$ and $2 \int_0^1 x^{m+n} dx$, for the particular case $m = n$. She separately calculated each diagonal entry one by one, and drew the matrix model as shown (fig. 23a). She then switched to MATLAB and calculated all the entries (including off-diagonal ones) to confirm the orthogonality condition (fig. 23b). This approach helped her go back to her written work and fix some of the incorrectly calculated fractions.

3.5. Visualizing Further Inner Product Properties : Cauchy-Schwartz, Triangle Inequalities, Pythagorean Theorem, Parallelogram Law

3.5.1. Cauchy-Schwarz Inequality

Students found the Cauchy-Schwarz Inequality (CSI) $\langle P_n, P_m \rangle \langle P_n, P_m \rangle \leq \langle P_n, P_n \rangle \langle P_m, P_m \rangle$ straightforward to explain. Especially after establishing parity relationships involving inner products $\langle x^n, x^m \rangle$ and $\langle P_n, P_m \rangle$ for the case $m + n$ even / odd, students analyzed the CSI in two cases: (i) $m \neq n \Rightarrow P_n \perp P_m \Rightarrow \langle P_n, P_m \rangle = 0$ so the LHS would be of the form $0 \cdot 0$ and the RHS would be the product of positive numbers so CSI is verified; (ii) $m = n \Rightarrow$ the LHS would equal the RHS so the CSI is once again verified. A typical student work looked like the numerical verification for the case where $m \neq n$ (fig. 24a).

$$\begin{aligned} \langle P_2, P_3 \rangle &= 0 \\ \langle P_2, P_2 \rangle &= 0.4 = \frac{2}{5} \\ \langle P_3, P_3 \rangle &= 0.28571 = \frac{2}{7} \\ 0.0 &\leq \frac{2}{5} \cdot \frac{2}{7} \\ 0 &\leq \frac{4}{35} \quad \checkmark \end{aligned}$$

(a)

$$\begin{aligned} \|P_3 + P_4\| &= \sqrt{\int_{-1}^1 (P_3 + P_4)^2 dx} \\ &= \sqrt{\int_{-1}^1 (P_3^2 + 2P_3P_4 + P_4^2) dx} \\ &= \sqrt{\int_{-1}^1 P_3^2 dx + \int_{-1}^1 P_4^2 dx} = \sqrt{\frac{2}{7} + \frac{2}{9}} \\ &= \sqrt{\frac{2}{7} + \frac{2}{9}} \leq \sqrt{\frac{2}{7}} + \sqrt{\frac{2}{9}} \end{aligned}$$

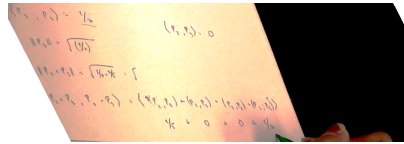
(b)

$$\begin{aligned} \|P_1 + P_2\| &\leq \|P_1\| + \|P_2\| \\ \left\| x + \frac{3x^2-1}{2} \right\| &\leq \|x\| + \left\| \frac{3x^2-1}{2} \right\| \\ \sqrt{\int_{-1}^1 \left(x + \frac{3x^2-1}{2} \right)^2 dx} &= \sqrt{\int_{-1}^1 \left(x^2 + x(3x^2-1) + \frac{(3x^2-1)^2}{4} \right) dx} \\ &= \sqrt{2 \int_0^1 \left(x^2 + \frac{(3x^2-1)^2}{4} \right) dx} \quad 9x^4 - 6x^2 + 1 \\ &= \sqrt{\frac{8}{15}} \leq \sqrt{\frac{2}{3}} + \sqrt{\frac{2}{5}} \end{aligned}$$

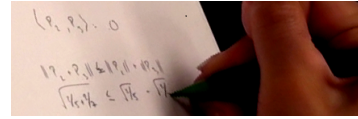
(c)

Figure 24. Students' Verification of Cauchy-Schwarz (a) and Triangle (b-c) Inequalities

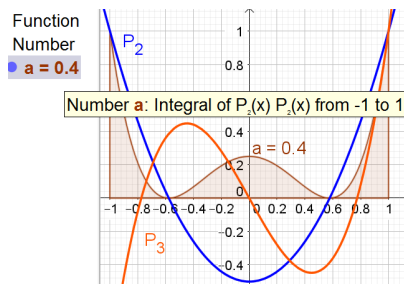
In the verification of inner product properties such as Triangle Inequality (TI), Pythagorean Theorem (PT), and Parallelogram Law (PL), most students dropped the long way of calculations and referenced the inner products involving powers of x . As was the case for the CSI, students thought of TI as a trivial property of inner products to verify. Andy was one of the few who verified TI both via the short way (fig. 24b) and the long way (fig. 24c), algebraically.



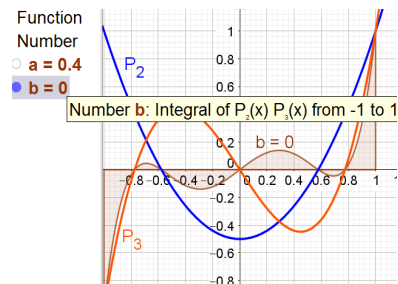
(a)



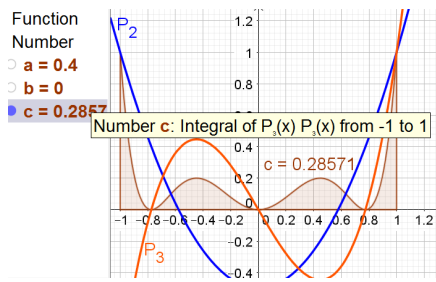
(b)



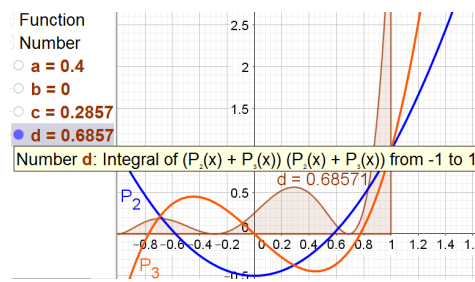
(c)



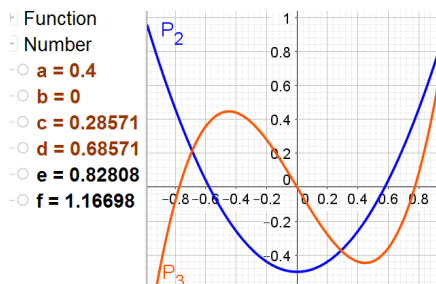
(d)



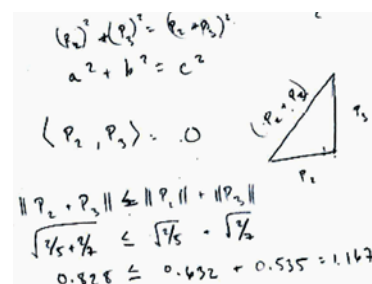
(e)



(f)



(g)



(h)

Figure 25. Kyra's Verification of Triangle Inequality $\|P_2 + P_3\| \leq \|P_2\| + \|P_3\|$

3.5.2. Triangle Inequality

Kyra explored Triangle Inequality in four main steps: (i) Writing the inner product $\langle P_2 + P_3, P_2 + P_3 \rangle$ as a sum of inner products as $\langle P_2, P_2 \rangle + \langle P_2, P_3 \rangle + \langle P_3, P_2 \rangle + \langle P_3, P_3 \rangle$; (ii) Evaluation of the inner product pieces $\langle P_2, P_2 \rangle, \langle P_2, P_3 \rangle, \langle P_3, P_2 \rangle, \langle P_3, P_3 \rangle$; (iii) Evaluation of the norms $\|P_2 + P_3\|, \|P_2\|, \|P_3\|$; (iv) Verification that $\|P_2 + P_3\| \leq \|P_2\| + \|P_3\|$ (fig. 25a-b). In the verification of the inequality involving square roots, she made use of the software and visually obtained these values as definite integrals (fig. 25c-g). Her thoughts about the Triangle Inequality was: “*The inequality is saying that the length of this vector here [highlighting the hypotenuse of the right triangle she just drew] is shorter than the sum of these two [highlighting the legs of the right triangle (fig. 25h)].*” In the calculation of $\langle P_2, P_3 \rangle$ and $\langle P_3, P_2 \rangle$ via the analytic and visual approaches, she only calculated $\langle P_2, P_3 \rangle$ as she explained that $\langle P_2, P_3 \rangle = \langle P_3, P_2 \rangle$ by the defining property (symmetry) of inner product. To verify the inequality, she respectively typed **e=sqrt(d)** and **f=sqrt(a)+sqrt(c)**, which she recorded numerically as well (fig. 25g). Upon the interviewer’s probing why she drew a right triangle and not another type of triangle, she responded: “*it has to be a right triangle in this case because P_2 and P_3 are orthogonal.*”

3.5.3. Pythagorean Theorem

Common approach used to explain the Pythagorean Theorem $\|P_n + P_m\|^2 = \|P_n\|^2 + \|P_m\|^2$ was a quick reference to the previously explored Triangle Inequality $\|P_n + P_m\| \leq \|P_n\| + \|P_m\|$. To verify the Pythagorean Theorem, most students simply referred to the right triangles they had previously drawn in the modeling of Triangle Inequality (fig. 26a). As an example, to explain the Pythagorean Theorem $\|P_2 + P_3\|^2 = \|P_2\|^2 + \|P_3\|^2$, Gaby stated: “*It’s the same equation except no square roots and it’s equal to instead of being less than or equal to.*” For further clarification, she also made use of a familiar example, the 3-4-5 right triangle, to verify both the Triangle Inequality and the Pythagorean Theorem (fig. 26b).

All students basically offered a statement similar to Gaby’s to explain how they deduce Pythagorean Theorem from Triangle Inequality for a pair of Legendre polynomials P_n and P_m where $n \neq m$. In an attempt to come up with a generalization for the Pythagorean Theorem for any pair of Legendre polynomials, Andy first worked out the inner products for the LHS and the RHS and verified the Pythagorean Theorem for $n \neq m$ by concluding $\frac{2}{2n+1} + \frac{2}{2m+1} = \frac{2}{2n+1} + \frac{2}{2m+1}$ (fig. 26d). Curious about the case $n = m$, he showed that the sum $\|P_n\|^2 + \|P_m\|^2$ could no way equal $\|P_n + P_m\|^2$ for the case $n = m$ – otherwise he would obtain the impossibility $\frac{2}{2n+1} \cdot 2 = 4 \left(\frac{2}{2n+1} \right)$. He further concluded that this impossibility would correspond to not having a triangle by drawing two congruent parallel vectors: “*You can think of it as*

this same vector [draws two congruent parallel vectors] and since they are parallel there is no triangle made (fig. 26c)."

Handwritten work for part (a):

$$\|P_2\| = \sqrt{\frac{2}{5}} \quad \|P_3\| = \sqrt{\frac{2}{7}}$$

$$\langle P_2 + P_3, P_2 + P_3 \rangle = \langle P_2, P_2 \rangle + \langle P_2, P_3 \rangle + \langle P_3, P_2 \rangle + \langle P_3, P_3 \rangle$$

$$\frac{2}{5} + 0 + 0 + \frac{2}{7} = \frac{24}{35}$$

$$\sqrt{\frac{24}{35}} \leq \sqrt{\frac{2}{5}} + \sqrt{\frac{2}{7}}$$

$$0.82303 \leq 1.16698 \quad \checkmark$$

$$\frac{2}{5} + \frac{2}{7} = \frac{24}{35}$$

$$\frac{24}{35} = \frac{24}{35} \quad \checkmark$$

Diagram: A right triangle with sides labeled $\|P_2\|$, $\|P_3\|$, and $\|P_2 + P_3\|$.

(a)

Handwritten work for part (b):

Diagram: A right triangle with sides 3, 4, and hypotenuse 5.

$$5 < 4 + 3$$

$$5 < 7$$

(b)

Handwritten work for part (c):

$$\left(\frac{2}{2n+1}\right) \cdot 2 = \frac{2}{2n+1} + 2 \cdot \frac{2}{2n+1} + \frac{2}{2n+1}$$

$$= 4 \left(\frac{2}{2n+1}\right)$$

Diagram: A triangle with sides represented by arrows.

(c)

Handwritten work for part (d):

$$\|P_n\|^2 + \|P_m\|^2 = \|P_n + P_m\|^2 \quad n \neq m$$

$$\int_{-1}^1 P_n(x)^2 dx + \int_{-1}^1 P_m(x)^2 dx = \int_{-1}^1 (P_n(x) + P_m(x))^2 dx$$

$$= \frac{2}{2n+1} + \frac{2}{2m+1} = \frac{2}{2n+1} + \frac{2}{2m+1}$$

(d)

Figure 26. Students' Modeling of the Pythagorean Theorem

In her exploration of the Triangle Inequality $\|P_2 + P_3\| \leq \|P_2\| + \|P_3\|$, Val first calculated all three norms she needed algebraically (fig. 27a). She then wanted to draw a triangle with sides corresponding to the norms she just evaluated, but she was not sure what type of triangle to draw so she first planned to draw a triangle with sides approximated as 0.5, 0.6, and 0.8. Upon drawing a triangle made of tick-marks, she first conjectured that it would be an acute triangle that is almost a right triangle

(fig. 27b). In this sense, Val's verification was similar to Gaby's with the slight difference that instead of the familiar 3-4-5 right triangle, she made use of a 5-6-8 triangle she obtained as an approximation of the sides corresponding to the three norms $\|P_3\|$, $\|P_2\|$, $\|P_2 + P_3\|$ respectively.

$$\langle P_3, P_3 \rangle = \int_{-1}^1 (P_3(x))^2 dx = \frac{2}{2(3)+1} \delta_{3,3} = \frac{2}{6+1} = \frac{2}{7}$$

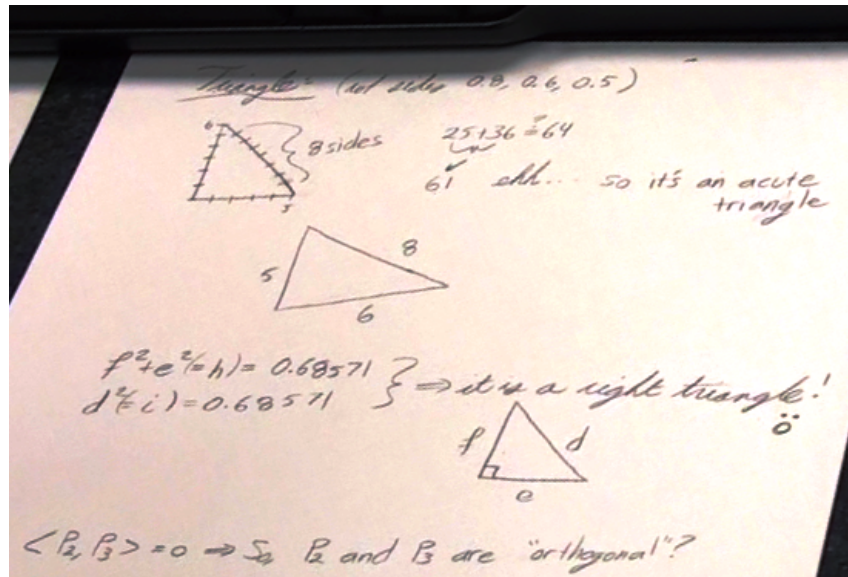
$$\|P_2(x)\| = \left(\int_{-1}^1 P_2^2(x) dx \right)^{1/2} = \left(\frac{2}{2 \cdot 2 + 1} \right)^{1/2} = \left(\frac{2}{5} \right)^{1/2} = 0.63246$$

$$\|P_3\| = \left(\frac{2}{6+1} \right)^{1/2} = \left(\frac{2}{7} \right)^{1/2} = 0.53452$$

$$\|P_2(x) + P_3(x)\| = \left(\int_{-1}^1 (P_2 + P_3)^2 dx \right)^{1/2} = 0.82808$$

$$0.82808 \stackrel{d}{=} 0.63246 \stackrel{e}{=} 0.53452 \stackrel{g}{=} 1.16698$$

(a)



(b)

Figure 27. Val's Verification of the Pythagorean Theorem $\|P_2 + P_3\|^2 = \|P_2\|^2 + \|P_3\|^2$

Comparative analysis of Gaby, Liz, and Val's approaches indicate the discrepancy between the orthogonality of the (P_n) s and the orthogonality of the sides of the 3-4-5 triangle: the visualization of the former involves algebraic areas representations (i.e. the standard geometrical interpretation of integrals) whereas the visualization of the latter relies on perpendicular segments representations. Hence, these semiotic

registers rely on very different concepts. In particular, the triangle in Figure 27b shows that Liz's interpretation of norms is not really satisfactory (since she writes inscriptions like $\|P_3, P_3\|$ instead of $\|P_3\|$ for the length of a side). An important question to address is the following: Do the students really get the common structure lying behind geometric orthogonality and functional orthogonality, or is it for them nothing more than a handy analogy, helpful to remind some theorems (like the Triangle Inequality) but without any fundamental importance? Whereas an awareness of the functional orthogonality, as shown above, is a product of a thorough understanding of a vanishing integral in the realm of a graphical semiotic register, an awareness of geometric orthogonality could be thought of as a result of a successful interpretation of the notational (inscriptive) semiotic register. Figure 27b is especially interesting in this context, with the quotation marks on the word "orthogonal" followed by a question mark that indicates Val's progress in the understanding of the theoretical definition of orthogonality. It can be postulated that at the expense of various registral clashes occurring at various instantiations do the students eventually appear to associate these two concepts (the geometric orthogonality and the functional orthogonality) in a reconciliatory manner via conversion between the graphical and the inscriptive semiotic registers.

3.5.4. *Parallelogram Law*

Parallelogram Law $\|P_n + P_m\|^2 + \|P_n - P_m\|^2 = 2\|P_n\|^2 + 2\|P_m\|^2$ required the calculation of distance between vectors $\|P_n - P_m\|$. Similar to Kyra's exploration of the Triangle Inequality (in which she had calculated the norm of vector sum $\|P_n + P_m\|$), Ted followed a step-by-step approach in which he made use of the defining properties of inner products (symmetry and linearity) to calculate $\|P_0 + P_1\|^2$ and $\|P_0 - P_1\|^2$. In his calculation of $\|P_0 + P_1\|^2$, $\|P_0 - P_1\|^2$, $\|P_0\|^2$, $\|P_1\|^2$, Ted used f and g to denote P_0 and P_1 , respectively (fig. 28a-d). To verify the equality $\|P_0 + P_1\|^2 + \|P_0 - P_1\|^2 = 2\|P_0\|^2 + 2\|P_1\|^2$ at the last step, he respectively typed **a+b** and **2c+2d**, which appeared on the Algebra Sideview as **e=5.333** and **h=5.333**, respectively. He explained that the equality $\|P_0 + P_1\| = \|P_0 - P_1\|$ would occur only for a pair of orthogonal vectors. When asked to draw a parallelogram illustrating the Parallelogram Law, he drew a rectangle with sides representing the norms $\|P_0\|$ and $\|P_1\|$. He concluded his verification by stating that the two diagonals of the rectangle correspond to the norm of vector sum $\|P_0 + P_1\|$ and the distance between vectors $\|P_0 - P_1\|$, respectively.

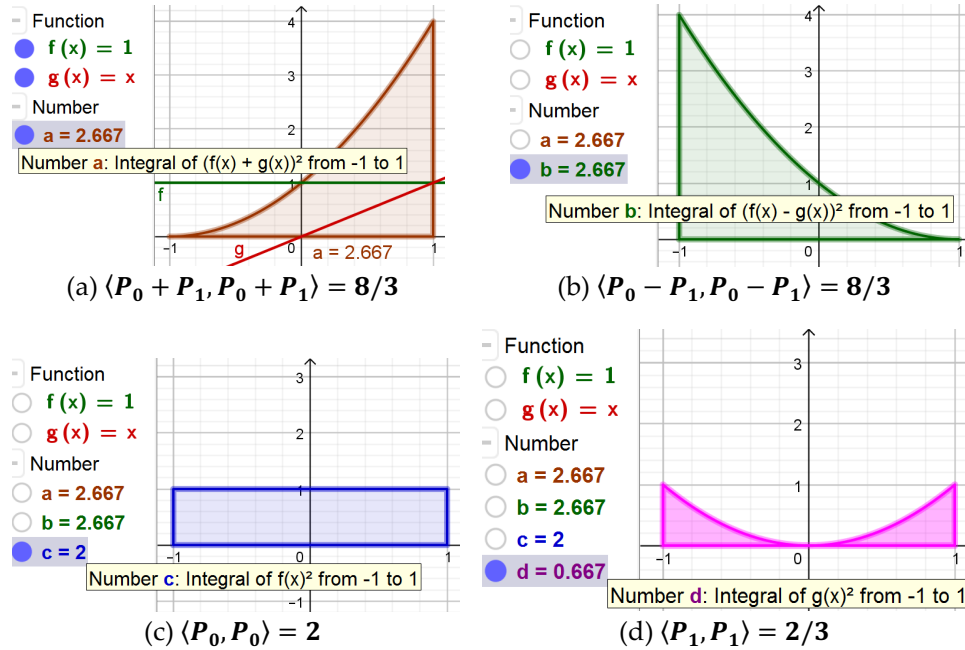


Figure 28. Exploring Parallelogram Law

4. Discussion and Conclusions

4.1. Teaching and Learning Legendre Polynomials

Coordination of multiple representation registers, the visual and the analytic registers in particular, played a significant role in university linear algebra students' learning and understanding of orthogonal Legendre polynomials in the polynomial inner product space with the associated integral inner product, along with its interrelationships with the previously mastered concepts of linear algebra, analysis, or even geometry. In such a multi-representational setting, students not only strengthened their own understanding of inner product properties and inner product spaces in general, but came up with certain other conjectures and properties in a high-tech environment as well. Such practices include but not limited to: (i) Adoption of a simplified definite integral representation within the symbolic register, which helped in understanding why a certain inner product of the form $\langle x^m, x^n \rangle$ should vanish, or be simplified as $2 \int_0^1 x^{m+n} dx$; (ii) Ability to expand inner products $\langle P_m, P_n \rangle$ involving Legendre polynomials as a linear combination of elementary inner products $\langle x^m, x^n \rangle$ involving powers of x within the symbolic and algebraic registers; (iii) Awareness of the two important notions about Legendre polynomials family (the geometric orthogonality and the functional orthogonality); (iv) Distinction between the geometric orthogonality and the functional orthogonality by

way of representation registers within which these notions are interpreted (the geometric orthogonality within the geometric register vs. the functional orthogonality within a combination of graphical, visual, algebraic representation registers); (v) Ability to overcome potential cognitive conflicts (e.g., as a result of registral clash) by forming a synergy of multiple representation registers.

4.2. Implications and Directions for Future Research

The present report proposes the visual-analytic strategy coordination as an appropriate strategy for the teaching and learning of Legendre polynomials and the integral inner product properties in linear algebra. Moreover, the findings of the present report proposes DGS besides MATLAB as an important environment for visualizing linear algebraic structures in action, in particular, inner product spaces in linear algebra. An important inference from the above analysis concerns the availability of multiple semiotic registers of representations. The constant interplay among these registers could be thought of as having a great impact on students' learning and understanding of the orthogonal family of Legendre polynomials as vectors of the polynomial inner product space, along with a thorough and meaningful interpretation of certain geometric type inequalities and theorems coming to interplay. As Duval (1993) put it, the mathematical object and its representation are not the same thing – only by way of multiple representations and by way of richness of the interplays taking place among them could students improve their understanding in such a way that this understanding will eventually converge to knowledge. An awareness of all these pieces satisfactorily working in different semiotic registers and meaningfully connecting to each other could be thought of evidence of learning and knowledge in the framework of representation registers (Duval, 1993, 2006).

The study revealed important findings on university mathematics majors' understanding and sense-making of Legendre polynomials and integral inner product properties. Having illustrated many properties inherent in inner products by exploring the inner product space $\mathbb{P}_n(\mathbb{R})$ with the integral inner product $\langle p, q \rangle = \int_{-1}^1 p(x)q(x)dx$ offered the linear algebra students opportunities to explore polynomials defined on a closed interval as vectors, and to make up for and strengthen, where appropriate, vector space and subspace properties associated with polynomials. In almost every situation were students able to investigate such properties in analytic and visual modes in a coordinative manner. As shown above, students came up with important discoveries as a consequence of successful coordination of visual-analytic approaches in the DGS-MATLAB environment. There still seems to be need for further research to clarify certain aspects of inner product spaces, in particular, the integral inner product in the vector space $\mathbb{P}_n(\mathbb{R})$ of polynomials of degree $\leq n$ with real coefficients. Potential avenue for future

investigation could focus on math majors' understanding and interpretation of $\mathbb{P}_n(\mathbb{R})$ and the inner product space properties by using an improper integral inner product $\langle f, g \rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x)g(x)dx$ with particular emphasis on Hermite polynomials, just as the same way Legendre polynomials with the integral inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$ have been investigated in a VA-pedagogy.

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Appendix

Item/Task	Generating Legendre Polynomials: (i) Recursive Approach: $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$ with $P_0 = 1$ and $P_1 = x$ to determine the polynomial sequence $\{P_0 = 1, P_1 = x, P_2 = \frac{3x^2-1}{2}, P_3 = \frac{5x^3-3x}{2}, \dots\}$ for $n \geq 2$; (ii) Rodrigues' Formula: $P_n = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$ for all $n \geq 0$.
Sample Probing Questions	What does it mean for P_n to be an odd / even function? Does this have anything to do with n being odd / even? How is this related to the fact that for the case n odd / even, P_n contains only odd / even powers of x ?
Item/Task	Evaluating and Interpreting Inner Products $\langle P_m, P_n \rangle$ for $m = n$ and $m \neq n$:
Sample Probing Questions	What does it mean for Legendre polynomials to be of same / different parity? How does this manifest in the calculation of inner products $\langle P_n, P_m \rangle$ for Legendre polynomials of different parity, analytically? visually? How do you reconcile the orthogonality condition $\langle P_n, P_m \rangle = \frac{2}{n+m+1} \delta_{nm}$ with the visual approach? What does it mean for the inner product $\langle P_n, P_m \rangle$ to be zero or nonzero, analytically? visually?
Item/Task	Parity Relationships: Establishing Orthogonality via Analytic and Visual Approaches
Sample Probing Questions	What are the conditions for $\langle P_n, P_m \rangle$ to result in a zero / positive inner product? What does this have to do with P_n and P_m being of different / same parity, analytically? visually? How do elementary Calculus properties play out in such analyses and visualizations?
Item/Task	Geometry Connections : Cauchy-Schwartz, Triangle Inequalities, Pythagorean Theorem, Parallelogram Law
Sample Probing Questions	How do you prove / verify the Cauchy-Schwartz Inequality / Triangle Inequality / Pythagorean Theorem / Parallelogram Law? How do you reconcile your analytic and visual approaches? How does orthogonality manifests for such geometric inequalities and theorems, analytically? visually? How would you model the Cauchy-Schwartz Inequality / Triangle Inequality / Pythagorean Theorem / Parallelogram Law in a drawing? Why did you draw a right triangle? How would you label / interpret the sides of your triangle? How does orthogonality manifest itself in your model?
Item/Task	Orthogonal / Orthonormal Bases for $\mathbb{P}_n(\mathbb{R})$
Sample Probing Questions	What do you propose as an orthogonal / orthonormal basis for $\mathbb{P}_2(\mathbb{R})$ / $\mathbb{P}_3(\mathbb{R})$ / $\mathbb{P}_4(\mathbb{R})$ using the Legendre polynomials? What does it mean for a polynomial in $\mathbb{P}_2(\mathbb{R})$ / $\mathbb{P}_3(\mathbb{R})$ / $\mathbb{P}_4(\mathbb{R})$ to be written as a linear combination of Legendre polynomials (or normalized Legendre polynomials), visually? analytically? How would you analyze / visualize inner product of Legendre polynomials (or normalized Legendre polynomials)? How would you reconcile the two approaches? How does coordinates relative to an orthogonal / orthonormal basis manifest themselves analytically? visually? How would you reconcile the two approaches?