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ANNALES de DIDACTIQUE et de SCIENCES COGNITIVES
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English-French use of theories in mathematics teaching, teaching development and teacher education

MAHA ABBOUD, ALF COLES (Guest editors)

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Maha Abboud, Alf Coles

Le theme des pratiques d'enseignement en didactique des mathematiques : developpement d'une collaboration franco-anglaise sur le role des theories

Résumé. Ce numéro spécial est le résultat d'une collaboration de trois ans entre didacticiens français et européens. Nous y présentons comment cette collaboration a vu le jour et s'est développée avec au cœur du travail, le rôle des théories dans les recherches menées par les participants. Nous exposons les thèmes principaux des articles et utilisons le concept d'objet-frontière pour rendre possibles la comparaison et l'enrichissement des différentes perspectives. Ce travail de collaboration dans la durée, de chercheurs venant de traditions diverses nous semble important à l'heure actuelle dans un contexte où l'institution s'oriente vers la recherche de solutions aux problèmes éducatifs en se tournant vers des pays ayant de meilleurs résultats dans les évaluations internationales. Nous concluons par des perspectives de travail, aussi bien pour notre propre groupe de travail que pour des collaborations plus larges en didactique des mathématiques.

Mots-clés. Didactique des mathématiques, approches franco-anglaises, théories, pratiques.

Introduction

Ce numéro spécial est le résultat d'une collaboration, qui a duré trois ans, entre des chercheurs travaillant dans le milieu français de la didactique des mathématiques et celui anglo-saxon de '*mathematics education'* (de nationalités britannique, norvégienne et grecque). La raison initiale de ce rassemblement de chercheurs européens est une volonté commune de discuter et comparer les cadres théoriques utilisés pour l'étude des pratiques enseignantes et de la formation des enseignants. Nous avions dès le début conscience de l'existence dans les approches franco-anglaises de domaines d'intérêt commun, mais avec peu de connaissances mutuelles suffisamment approfondies des détails de ces approches. Le voyage que nous avons entrepris et qui a abouti à ce numéro spécial a nécessité d'abord de trouver des 'façons' de travailler ensemble et ensuite de délimiter les questions de recherche et préciser les données qui permettraient à nos 'conversations' de prendre corps. Dans cet article introductif, notre objectif est de présenter le cheminement de ce voyage afin d'illustrer les éléments qui ont, ou non, rendu cette collaboration raisonnée, utile et qui ouvrent des perspectives pour d'autres types de collaborations. Pour ce faire, nous utilisons le concept d'objet-frontière (*boundary object*) défini par Star et Griesemer (1989), nous présentons succinctement les thèmes et théories travaillés à travers les articles de ce numéro et nous concluons

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par une discussion sur l'intérêt d'une telle collaboration en ouvrant des perspectives pour l'avenir.

1. Voyage vers un numéro spécial

Notre première réunion a eu lieu en janvier 2014. Nous avons commencé par partager nos idées et questions sur la façon dont les théories sont ou peuvent être utilisées pour l'étude des pratiques d'enseignement et de formation. Nous ne voulions pas nous contenter d'exposer nos travaux et nos idées les uns aux autres, mais plutôt construire un espace de travail dans lequel des idées peuvent être partagées et où d'autres peuvent émerger. Trois dimensions de travail étaient alors en jeu. La première est centrée sur les théories, permettant ainsi aux participants de mutualiser leurs cadres et discuter leurs perspectives théoriques. La deuxième correspond à l'objectif principal de nos recherches respectives : l'étude des pratiques d'enseignement. Enfin, la troisième dimension est celle de l'intérêt que nous portons à l'étude du développement professionnel des enseignants et à l'impact de nos approches théoriques sur la formation des enseignants.

Les rencontres qui ont suivi cette première réunion, ainsi que le travail continu de collaboration, nous ont donné en premier lieu la possibilité 'd'apprendre' les uns des autres. Cette collaboration a ensuite évolué vers la constitution de petits groupes autour de thématiques ou d'approches spécifiques. Le travail de ces petits groupes a donné lieu à son tour à des écrits communs dont l'aboutissement est ce numéro spécial avec les différents articles qui le composent[1].

Inscrire ce travail collaboratif dans la durée s'est révélé nécessaire, notamment pour passer de l'écoute mutuelle au travail sur des objets communs supports de la co-construction d'analyses comparatives. Ces objets peuvent être qualifiés d'objets-frontière, et dans certains cas d'objets-frontière améliorés par la technologie[2] (Hoyles et al., 2010). En effet, Star & Grieseman (1989) définissent les objets-frontière comme étant des objets qui permettent la communication entre des groupes sociaux et facilitent la confrontation de points de vue ou la résolution des conflits de manière créative. Ces objets sont « à la fois suffisamment flexibles, pour s'adapter aux contraintes et besoins locaux des différents groupes qui les utilisent, et assez robustes, pour maintenir une identité commune au-delà des

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1 Sans oublier certains collègues qui, à un moment ou un autre, ont abandonné l'aventure, n'ayant pas trouvé au sein des groupes assez de synergies qui rejoindraient leurs propres préoccupations. Ils ont cependant joué un rôle dans les débats collectifs et ont permis de faire avancer la réflexion de notre grand groupe.

2 L'expression anglaise utilisés par Hoyles et al. est : Technology enhanced boundary objects

spécificités de chacun[3] » (p. 393). En reprenant cette citation à notre compte dans le contexte du travail des groupes d'auteurs de ce numéro, les objets-frontière peuvent, par exemple, être analysés dans différentes traditions de recherche, mais malgré cela garder un sens et une identité commune à travers ces analyses.

Une des difficultés de communication entre des groupes sociaux, ou dans le cas de notre travail de chercheurs issus de différentes traditions est que les objets utilisés de façon routinière sont 'naturalisés' (Bowker & Star, 1999). Autrement dit, certains termes et concepts sont utilisés si couramment dans notre tradition de recherche que leur signification est considérée comme allant de soi. De plus, dans le cadre d'un même domaine de recherche (la didactique des mathématiques) certains de ces termes sont utilisés pour des idées 'naturalisées' différentes. Par exemple, un des constats faits par les participants est celui de l'usage par les chercheurs français du terme 'adaptations' lorsqu'il s'agit d'analyser les connaissances mises en fonctionnement par les élèves lors de l'exécution d'une tâche mathématique donnée ; terme qui est vraisemblablement naturalisé pour ces chercheurs. Il a fallu du temps, et parfois des malentendus, pour que les chercheurs anglais se rendent compte que cela correspondait à ce qu'ils désignent, eux, par 'prior knowledge' et non 'adaptations' dans le sens d'ajustements à faire par rapport à des situations déjà rencontrées, permettant la mise en fonctionnement des connaissances.

Un objet-frontière est celui qui n'est naturalisé dans aucun groupe, mais émerge lorsque des mondes sociaux, dans notre cas des traditions de recherche, se croisent. Le statut de ces objets n'est pas nécessairement fixé définitivement puisqu'un objet-frontière peut évoluer pour devenir naturalisé dans les deux traditions et perdre ainsi son statut d'objet-frontière (Star, 2010). Ainsi, l'expression 'teachers' professional learning'[4] qui a souvent été employée utilement dans les discussions des groupes sans qu'aucune théorisation correspondante ne soit mentionnée. De plus, un regard *a posteriori* sur l'ensemble des textes laisse penser que certains termes sont utilisés régulièrement dans une acception commune, sans pour autant que cette acception soit clairement explicitée. On observe par exemple, qu'il existe, sans que cela soit clairement mentionné, une conception constructiviste de l'apprentissage des élèves dans toutes les études présentées ; peut-être s'agissait-il ici d'une hypothèse 'naturalisée' pour la plupart des collègues des deux traditions anglaise et française.

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3 Traduction faite par les auteurs de cet article.

4 Une traduction littérale serait : « apprentissage professionnel des enseignants », mais cette terminologie n'est pas couramment utilisée dans le contexte français.

À ce stade de notre travail collaboratif, nous sommes conscients de l'existence de différences significatives dans nos approches de l'enseignement, l'apprentissage et la recherche en didactique, ainsi que des ressemblances évidentes dans nos questions de recherche et nos centres d'intérêt. Ce numéro spécial offre au lecteur, anglophone ou francophone, une vue unique sur une tradition qui lui est peu familière.

Les chercheurs anglo-saxons constatent que des analyses détaillées des moments d'enseignement et d'apprentissage sont mises en avant dans chaque article de ce numéro par les chercheurs français. Nous y trouvons régulièrement une analyse *a priori* des tâches mathématiques. Il s'agit donc d'une prise en compte systématique des connaissances mathématiques enjeux des apprentissages. Cette idée centrale qui est employée à travers les différents articles permet d'avoir un aperçu d'une certaine perspective française de recherche. Elle fait également ressortir une préoccupation récurrente, dans cette perspective, de l'utilité qu'aurait cette analyse dans la façon dont l'enseignant négocie l'imprévisible inévitable de la salle de classe, lorsqu'il tente de mettre les élèves en contact avec des concepts mathématiques.

En revanche, d'une idée influente datant des années 1960 et 1970, dans le contexte du Royaume-Uni, vient la conception que les élèves s'engagent dans une démarche d'investigation de 'leurs propres mathématiques' - expression qui est, bien sûr, interprétée de façons très différentes dans ce contexte. Les objectifs actuels de l'une des associations thématiques du Royaume-Uni (Association of Teachers of Mathematics - ATM) comprennent la déclaration suivante : « Le pouvoir d'apprendre appartient à l'apprenant. L'enseignement y a un rôle subordonné.[5] » (ATM, n.d.). Nous voyons ici l'idée que l'enseignant aurait, dans certaines phases de l'enseignement, à suivre la direction dans laquelle les élèves choisissent d'amener la tâche. De ce fait, une analyse *a priori* détaillée sensibiliserait les enseignants aux différentes possibilités, mais peut-être pas avec le sens de prédire les résultats probables de l'activité de la classe. Les modes de travail en classe de mathématiques peuvent être aussi bien une préoccupation pour un enseignant que le contenu conceptuel lui-même. Une question, d'un point de vue anglophone, au sujet de l'analyse a priori pourrait donc être de savoir s'il existe toujours un itinéraire prévu d'apprentissage pour les élèves et, par conséquent, un 'décalage' inévitable entre les prévisions de l'enseignant et les activités effectives des élèves.

2. Thèmes et théories

Ce numéro spécial est structuré de façon à rendre compte du processus de collaboration et de son évolution au fil du temps. L'article 2 est le produit du

5 Traduction faite par les auteurs de cet article.

besoin ressenti dès le début du travail collaboratif de connaitre et comprendre l'historique du développement des théories en France et au Royaume-Uni, notamment dans le domaine de l'étude des pratiques d'enseignement. Les auteurs de cet article retracent l'évolution des perspectives théoriques dans les cultures française et anglaise de la recherche sur l'enseignement des mathématiques depuis les années 1960-70 jusqu'à nos jours.

Les articles 3 à 6 présentent les différentes manières dont les groupes de coauteurs ont croisé leurs regards sur leurs approches multiples : juxtaposition, mise en réseau, analyse des mêmes données… et, comment ils ont parfois créé leurs propres objets-frontière, à l'intérieur de chaque groupe. Nous expliquons brièvement dans ce qui suit les similarités et les différences entre ces quatre articles.

Les articles 3 et 6 partent de la même théorie tout en montrant les différentes interprétations qui en sont faites et comment celles-ci sont utilisées dans différentes recherches.

L'article 3 (Abboud, Goodchild, Jaworski, Potari, Robert, Rogalski) est centré sur l'utilisation de la Théorie de l'Activité (TA) pour analyser le discours de l'enseignant et ses interactions en classe. Chacun des groupes français et anglais utilise ses propres données et les analyse en utilisant sa propre compréhension de la TA, en montrant les différences et en pointant des interrogations qui ne pourraient pas émerger dans chacune, seule, des deux traditions. L'interprétation et la compréhension de la TA semble être influencée par le contexte culturel de la recherche, comme c'est par exemple le cas des différences dans la prise en compte du rôle des connaissances mathématiques dans la TA appliquée aux pratiques de classe.

Dans l'article 6 (Mangiante-Orsola, Perrin-Glorian, Stromskag), l'objectif est de discuter l'utilisation de la Théorie des Situations Didactiques (TSD) pour répondre à des questions de recherche communes. Ces questions portent sur la façon dont le chercheur peut utiliser la théorie comme outil à la fois pour comprendre les pratiques enseignantes et pour contribuer au développement de ces pratiques. L'article invite à réfléchir aux différences entre les connaissances pour enseigner, les connaissances pour apprendre, les connaissances pour agir, et à la façon dont la TSD aide à réfléchir à ces questions.

Les articles 4 et 5 ont les mêmes objets d'étude, mais utilisent des cadres théoriques différents pour les traiter.

Les auteurs de l'article 4 (Abboud, Clark-Wilson, Jones, Rogalski) s'intéressent à l'étude des pratiques d'enseignement avec les technologies numériques tout en cherchant à développer des outils (théoriques et méthodologiques) qui pourraient être utilisés dans la formation des enseignants. Ils présentent deux concepts théoriques (tensions-perturbations et *hiccups*[6]) et montrent que malgré la différence des contextes étudiées et des choix méthodologiques adoptés, les résultats en matière d'analyses des pratiques enseignantes sont très proches. Ils qualifient leurs approches comme des façons d'observer et de comprendre les deux faces d'une même pièce, à partir de deux perspectives culturelles différentes. En travaillant ensemble, leur objectif commun était de voir si la connaissance de chacune des faces conduit à une compréhension plus profonde de la pièce dans son ensemble.

L'article 5 (Coles, Horoks, Chesnais) aborde la question de l'utilisation de la vidéo en formation d'enseignants. Les auteurs s'intéressent au rôle du didacticien-formateur utilisant des vidéos pour la formation des enseignants et aux différents usages de ces vidéos. À travers la narration et le partage de différentes pratiques de formation (avec des enseignants en formation initiale et continue), les auteurs développent des idées et des questions pour aider à cerner le rôle de la théorie dans le travail d'un didacticien-formateur : quelles sont les théories adoptées ? Quelles sont les théories explicitées lors d'une séance de formation ? Quelles sont les théories destinées à être utilisées par les enseignants (tant pour analyser que pour informer leur enseignement) ?

Dans le dernier article (7) de synthèse, les auteures présentent un regard global sur les quatre articles précédents en reprenant certains des fils conducteurs qu'elles avaient développés lors de leur rédaction de l'article 2. Cela leur permet de synthétiser les similarités, complémentarités et différences observées entre les deux perspectives, française et anglaise, de recherche. Elles ont essayé de montrer que la collaboration des auteurs a servi en partie à créer de nouveaux objets-frontière en forçant les termes et hypothèses naturalisés à être questionnés, soit sur l'utilisation de la théorie (TA et TSD), soit sur les pratiques des enseignants en classe et en formation (utilisation de la vidéo ou des technologies numériques).

Un autre thème plus ou moins explicite dans les quatre articles (3 à 6) est celui des 'contradictions' et des 'tensions'. L'article 3 met explicitement l'accent sur les tensions et les contradictions au sein des représentations mathématiques, qui peuvent être en fonctionnement en classe. L'article 4 théorise la notion de tensions cognitives, pragmatiques et temporelles dans l'utilisation des technologies en classe, ainsi que celle des *hiccups* qui peuvent se produire pour perturber le bon fonctionnement d'une leçon. Les tensions de l'article 5 se manifestent à différents niveaux de théorie, par exemple, les différences potentielles entre la théorie adoptée par un formateur et ce qui est mis en pratique dans ses sessions de formation. Dans l'article 6, il est clair qu'au sein de la TSD, le milieu mis en place dans la salle de classe est conçu pour provoquer des conflits et des contradictions

6 Hoquets ou à-coups

parmi les élèves, conduisant à de nouvelles connaissances. Nous pensons que de telles ‘tensions’ constituent un autre ensemble d’objets-frontière dans ces articles, qui a permis aux auteurs de comprendre en quelque sorte les détails des pratiques des uns et des autres en tant que chercheurs et formateurs d’enseignants.

Dans l’ensemble, les inévitables moments de contradictions et de tensions qui se sont produits au cours de nos réunions pourraient eux-mêmes être considérés comme les objets-frontière, qui ont provoqué une communication raisonnée entre les auteurs et qui ont mené à la rédaction de ce numéro spécial.

Conclusion

À l’heure où nous écrivons ces lignes (2018), nous sommes interpelés par le fait que les décideurs des politiques éducatives en France et en Angleterre semblent se diriger vers des approches pédagogiques inspirées par les méthodes d’Asie de l’Est, sans tenir compte de près de 50 ans de recherche sur l’enseignement des mathématiques dans nos deux traditions. En Angleterre, il y a une poussée en faveur de l’enseignement de la "maîtrise des mathématiques". Le terme ‘maîtrise’ est, inévitablement, contesté, mais semble mettre davantage l’accent sur les détails du développement conceptuel des élèves pendant une leçon, au cours d’une année et d’une année à l’autre. Il nous semble qu’il existe ici des liens avec l’attention portée dans la tradition française, à l’analyse a priori et à l’enchaînement des tâches mathématiques conçues pour provoquer le développement des connaissances mathématiques, en particulier au niveau de l’enseignement dans le primaire. Un aspect important de l’analyse a priori est celui de mettre l'accent sur les formes cohérentes de représentations en jeu dans une situation d’apprentissage. Par exemple, il est vrai que la manipulation d’objets concrets, au niveau primaire, joue un rôle important dans la création du milieu dans les situations d’action en TSD (cf. l’article 6 dans ce volume). Toutefois, dans cette théorie les situations d’action sont aussi conçues pour que les élèves puissent mettre en action des connaissances préalables et les faire évoluer : changer, rejeter ou remplacer par les connaissances dont l’apprentissage est visé. De plus, dans les situations de validation (en TSD), ce sont les connaissances mathématiques au cœur de la situation qui déterminent les types de validation, qui, à leur tour, requièrent des compétences langagières et des représentations, et non le développement des représentations qui va déterminer les connaissances mathématiques.

Dans un tel contexte, qui n’est sans doute pas propre à l’Angleterre et à la France, de pays à la recherche d’autres pratiques d’enseignement et de formation, nous soutenons que les collaborations entre chercheurs de traditions différentes sont plus importantes que jamais. De telles collaborations, qui mutualisent et capitalisent les résultats de la recherche, sont elles-mêmes un processus de transformation-adaptation, conduisant à des propositions à la fois pour la formation initiale et continue des enseignants et pour les classes de mathématiques.

Pour conclure, nous citons ci-dessous deux réflexions de participants à ce voyage dans le monde des théories. Nous ne les traduisons pas afin de ne pas perdre l'essence et les nuances de la réflexion. L'un d'eux, Anglais, pour donner suite à une discussion avec un participant, Français, a synthétisé la teneur de cette discussion en exprimant ce qu'il a acquis à travers son engagement dans ce travail collaboratif :

> 'I could explain what the other group means by 'Double Approach' (DA), but the adherents of DA would say that, no that's wrong it is not what we mean. This has been our experience – we discuss, we hear, we interpret, and we test the meanings we make by feeding back with our own words and the meaning we test is not that intended. Because we interpret from our own cultural immersion, and feedback in the language of our cultural immersion, and the culture and language are not shared, we hear differently, we interpret differently, and we express differently because we come to the discussion from different cultural positions. The discussion then produces a productive tension – tension because of the explicit disagreement, productive because it challenges the dispositions we have because of our immersion within a culture, of which we are unaware until the disagreement becomes evident.'

Nous pouvons voir ici encore une fois combien la communication peut être difficile entre deux traditions de recherche, mais aussi combien elle peut être enrichissante quand nous trouvons des objets-frontière qui la soutiennent. Un autre participant a déclaré :

> 'After many sessions discussing what we did with video, including doing some writing together, it was when we came to co-plan a session using video that there was a shift in our understanding of what each other does … because we could not find a video that we could both use! This dissonance felt highly productive and allowed access to some of the words each of us were using to describe what we did.'

De nouveau, il nous semble voir ici le développement d'objets-frontière à travers le fait que nous avions émis des suppositions sur des significations qui n'étaient pas partagées – permettant ainsi l'accès à une communication plus profonde et le développement ultérieur de la réflexion de chacun sur ses objets de recherche.

En regardant vers l'avenir, nous sommes déterminés à poursuivre et à élargir notre collaboration ; il s'agit de questions ouvertes quant à ce qui pourrait être dès lors de nouveaux domaines d'intérêt, productifs, et sur des thèmes ayant le potentiel de réunir les chercheurs et les enseignants, nécessitant pour cela des objets-frontière qui restent à déterminer. Nous accueillons avec plaisir les idées de tous les lecteurs et les propositions de pistes sur lesquelles nous pourrons nous engager ensemble.

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THE PRACTICE THEME IN MATHEMATICS EDUCATION : DEVELOPMENT OF A FRENCH-ENGLISH COLLABORATION ON THE ROLE OF THEORIES

Abstract. In this introductory article, we describe how the Special Issue came about, as a result of three years of collaboration. We show how this collaboration developed with a focus on the role of theory. We draw out themes across the individual papers, making use of the concept of 'boundary objects' that have allowed the comparison and enrichment of our different perspectives. We suggest that the work of bringing together researchers, from different traditions and on a sustained basis, is more important now than ever, in a context in which policy makers are increasingly looking at 'solutions' to problems in education from the other, seemingly more successful, countries (e.g. as measured by international comparison tests). We conclude with questions for further research, both for our particular collaborative group and more widely.

Keywords. Mathematics education, French and English approaches, theories, practices.

Introduction

This Special Issue is the result of a three-year collaboration between mathematics educators researching within French and English traditions (with French, British, Norwegian and Greek nationalities represented across the group). The initial reason we came together was to discuss and compare our theoretical approaches related to teachers' practices and teacher education. There was a sense that, across French and English traditions, there were similar areas of concern but with varied awareness of the detail of each other's work. The journey of arriving at the Special Issue has necessitated finding ways of working and questions or data allowing meaningful conversation. In this introductory paper our aim is to describe some of this journey in order to illustrate what has, and has not, occasioned useful collaboration. We use the theoretical notion of a boundary object (Star and Griesemer 1989) to allow us to analyze our work. We then offer some of the themes we observe across the papers within this Special Issue, leading us to implications for the future.

1. Journey to a Special Issue

The first meeting took place in January 2014, gathering European researchers having common interests but with different theoretical perspectives. We brought

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p. 17 - 24.

together ideas and issues relating to how theories are, or can be, used to investigate teaching practices and teaching development in mathematics. The aim was not just to present each participant's own work and ideas but to provide a forum in which ideas can be shared and new ones can emerge. Three work dimensions were at stake. The first was a focus on theories, enabling participants to bring to the table a range of theoretical perspectives and theoretical frameworks. The second was the substantive focus of our theoretical work; that is, teaching practices. The third was the interest in the use of theory to describe and understand how teaching develops and to question and inform the professional development of teachers.

Our meetings, together with the collaborative work in-between, first gave us opportunities to learn from each other. It also made possible groupings around specific interests and expectations. These groupings gave rise to common writings that are the articles in this Special Issue[1].

The duration of our collaboration turned out to be necessary, notably in order to shift from listening to each other to trying to find 'objects' that supported debate and the co-construction of comparative analyses. These objects might be qualified as boundary objects and, in some cases, technology enhanced boundary objects (Hoyles et al., 2010). Star & Grieseman (1989) define boundary objects as objects that allow communication across social groups and facilitate the resolution of different view points or conflicts in a creative manner. They "are objects which are both plastic enough to adapt to local needs and the constraints of the several parties employing them, yet robust enough to maintain a common identity across sites" (p. 393). Translating this quotation into the context of the work of the group of authors of this Special Issue, boundary objects can, for example, be analyzed across different research traditions and yet retain a sense of a common identity across these treatments.

One of the difficulties of communication across social groups or, in the case of this Special Issue, researchers coming from different traditions, is that objects used routinely within our respective traditions become naturalized (Bowker & Star, 1999), i.e., there are words or concepts used so routinely that their meaning becomes taken for granted. The further problem, within mathematics education, is that the same word may be used for different 'naturalized' ideas. A pertinent example within the group of authors of this Special Issue is the word 'theory' when talking about a mathematics lesson implementation. It took some time of working together to recognize the difference across traditions of what might be taken to be a

[1] We want to also acknowledge the role of colleagues, not represented in these pages, who joined meetings and attempted collaborations (e.g. around 'knowledge for teaching' and assessment) and found insufficient synergy to develop new insights. They each played an important part in discussions and in moving forward our collective thinking.

‘theory’. A naturalized word for French researchers is that of the ‘adaptation’ of mathematics knowledge required in a lesson (which is subject to analysis, prior to teaching). While in the English tradition, this ‘adaptation’ is more likely, perhaps, to be conceptualized in terms of ‘prior knowledge’, possible ‘misconceptions’ and learning ‘objectives’.

A boundary object is one that is not naturalized within any system or group but instead arises where social worlds, in our case research traditions, overlap. These objects are also not fixed over time and things may come into being as boundary objects between groups and then cease to function in such a way if they become naturalized in both settings (Star, 2010).

We can take here the example of ‘teachers’ professional learning’. This ‘expression’ often recurred in group discussions, about what the teacher learns after a lesson has been implemented. But no corresponding concepts seem to be theorized related to different approaches to learning or cognition. For instance, without being explicitly mentioned, there seems to be a largely constructivist conception of ‘school’ learning across the studies; perhaps this was a ‘naturalized’ assumption for most (not all) colleagues across English and French traditions.

At this point in our collaboration we are now aware of significant differences in our approaches to teaching, learning and research in the context of mathematics and mathematics education, as well as strong similarities in our concerns and overall interests. This Special Issue potentially offers readers a unique insight into a tradition of thinking with which they may be unfamiliar.

From an English perspective, the detail of the French analysis of teaching and learning moments comes through in every article. There is a recurrent pattern, more or less explicit, of the *a priori analysis* of mathematical tasks, i.e. the deep consideration of what mathematical knowledge needs to be brought to solve a particular task. Reading how this key idea is put into practice across the different articles allows insight into a French perspective and points to a recurring concern, within this perspective, about how such an a priori analysis could allow the teacher to negotiate the inevitable unpredictability of the classroom, when he/she attempts to bring students into contact with mathematical concepts.

In contrast, an influential idea from the past in the United Kingdom, perhaps from the 1960s and 1970s, has been the notion of students ‘investigating’ their own mathematics – a word that is, of course, interpreted in widely different ways. The current aims of one of the UK’s subject associations (the Association of Teachers of Mathematics - ATM) include the statement : “The power to learn rests with the learner. Teaching has a subordinate role.” (ATM, n.d.). We see here the idea that the teacher might, in some phases of teaching, be following the direction in which students choose to take a task, hence a detailed *a priori* analysis will sensitize teachers to possibilities but perhaps not with the sense of predicting likely outcomes for a lesson. *Ways of working* on mathematics might be as much a focus for a teacher as the conceptual content. A question, from an English perspective, about *a priori* analysis might, therefore, focus on the extent to which there is always an intended path of learning for students and hence an inevitable 'gap' between teacher intention and student activity.

2. Themes and theories

This Special Issue is structured in a way that reflects the collaboration processes and their development. Article 2 (Jaworski, Lerman, Robert, Roditi, Bloch) is derived from the need felt at the beginning of the collaboration to understand the history of theory development in the French and English teacher education domain. It traces the development of theoretical perspectives in the English and French mathematics education research cultures from the 1960 and 70s to the present day.

Articles 3 to 6 present different ways in which groups of co-authors brought multiple perspectives into conversation with each other: juxtaposing, networking, analyzing the same data, meta-theorizing and, within each group, creating their own boundary objects. We elaborate briefly on the similarities and differences across these four Articles.

Articles 3 and 6 start from the same theory and show different interpretations and their use in different research.

Article 3 (Abboud, Goodchild, Jaworski, Potari, Robert, Rogalski) focuses on using Activity Theory to analyze classroom dialogue. Each of the French and English-speaking group uses their own data and analyzes it using their own understanding of Activity Theory, drawing out differences and allowing for reflections that would arise neither from English nor French traditions alone. The cultural context of researching seems to influence the interpretation of an understanding of Activity Theory, e.g. differences in view of the role of mathematical knowledge in Activity Theory applied to classroom practises.

The common interest in Article 6 (Mangiante-Orsola, Perrin-Glorian, Stromskag) is in the use of the Theory of Didactical Situations (TDS) to help structure classroom tasks. Common questions include how to design a-didactical situations (i.e., situations or tasks that hardly require the teacher's involvement, once set up) which, it is hoped, come close to 'guaranteeing' specific knowledge outcomes. The article leads to thinking about differences in: knowledge to teach; knowledge to learn; knowledge to act. How does TDS help thinking about these elements?

Articles 4 and 5 have the same focus but use different theories to address it. Article 4 (Abboud, Clark-Wilson, Jones, Rogalski) is based on the authors' mutual interest in investigating teachers' uses of, and practices with, digital technologies, alongside the need to develop tools that could be used within teacher education programmes. They qualify their different approaches as ways of looking at the two different sides of the same coin, teachers' classroom practises with digital technology, from two different cultural perspectives. By working together, their aim is to see whether a knowledge of each side's facets leads to a deeper understanding of the coin as a whole.

Article 5 (Coles, Horoks, Chesnais) addresses the issue of making effective use of video for teacher development. The authors are interested in the role of the didactician-educator in working with video and, in particular, how theory is used in different ways by didactician-educators in the context of working with video. Through narrating and sharing different practices (with pre-service and in-service teachers), the authors develop ideas or questions to help tease out the role of theory in the work of a didactician-educator: What are the theories espoused by the didactician-educator? What theories are made explicit in any training session? What theories are intended to be used by the teachers (both for analysis and to inform their teaching)?

In the closing article, Jaworski and Robert offer a global overview of these last four papers, picking up some threads from Article 2, in which the authors presented key aspects of the English and French perspectives, to synthesize similarities, complementarities and differences. In part, then, the collaboration of authors has served to create new boundary objects by forcing naturalized words and assumptions to be questioned, both about the use of theory (Activity Theory or TDS) and, about practices with teachers and in classrooms (using video or using ICT).

Another theme more or less explicit in all four Articles (3-6) is that of 'contradictions' and 'tensions'. Article 3 includes an explicit focus on tensions and contradictions within mathematical representations, that can be put to use in a classroom. Article 4 theorizes cognitive pragmatic and temporal tensions in the use of ICT in a classroom, as well as the 'hiccups' that can occur to disrupt the smooth functioning of a lesson. The tensions in Article 5 occur across different levels of theory and, e.g. potential differences between an educator's espoused theory and what is enacted in their training sessions. In Article 6, it is clear that within TDS, the milieu enacted in the classroom is designed to provoke conflict and contradictions among students, leading to new knowledge. We suggest that such 'tensions' are another set of boundary objects across these articles, which have allowed authors to understand something of the detail of each other's practices as researchers and educators.

In fact, the inevitable moments of contradiction and tension that were generated during symposium meetings might themselves be viewed as the boundary objects, that provoked our communications, leading to the writing in this Special Issue.

Conclusion

We are intrigued that, at this moment (2018), policy makers in both France and England seem to be moving towards pedagogical approaches inspired by East Asian methods, seemingly taking less account of nearly 50 years of research in mathematics education in our two traditions. In England, there is a push to move towards 'mastery' teaching. The meaning of the term 'mastery' is, inevitably, contested but seems to capture a greater focus, than has been historically the case, on the details of conceptual development of students during a lesson or over the course of a term, year, school career. There are links here to the French research practice of *a priori* analysis of mathematics and attention to the careful sequencing of tasks, designed to provoke the development of mathematical knowledge, especially throughout the primary years. One important aspect of a priori analysis is the focus on consistent forms of representation at stake within a learning situation. For example, it is true that hands-on experiences at a primary level, using concrete objects, has an important role in an 'action situation' in the TDS (cf. Article 6 in this volume). However in this theory, 'action situations' are also conceived to engage students' initially available knowledge, with the aim that such knowledge will evolve, change or be rejected and replaced by targeted mathematical knowledge. Moreover, in the later 'validation situations' of TDS, it is this targeted mathematical knowledge which determines the types of validation, that themselves require language skills and representations, and not the representations which determine the mathematical knowledge.

In such a context, which we suspect is not unique to England and France, of countries looking to practices elsewhere, we argue that collaborations across researchers from different traditions are more important than ever. Such collaborations, merging and capitalizing on the results of research, are themselves a process of transforming and adapting, leading to proposals for initial or in-service teacher education as well as for the mathematics classroom.

To conclude, we quote here two thoughts from participants in this journey in the world of theories. One participant expressed their experience of what they gained from our collaboration as follows:

> 'I could explain what the other group means by 'Double Approach' (DA) , but the adherents of DA would say no, that's wrong; it is not what we mean. This has been our experience – we discuss, we hear, we interpret and we test the meanings we make by feeding back with our own words and the meaning we test is not that intended. Because we interpret from our own cultural immersion and feed back in the language of our cultural immersion, and the culture and language are not shared, we hear differently, we interpret

> differently and we express differently because we come to the discussion from different cultural positions. The discussion then produces a productive tension – tension because of the explicit disagreement, productive because it challenges the dispositions we have because of our immersion within a culture, of which we are unaware until the disagreement becomes evident.'

We see here again evidence for just how difficult communication is across research traditions and also how enriching it is, as we find the (boundary) objects and artifacts that allow us to collaborate. One other participant reflected:

> 'After many sessions discussing what we did with video, including doing some writing together, it was when we came to co-plan a session using video that there was a shift in our understanding of what each other does … because we could not find a video that we could both use! This dissonance felt highly productive and allowed access to some of the words each of us were using to describe what we did.'

Again, what we see here is the development of boundary objects, through a recognition that we had been making assumptions about meanings that were not shared - allowing access to deeper communication and the further development of thinking.

As we look to the future, we are committed to continuing and widening our collaboration; it is an open question as to what might now be productive areas of focus, what themes might bring together researchers and teachers and what new boundary objects might be needed. We welcome ideas, comments and communications from all readers.

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THEORETICAL DEVELOPMENTS IN MATHEMATICS EDUCATION RESEARCH: ENGLISH AND FRENCH PERSPECTIVES IN CONTRAST

Abstract. This article traces the development of theoretical perspectives in the English and French mathematics education research cultures from the 1960 and 70s to the present. The main parts of the article are the separate accounts of development in the two domains. The two areas are presented separately since they are very different both in terms of what is in focus at different times and in terms of the theories originated, developed or appropriated. The place of *a priori* mathematical analysis (i.e. analysis of the mathematics to be taught, prior to teaching) seems a key difference, beyond the institutional and cultural differences. The final part of the paper draws attention to key areas of difference between the two domains and suggests key questions and issues in which there is common ground albeit addressed from the differing perspectives and cultures.

Keywords. Mathematics education, constructivism, socioculturalism, activity theory, didactical theories.

Résumé. Développements des recherches sur l'enseignement et l'apprentissage des mathématiques – regards contrastés sur les cas anglais et français. Cet article retrace le développement des perspectives théoriques des chercheurs concernés par les questions d'éducation mathématique en Angleterre (et dans les pays de tradition anglais) et d'enseignement et d'apprentissage des mathématiques en France (et dans les pays de tradition francophone), des années 60-70 à maintenant. C'est une présentation en deux volets successifs qui occupe la plus grande partie de l'article, tant les différences sont importantes – concernant aussi bien les origines des recherches que leurs fondements théoriques. La place des analyses mathématiques semble constituer une différence majeure, par delà les différences institutionnelles et culturelles. C'est ce que reprend la dernière partie de l'article, dégageant les principales orientations de chaque pays en les mettant en regard, et présentant des questions majeures communes qui restent néanmoins posées aux deux communautés.

Mots-clés. Didactique des mathématiques, constructivisme, socioconstructivisme, théorie de l'activité.

Introduction

This article traces the development of theoretical perspectives in the English and French mathematics education research cultures from the 1960s and 70s to the present. Initially, we deal with the two areas separately since they are very different

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both in terms of what is in focus at different times and in terms of the theories originated, developed or appropriated. Necessarily there is a strong historical dimension in each as theories are related to events, educational trends and developments across 50-60 years. The main parts of the article are the separate accounts of development in the two domains. These are necessarily lengthy in order both to cover the range of theories and address associated educational structures and issues. The final part of the paper draws attention to key areas of difference between the two domains and suggests key questions and issues in which there is common ground albeit addressed from the differing perspectives and cultures. We are aware that there may appear initially some inconsistency between the two parts, but it reveals a great difference between the two developments – mainly because the French one was developed in contrast to the education sciences, with the mathematical content analysis up front, although some concepts are borrowed or shared. In contrast the English development has to be understood as a part of these sciences, even where specific mathematical content is taken into account. These differences may not be reduced into a uniform presentation.

Two of the authors are associated with each of the two domains, Aline Robert and Eric Roditi with the French domain and Stephen Lerman and Barbara Jaworski with the English domain. We speak of the “English” domain in general rather than the “United Kingdom” domain to emphasize that international theoretical trends in English-speaking countries have influenced the domain, rather than developments only in the United Kingdom. However, the educational perspectives historically pertain mainly to the United Kingdom. In the French case, the developments discussed arise first within France and relate to the history of educational development in France. It will also be evident to readers that the English history evidences a wide range of theories whereas the French history is much more focused on a few overlapping theories.

1. Theoretical developments in English mathematics education research (Stephen Lerman and Barbara Jaworski)

1.1. Early influences

Research in mathematics education in the United Kingdom has a history dating back to the late 1800s. We can point to the founding of the Association for the Improvement of Geometry Teaching in 1871, renamed as The Mathematical Association (MA) in 1894, as perhaps the beginnings of the field. It has had many eminent mathematicians as its president, including A.N. Whitehead, co-author with Bertrand Russell of *Principia Mathematica,* from 1915 to 1916. The MA was dominated by secondary school teachers, mainly from private and grammar schools. A breakaway group, led by Caleb Gattegno (an influential figure in education at that time) founded The Association for Teaching Aids in Mathematics

(ATAM) in 1952 to focus on primary as well as secondary mathematics teaching. A decision to change the name of the association to the Association of Teachers of Mathematics (ATM) was initiated at the 1962 AGM and took effect in June 1962. As it says on their website, “An early aim of the Association was that all children should learn mathematics through lively and interesting experiences”, an egalitarian direction that has formed a feature of research in mathematics education in the United Kingdom since then. Both associations, the MA and the ATM, have remained active in the field of research, holding conferences and being productive in publications, and incorporating teachers and researchers as members.

As we indicate in the next section, we can perhaps take the early years of the ATM as the beginning of the modern era in mathematics education research in the United Kingdom. Gattegno’s work, which viewed working mathematically as a central part of all human functioning, was and remains a huge influence on teachers of mathematics. In these early years a strong influence also came from the work of Jean Piaget, particularly his clinical interviews and stages of intellectual development (Gruber and Vonech 1977). These influenced a Government report on primary education, whose committee was headed by Lady Plowden, published in 1967. The Plowden report led to a revolution in primary education, introducing the concept of *child centeredness* into the language of teaching and curriculum, highly consistent with the thinking within ATM. These theoretical beginnings can be seen as a forerunner of the practically focused and wide-ranging theoretical orientations discussed below.

1.2. Wide-ranging orientations

In our brief survey of theoretical orientations in United Kingdom (henceforth UK) mathematics education research since the 1950s and 1960s which follow, we suggest that a range of perspectives have been drawn upon by researchers, evolving and developing over those years. We have set out a broad timeline (see Figure 1) and will expand on the developments below. It is perhaps typical of the rather eclectic and practically focused approach of British intellectual thought, possibly even across all the English-speaking world, that there should be a range of orientations, rather than a strong and unified set of theories common to nearly all researchers as is the case in France.

Furthermore, we would emphasize that there is no sense in which we can speak of a ‘progression’ across the decades. It is a phenomenon of the social sciences in general and education in particular that new languages of research emerge and sit alongside existing ones, a phenomenon that the UK sociologist of education Basil Bernstein called a horizontal knowledge structure (Bernstein 2000). Thus, Piaget’s child development theories did not replace behaviourism, nor did the emergence of Vygotsky’s work in mathematics education in the late 1980s lead to a move away from Piagetian theories. More recently, postmodern critiques and methodologies have been developed that, once again, sit alongside existing theories (see Lerman 2000, for a more developed account) developing a language of research within their own set of theoretical structures and over time continue to proliferate.

We might suggest that this proliferation is at least in part responsible for the lack of a progression. What one researcher or group might consider progress may well be criticized by another group with a different orientation. We leave further discussion on these matters to a later part of this paper where we contrast and compare the English with the French research traditions in mathematics education. In regard to references, at an earlier stage of the writing we began to reach a bibliography that covered ten pages. This is not possible for a journal paper. We have decided, therefore, to restrict the references severely, and will list just those that we consider to be essential. In many places the names of scholars associated with developments in research will be mentioned, giving readers leads to further references.

1.3. Beginnings of modern developments in English research and influences in the United Kingdom

Piaget's developmental psychology and practical orientations in United Kingdom research

The *British Society for Research into Learning Mathematics* (BSRLM, originally BSPLM where P means Psychology) was founded in 1976, the same year as the International Group for the Psychology of Mathematics Education (PME), by Richard Skemp, Celia Hoyles, Kath Hart, Alan Bell, Margaret Brown, David Tall, and others. The early field was extremely influenced by the *Psychology* of Mathematics Education and this influence continues to an extent to the present day.

This psychological tradition in empirical research in the UK derives strongly from the work of Piaget, himself both a theoretician and empirical researcher in psychological traditions. A leading exemplar of this orientation, in the late 1970s, was the "Concepts in Secondary Mathematics and Science" (CSMS) project, led by Kath Hart at the London University Chelsea College. Based on hierarchies of biological development the CSMS team surveyed students across the UK and developed levels of progression across a range of topics of school mathematics (e.g. Hart 1981). The findings of this study have permeated teacher education courses and influenced teaching and curricula over 20-30 years. Also influential has been Mellin-Olsen and Skemp's distinction between forms of understanding which they classified as *instrumental* and *relational* (Skemp 1971) : the *relational* being understanding in which concepts and their use are understood as a basis for mathematical activity, whereas *instrumental* understanding implied a use of rules or procedures, often without a conceptual underpinning. This distinction was seen by Skemp as an essential extension of Piaget's work on understanding. The influence of Piaget can also be seen in extensive work on diagnostic assessment, on cognitive conflict and conflict discussion, much of it taking place at the Shell Centre in Nottingham and at Kings College London (which absorbed Chelsea College in the early 1980s). We can now see these areas of more local theory, within the Piagetian perspectives on intellectual development, as forming a practically rooted theoretical base for modes of classroom activity.

A philosophical turn emerged in the early to mid-1980s, developed, in particular, by Paul Ernest and Stephen Lerman, with Ernest continuing that body of work until today. That work saw itself drawing particularly on Imre Lakatos's fallibilistic philosophy of mathematics (Lakatos 1976) and was strongly associated with the Radical Constructivist tradition, based on Piaget's theoretical ideas on learning that was growing in strength in the USA through pioneering work of Von Glasersfeld, Cobb, Confrey, Steffe and others (see Glasersfeld 1991; Cobb and Steffe 1983). The UK community was not swept along with Radical Constructivism to the same extent as USA colleagues; however, concepts from constructivism and radical constructivism became useful to some researchers in the UK. Nevertheless Piaget's developmental psychology was hugely influential in schools and broadly a firm theoretical background for UK mathematics education researchers. The hierarchy of knowledge in mathematical topics based on stages of intellectual development developed in the CSMS project, and the attention given to common errors and misconceptions, influenced the development of the first National Curriculum for Mathematics in the UK, in 1988.

As we have suggested above, at the roots of theory development in the UK, and influencing its diversity, is an exploratory, investigative tradition in classroom practice and its development, with teachers engaging in classroom research alongside teacher-education researchers from university education departments. Historically and significantly, this investigative tradition in teaching and learning mathematics was represented in the work of the ATM with its influential journal *Mathematics Teaching*, and annual conference including workshops for teachers and researchers to explore mathematical ideas. This activity was complemented by the early days of the Open University mathematics programme in which all mathematics students, many of whom were teachers, had to attend a summer school during which they engaged in investigative activity. In classrooms, an investigative approach to learning mathematics was encouraged through curriculum support materials such as the Kent Mathematics Project (KMP) text books, the SMILE series of work cards for students (see http://www.greatmathsteachingideas.com/smile-mathematics-resources/) and the School Mathematics Project (SMP) series of books (some of which are available here: https://www.stem.org.uk/resources/collection/283319/school-mathematics-project).

A practical tradition was established in which classroom mathematical activity developed through the work of inspired teachers and educators (such as Dick Tahta, John Mason, Eric Love). Love wrote a seminal article in the book *Mathematics, Teacher and Children* (Pimm 1988) called “Evaluating Mathematical Activity”. Jaworski’s study of investigative practices (Jaworski 1994) linked the investigative tradition in classrooms with the theory of radical constructivism. The work at the Shell Centre in Nottingham on cognitive conflict discussion (introducing conflicts into classrooms dialogue to promote accommodation of mental schemas) fitted with the exploratory ambience as did John Mason’s “*Theory of noticing*” (Mason 2002). Mason’s theory encouraged teachers to ‘notice’ aspects of their practice relating to tensions or issues in teaching/learning and to reflect on them, both *after* teaching and *in* teaching. Reflection *in* teaching could then lead to opportunity to change the action ‘in practice’ rather than in future planning. Thus inquiry within teaching practice itself was both theorized and promoted. Critiques of constructivist theory, and particularly of radical constructivism, suggest its dualistic nature - a paradox of positing an inner subject experiencing an outer world, resulting in the human subject constructing a representation of the world. Seeking to avoid this claimed dualism, the theory of *enactivism* avoids the inside-outside dilemma. Using a metaphor of « a path laid while walking » (e.g. Dawson 2008) in which “all knowing is doing and all doing is knowing” (Maturana and Varela 1987, p. 27) enactivism is essentially a non-representationalist view of cognition. In other words, our knowing is in our action and vice versa, or to quote Maturana and Varela (1992, p. 29). “Knowing is effective action, that is, operating effectively in the domain of existence of living beings”. Laurinda Brown and Alf Coles are UK scholars working with enactivism (Brown and Coles 2011). All of the work referred to above was very much in the practical tradition with research being closely associated with ‘activity’ in teaching and learning.

This practical tradition was also seen in the early days of the UK National Curriculum (introduced for the first time in 1988) which had a strand on “Learning and Doing Mathematics”. The inclusion of assessed coursework for students which was investigative in style in the national examinations at age 16 led to all schools focusing on investigations in mathematics classrooms. Attention to issues of equity and diversity grew through this practical tradition, with practices of differentiation and inclusion growing through in-service work with teachers, and in initial teacher education programmes (people such as Laurinda Brown, Anne Watson, Peter Gates). Research in teaching became important in order to conceptualize teaching beyond anecdotal practice. Through PME, research in teaching was made more public with working group publications – collections of papers from research into teaching around the world (e.g. Vicki Zack, Judy Mousley and Chris Breen; Barbara Jaworski, Terry Wood and Sandy Dawson). Since then theories of teacher knowledge and practice have extended and grown, as work by Tim Rowland, Kenneth Ruthven and others demonstrates.

Digital technology in mathematics teaching and learning.

To add to what we have written above, we need to address an important, although somewhat separate dimension of mathematics education research, that of the integration of digital technologies in the teaching and learning of mathematics. It seems fair to say that early activity in the UK drew on two important dimensions:

1) An interest in computer programming led by mathematics teachers and researchers in the MA and ATM;
2) The work of Seymour Papert at MIT, focusing on the theory of constructionism (different from constructivism in several important respects, including dualistic imputations and the importance of language and discourse).

Activity deriving from (1) was almost entirely practical rather than theoretical. It coincided with an era of technological development in which schools started to use microcomputers (e.g. the BBC micro) and started to teach Computer Studies/Science.Students were encouraged to write simple programs (in the language BASIC) and to understand the working on computers in a range of applications.

Activity deriving from (2) also involved computer programming, largely in the language LOGO, or simplified versions of it involving Turtle Geometry, as developed through the work of Papert (Papert 1980). Scholars in the UK using Papert's theoretical perspectives in researching the use of LOGO included Celia Hoyles, Richard Noss, Ronnie Goldsten and Janet Ainley. From this early work, Hoyles and Noss developed their theory of *Windows on Mathematical Meanings* which was an extension of constructionism (Noss and Hoyles 1996). Their work led to further developments within the UK in which students were encouraged to work within technological micro-worlds constructing their own computer-based models in solving mathematical problems.

In parallel with this work in the UK, and consistent with Papert's philosophy, colleagues in France were developing dynamic software to support the teaching and learning of geometry. Colette and Jean-Marie Laborde introduced the software Cabri-Geometre, which was designed to engage students in collaborative exploration of geometrical concepts (e.g. Laborde 1995). This was highly influential on geometry teaching worldwide and the forerunner of other such software (such as GeoGebra). Also in France, a theory of *Instrumental Genesis* emerged through the work of Luc Trouche and Ghislaine Gueudet, capturing relationships between the digital medium and the user of this medium in an educational context. While the impacts of this work were international, they were also significant for scholars working with computer-based media in the UK.

Research into mathematics teaching and learning in higher education

Most of the research referred to in this section above has taken place in primary and secondary education; theoretically-based research in higher education in mathematics has been less visible during these times. An exception has been research into so-called 'Advanced Mathematical Thinking', largely rooted in Piagetian or constructivist ideology and developing from the seminal book edited by David Tall (Tall 1991). David Tall has been a key figure in the field since the 1970s. Most recently he has developed a comprehensive account of human development and of teaching, built around both psychology and the nature of mathematical thinking (Tall 2013). This work is probably unique internationally, in that such a comprehensive account, which also attempts to incorporate all the substantial developments in the field, cannot be found elsewhere.

Tall's work has been particularly influential on research on university-level mathematics education (e.g. Tall 2008). Spurred by the publication of *Advanced Mathematical Thinking* (Tall 1991), a number of researchers have aimed to understand the cognitive processes involved in advanced mathematics. Particular focuses have included the construction and evaluation of mathematical proofs (e.g. Weber and Alcock 2004), students difficulties with definitions (e.g. Alcock and Simpson 2017), mathematicians' epistemic cognition (e.g. Weber, Inglis and Mejia-Ramos 2014), as well as detailed analyses of students' difficulties with particular concepts in undergraduate mathematics (e.g. Pinto and Tall 2002).

During the period from 2000, research activity at the higher education level has become more diversely theoretical. As more mathematics educators have started to study teaching and learning within the university, other theories have been used to make sense of educational practices in the UK – notably Commmunity of Practice and Community of Inquiry (Jaworski 2014) and Commognition (Nardi, Ryve, Stadler, and Viirman 2014), introduced by Anna Sfard (Sfard 2008) and focusing particularly on language and discourse in mathematical learning and teaching. We see these new theoretical directions to be influenced by moves away from Piagetian constructivism towards sociocultural perspectives on knowledge, drawing extensively on the work of Vygotsky and other theorists in sociological domains as we address in Section 1.4 below.

1.4. Sociocultural and sociological approaches

During the late 1980s, Vygotsky's cultural developmental psychology, with its intellectual roots and theory of learning and teaching, radically different from those of Piaget, became known in the UK in mathematics education, and around the world, influenced by Jerome Bruner's seminal talk in Geneva, "Celebrating Divergence : Piaget and Vygotsky", (Bruner 1997). Its knowledge and influence began to permeate thinking and practice from the mid-1990s. The notion of scaffolding, a popularized but, we would argue, also inappropriate term for the zone of proximal development, became ubiquitous in the education world, including Government documents for education. Mediation, activity theory, and the zone of proximal development (Wertsch 1991) became research foci amongst some parts of the mathematics education research community in the UK. An early example of this in the UK can be seen in the work of Simon Goodchild who analyzed 'Students' Goals' in the mathematics classroom using activity theory concepts and Jean Lave's cultural psychology (Goodchild 1995, 2001; Lave 1988).

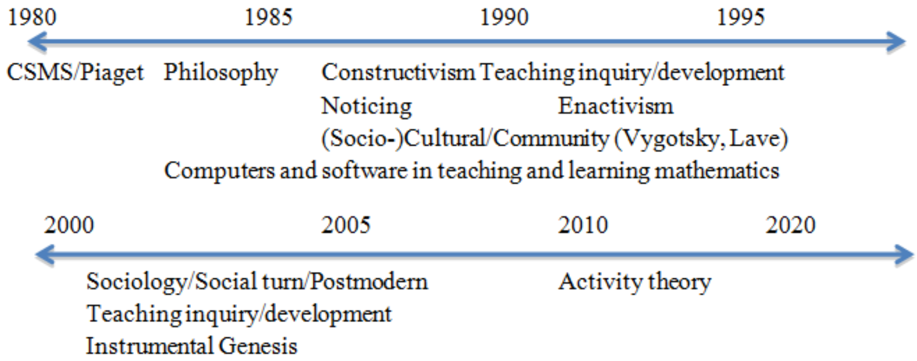

Figure 1 : Suggested timeline for theoretical development in the English World

Activity theory, either in its first generation form from Vygotsky of the mediation triangle, the second generation form from Leont'ev of activity, action and operation, or the third generation form from Engeström, has become a growing tradition of research in the UK beginning late in the 20th century (Leont'ev 1981 ; Engeström 1999). It is certainly as a consequence of these Vygotskian developments that researchers very often refer to sociocultural aspects in their research. This is often taken to mean a recognition of the need to pay attention to students' or teachers' wider social context when accounting for learning, teaching or both. Thus issues of social class are addressed: such as whether those being researched are from advantaged backgrounds or are what has been called poorly served students in those schools in lower socioeconomic settings may have greater changeover of teachers, less qualified teachers and so on. Language and culture may be discussed and examined. It is not always the case, however, that learning–teaching as a cultural-historical process feeds into research questions, design or analysis. Work is required on the part of the researcher to draw on the literature of sociocultural theory in such a way that the significance of Vygotskian theory underlies and informs the research studies. Such issues and concerns have become

the focus of the conference Mathematics Education and Society, discussed further below.
Many researchers attempt to construct a combination of Piagetian and Vygotskian ideas to inform their work. They often tend to use Vygotskian concepts for focusing on the group, class or school as a whole when constructivist learning theories, used for analyzing classroom interactions, do not provide a theoretical basis for the wider settings. For example, Potari and Jaworski (2002) used constructivist theory in elaborating the Teaching Triad through analysis of teacher-student interactions in classrooms, a *micro* analysis. However, they found it difficult to include *macro* considerations, such as social and cultural issues in lives outside the classroom. To include these factors in analysis, they turned to activity theory (Jaworski and Potari 2009). Notions such as the negotiation of meaning and knowledge, and opportunities set up by the teacher to support students' constructions of mathematical concepts are common in UK research today in mathematics education, and beyond (Lerman 2013). Group problem solving or an examination of the rules and goals of an activity are also taken to be features of what is often called social constructivist research. Constructivist, social constructivist and sociocultural theories and their differences continue to be discussed. Many of the important theoretical issues were presented and discussed in a special issue of the journal NOMAD (Volume 8 No. 3, 2000).

A development of Vygotsky's sociocultural programme, that of situated cognition and communities of practice, emerged in the 1990s, following the publication of Lave's 1988 book and her book with Wenger in 1991 (Lave 1988; Lave and Wenger 1991). Jean Lave visited the UK in 1996/7 and an influential seminar was held in Oxford, followed by the publication of a group of papers (Watson 1998) and a further reflective collection some years later (Watson and Winbourne 2008). Communities of practice (CoPs) as described by Wenger (Wenger 1998), based on concepts of participation and reification, along with identity, community, practice and meaning, seem to offer researchers a way of focusing on the group when studying teaching and learning. Learning as participation in the community is a Vygotskian idea developed through Wertsch and Lave, the latter through her studies of learning in cultural contexts in West Africa. In early studies, deriving from CoP theory, there was a danger that CoPs were seen to be everywhere: all kinds of situations in classrooms were described as communities without either examining and identifying Wenger's main components of learning.as mentioned above, and without the complex but very important ideas of legitimate peripheral participation as worked out in Lave and Wenger's book (Lave and Wenger 1991)

Inequalities

It is said that people in the UK have always been conscious of social class, and inequality has therefore formed a strong direction of study and action for many decades. This concern was expressed in ATM's goals as described above. Work in mathematics education research with a concern for equity took a major step when the first *Mathematics, Education and Society* (MES) conference was held in Nottingham in 1998, though predecessors include the two groups *Social Perspectives of Mathematics Education* (Nickson and Lerman 1992) and *Political Dimensions of Mathematics Education* (Julie, Angelis and Davis 1994). Its goal was to support the research in such perspectives in the UK and internationally, the founders Gates and Cotton (1998) arguing that the leading international research group, PME, was too restrictive in its theoretical demands on contributors to allow alternative research paradigms, the sociological, political, and sociocultural at least. If a research contribution did not refer to psychology in some way, it would not be accepted for presentation at the PME conference. That changed formally in 2005 when PME's constitution changed to allow a much wider range of theories to constitute the framework for the research, although the vast majority of research reports presented at PME remain informed by psychology. MES, however, has taken on a very substantial and important life of its own, and it continues to grow in size and influence. Its ninth conference was held in April 2017 in Greece.

Sociology

A different direction for research in mathematics education in the UK has come from sociology. Sociology of education is a well-established field, drawing mainly on Durkheim and Marx, and there are many international journals of sociology of education. Basil Bernstein (2000) has been the major influence in the UK, South Africa and many other countries. His work draws connections directly between the macro features of society, in particular power and control, and the micro issues of the relationships between teachers and learners and who has access to what forms of knowledge. Dowling, Brown, Evans, Tsatsaroni, Morgan and Lerman are just some of those whose work has been located in sociology since the 1990s. Bernstein's theoretical framework enables insights into how curriculum, schools, Government policies and social class pressures lead to maintaining privilege and denying access to success in mathematics to those from disadvantaged backgrounds. Revealing where these policies come from and how these processes take place to allow and deny access are the first steps in being able to make a difference in classrooms, though social structures of society are, of course, not available to us to change. The Marxist origin of sociology of education, and Bernstein in particular, means that there is a strong overlap with Vygotsky's work, his theoretical framework being inspired by Marxism too. Indeed Bernstein wrote the Preface to Daniel's 1993 book on Vygotsky (Bernstein 1993), indicating clearly there that although he, Bernstein, was a structuralist nevertheless his work did not align with Piaget, also a structuralist, but with Vygotsky.

Semiotics

A well-developed sub-field of research in mathematics education, both in the UK and beyond, is that of semiotics. Saussure's work provided a point of departure in the early 1990s into language and meaning by the Manchester Metropolitan University group (notably Tony Brown and Olwen McNamara), in Peircean semiotics (Adam Vile), and since then the chief UK proponents of mainstream Anglo-American linguistics have been David Pimm, Candia Morgan, Tim Rowland and more recently Richard Barwell (e.g. Pimm 1987).

1.5. Postmodern theories

A different orientation in teacher education research and also in mathematics learning in general has emerged from the poststructuralist/postmodern traditions, including Tony Brown, Heather Mendick, Margaret Walshaw and others (e.g. Brown 2011).

The forerunner of this direction is Valerie Walkerdine (1988, 1997) whose gender studies in mathematics education were informed by Foucault in particular. The move from structuralist work, such as Bernstein, to poststructuralist work has led to studies at a local level of the play of power through language. The two key features of these approaches, in the sense of aspects that have informed educational research, are the location of meanings in the local, and in the sources and effects of power.

Meanings in the local

Modernism is characterized by meta-narratives, including Marxism, religion, psychoanalysis, scientism (the notion that scientific research is value-neutral and a 'good' in itself), and capitalist values such as the free market. The break to postmodernism in cultural and social studies was marked in particular by the sense of failure of the meta-narratives to provide universal meaning. Meanings and values, it is argued, are to be found and developed at more local levels, including a recognition of multiple 'locals' of gender, race, ethnicity, religion, and social class that make up the multiplicity of social environments in which each of us moves. The turn to postmodernism points to methodology in particular and calls for ethnography to excavate meanings of students in the classroom, of student teachers in training or in school practice, of teachers in their own contexts, and other lived situations (e.g. Lather 2007).

Effects of power

Relations of power in educational contexts have always been in the consciousness of researchers in education. The high status of mathematics, in the social capital a mathematical qualification carries, in the intellectual status it seems to bestow on those successful in mathematics, and in the ubiquity of the applications of mathematics in society, perhaps singles out the contexts of mathematics education as especially implicated in power. For the most part, research narratives building on the disciplines of psychology, particularly Piagetian theories, the nature of mathematical knowledge, philosophy and others enable an analysis of the effects of power in quite limited ways. Foucault's identification of power with knowledge opened new and very fruitful dimensions for research in mathematics education (e.g. Walshaw 2004).

A new series of research meetings, the *Mathematics Education and Contemporary Theory* conferences, held in Manchester, has extended the semiotic and postmodern work nationally and internationally. Foucault's notions on power/knowledge, Derrida's deconstruction, Rorty's pragmatism and other theories are played with at that conference and in publications. Special issues of *Educational Studies in Mathematics* (ESM Vol. 80, 1/2) emerging from presentations at those meetings demonstrate the body of work developing.

Much has been done over decades in gender studies (see e.g. Burton 1991) and social class, on learning over time, and on analyzing accounts of participants. Postmodern theories have been central in these studies. In relation to these more recent developments, the introduction of postmodern and poststructuralist theories, a question to be asked is "how do they inform mathematics education?" Education can be seen as a *region* (Bernstein 2000), by which is meant that, unlike sociology, psychology, mathematics, and other fields, education has a face to theory and a face to practice. Medicine is another example of a region. Being informed by the disciplines of sociology, psychology, mathematics and others, mathematics education, a sub-field of education, seeks always to see how theories can be seen to shed light on practice, in this case the practice of teaching and learning mathematics. New theories appear and are applied in this way, as a lens on practice, seeing differently and interpreting differently. That these new ideas gain purchase in the sub-field depends on the usual 'gate-keeping' processes of journal review, research grant application, and PhD success or otherwise. It can certainly be said that these theories have provided new insights into power relations and equity issues. Just one example, Mendick's analysis (2006) of girls who do well in mathematics in school but choose not to take it on into the University entry level, the 'A' levels, is set within notions of identity formation and gender, an approach that arises out of postmodern theory.

2. Theoretical developments in French mathematics education research (Aline Robert, Eric Roditi in collaboration with Isabelle Bloch)

In this section, the French authors set out their perspective on the development of mathematics Education research in France from the 1970s to the present day in two parts (70s-80s, 80s-to the present). This research is called Didactics of Mathematics – and this is not anecdotal, as it reveals the intention of a split-up with the Education sciences. In fact, we have taken into account only research clearly identified as didactical research, even if there is in France other research streams such as psychology, sociology, and education sciences. Some of them may concern mathematics teaching or learning but without the focus on mathematical content that is a specific factor in didactical research resulting from the 1960s.

2.1. The development of research in mathematics education: the beginning and the first stage (70s-80s)

A brief reminder of the French context of the emergence of the specific research field called "Didactics of mathematics"

The institutional setting of the so-called "Modern math reform", from 1960 to 1970, brought a great need for mathematics teachers' education. At the same time the social conditions, tied to the students and other movements in 1968, gave rise to a real movement to provide education to all society levels (democratization of education), including university. Unfortunately it was followed by the beginning of the economic crisis from 1974 which changed the perspectives. It is interesting to notice that, probably according to this live context, some personalities revealed a great interest into issues in teaching mathematics (mathematicians, historians…).

The first institutional response was the creation of the network of the IREM (Research Institutes into Mathematics Education), the first three in 1969 and the others, 28 in the whole France, later. And, according to the new training needs for teachers, tied to the reform, and to the start up of this new structure (IREM), a lot of young mathematicians (recruited at university), began to train mathematics secondary teachers in the IREMs. Many pedagogic problems emerged from changes in the school curriculum and from the democratization it was expected to bring (even if results did not live up to these expectations). These mathematicians became quite naturally the first researchers in the field of didactics of mathematics, as developed by Guy Brousseau starting from the 1960s. Some older mathematicians joined them, as they had previously thought about mathematics teaching and empirically explored the new curriculum in some classes.

From research in Education to research in Didactics of mathematics.

It is important to notice that, at the same time in France, the constitution of the "Educational sciences" as an academic field was established, dominated by philosophers and sociologists at the beginning. But subject knowledge was not central to their inquiry. This orientation explains in a large part the need for another scientific approach, centered on subject-based knowledge, and, where mathematics was concerned for the didactics of mathematics. It is meaningful to notice that the university-based French researchers struggled for years to link institutionally to the Mathematics Department and not to the Educational Sciences Department.

First research and theories

In the 1970's, French research in didactics had been inspired by educational science research, referring first to Piaget (and later to Vygotsky) according to Bachelard (even if in his theory the obstacles did not concern mathematics specifically (Bachelard 2000, p. 26). But the need of a systemic analysis of mathematical knowledge – and the way it could be complemented or carried out – emerged, as a crucial tool to be able to understand *how* students learn mathematics. Here, cognitive models are limited, since they pay little attention to the subject matter. In fact, cognitive models might seem to suggest that issues related to learning relate personally to the learner and that difficulties may come from individual deficiency, rather than from the mathematics in focus.

For didacticians, the access to mathematical knowledge depends first on the epistemological analysis of mathematical objects: the specificity of mathematical knowledge is of great importance, as are also the conditions of teaching. Didactics aims at a systemic analysis of teaching and learning processes in an institutional context, and this leads to adapted theories and models. Didactic research does not deny the existence of cognitive operations within individuals, but didacticians aim at identifying the link between mathematics knowledge and, for instance, situations in which this knowledge can prove to be relevant – and then, effective to learn. This approach maintains a strong component of the specific nature of mathematical knowledge, with global and local mathematical analysis of the contents to be taught, leading to a conception of adapted learning situations, which have to be explored further. As difficulties in learning cannot come only from individuals, the complexity in the learning of mathematics recognizes that students can meet some common obstacles, which have to be explicitly studied, and to be taken into account in the elaboration of teaching situations. However, some of the obstacles might be created by the teaching itself, which needs to be avoided.

In this perspective, in the 70s, various researchers elaborated theories that are adjusted to different contexts. We can notice that some first studies focused on primary school mathematics, one reason being the desire to begin with the first development of the child, as Jean Piaget and Gérard Vergnaud did. Another reason may be that at this period, teacher training had not been currently developed at secondary level. Theories are rooted in *questions*, mathematical and professional; they have been elaborated to investigate these questions and, further, to build didactical engineering, not as an end in itself but as an experimental methodology.

The theory of conceptual fields (Vergnaud 1991) can be seen as a transition between cognitive models and didactics studies, as it is inspired by cognitive observations. However, it has a focus on mathematics (mainly at primary level) and it provides an interesting model concerning the way students can deal with a mathematical concept. This concept would be analyzed through three components:

- the collection of situations (problems) in relation to the concept;
- the operational invariant that take place into the resolution of a problem, since the concept is at stake in this problem; and
- the semiotic signs involved in the resolution.

As Vergnaud states:

> "It is a psychological theory of concepts, or better of the process of conceptualizing reality: it enables us to identify and study the continuities and discontinuities between different steps of knowledge acquisition from the point of view of their contents." (1991, p. 133)[1]

Vergnaud's model provides good analysis of the students' work and access to mathematical concepts, and in this way it has been a fruitful transition between a psychological approach and the intentions of building didactical situations for the learning and teaching of specific concepts.

From the 1960s, Guy Brousseau's ambition was both to build a broad model concerning the field of mathematics learning, and to develop situations involving mathematical concepts. As Brousseau had been a primary teacher, the first elaboration of TDS (Theory of Didactical Situations) concerned primary level education and the tools it offered were focused on basic concepts in mathematics, such as numbers, operations (addition, subtraction, multiplication, division of whole numbers), random probability and geometry. Nevertheless, the intentions of the TDS theory are wider and it aims at a global organization and analysis of the teaching and learning context, in the field of mathematics. This model was designed to include at least three dimensions of the teaching-learning problematic (Brousseau 1997, p. 33):

- The first point is the pertinence of the description provided by the model, and the ability make evident the relevant phenomena in the field of research and experience;
- The second ambition is that this theory aims at the exhaustiveness in this description;
- The third point is the consistency of the analysis: Brousseau argues that teachers are not responsible for the coherence of the different tools they use in a classroom, but a theory must assume this coherence in its analysis of the field.

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[1] Author's translation.

To study the knowledge to be taught, TDS introduced the concepts of *didactical transposition* and *fundamental situation* for a specific concept, supposed relevant for its emergence when adapted into sequences for a class. Then the difference between personal and institutional knowledge was introduced. From the beginning of the theoretical elaboration, the concept of "milieu" of a situation was used to characterize the mathematical and technical elements the students can actually use to solve the problem they face (alone or with the teacher's help). At the same time the concept of didactical contract was introduced by Brousseau, to specify the expectations about knowledge, explicit or not, of the teacher and the students towards each other. These concepts help the researcher both to understand deeply what occurs during the classroom, in terms of the teacher's and the student's work. For instance, the study of the contract makes possible the understanding of what could distort or reinforce the activities in which the pupils are engaged. These concepts also help to conceive new adapted situations, tied to fundamental situations and including some adidactical moments when the progress of the students' work may occur without the teacher's help (Brousseau ibid, cf. Article 6).

At the same time (the 70s up to the 80s), Regine Douady, working in the IREM of Paris Sud, elaborated her *tool-object dialectic model*, which is also focused on primary school. The concepts involved in her research are mainly tied to lengths, areas and decimal numbers for instance. Tool-object dialectic is a cyclic process organizing the role of the teacher and the pupils, in which mathematical concepts appear successively as tools for the solution of a problem for the students and as objects with a place in the construction of an organized knowledge, under the responsibility of the teacher. Douady proposed some tool-object situations. An interesting point of her work is also the idea that, to understand a mathematical concept, it is necessary to meet this concept in different settings, and to organize what she describes as an *Interplay Between Settings (IBS)*. As she says, "I.B.S are changes of settings (algebraic setting, numerical setting, geometrical setting) induced by the teacher in order to make the research of the pupils progress and their conceptions evolve." (Douady 1986, p. 5). It may be interpreted as a disequilibrium/re-equilibration introduced in the learning process.

All these theories have developed and been adapted to new contexts during the twenty-first century as we shall see below. But before that, a glance on the institutional frame seems important to better understand what occurs when it changes from 1993.

The first institutional structures of didactics of mathematics

Very important from the institutional point of view was the creation (starting from the 80s) of Didactics diplomas (for mathematics, and later for physics): DEA (equivalent of a master degree) and doctorate (PhD) with possible international collaboration. Moreover, research teams were created (Bordeaux, Paris, Strasbourg). Then, as informal teams became recognized laboratories, many researchers in didactics of mathematics were integrated in multidisciplinary laboratories. But there were no specific jobs for the didacticians in the university until 1990.

More precisely, starting from 1980, summer schools were organized every 2 years as also were, from 1977-78, national seminars (3 times a year). At the same time a research journal was created: RDM (from 1980), then another one: Annals of didactics and cognitive sciences. Some other reviews appear for educators and teachers: Grand N, Petit x.

International structures such as ICME, PME, CERME, EMF, etc., European Structures, English and Spanish journals have all come to enrich the diversity of this landscape.

Connections between research and school teachers

Generally speaking, school teachers are not directly involved as researchers in the didactical research, but there exist relations between teachers and researchers during the research involving some experiments in the classrooms or for in-service education. At the primary school level, the experimental school COREM (Centre of observation and research for the mathematics teaching) was created to enable research in classrooms and has been tightly associated with Guy Brousseau’s team’s research. Up until today, the COPIRELEM (Commission interIREM for the elementary schools) structure created in 1975 within the network of the IREM, enables educators to meet once a year and compare their experience and works.

At the secondary school level, along with the development of the IREM, training on new mathematics programmes allows even until today a collaborative work between some researchers, some educators and some teachers. However, the influence of the didactical research, which concerns all the education levels[2], is greater in the primary school than in secondary and in the university.

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[2] The first HDR in “pure” didactics (and not in mathematics) was held in 1981 and the subject was the acquisition of the concept of series’ convergence. After a first thesis (PHD), giving access to the associate professor, the HDR is a “second” thesis giving access to the university professor ship (twenty years ago it was called “thèse d’état”). In a word-for-word translation HDR means “ability to conduct research”.

2.2. Later stages in the development of the field of the Didactics of mathematics (80s-present)

New contexts

Beginning in the 1980s, developments of technology and software began to address the learning of mathematics, and some new issues emerged, tied to the integration of technology in class. Furthermore the expansion of the digital tools and computers led researchers to work on their integration in the mathematics classes. Their study as artefacts were introduced by some researchers (Artigue 2002; Trouche 2005). Taking into account the Internet, which changes the way teaching can be organized by introducing a greater part of collective work, led to the so-called Instrumental and Documentational approaches, that is, the study of the way teachers can have access to websites, documentation, etc. that can modify their way of preparing their lessons (Gueudet and Trouche 2009).

A favourable institutional context started in 1992: researchers in didactics were “welcomed” to participate in teacher education programmes for primary and secondary levels in the new institutional structure for educating young teachers (pre-service ones): the IUFM[3] in 1992 and then the ESPE since 2013. They could be recruited as associated professors and even full professors. For the secondary level it constituted a real extension of the training.

The development of international assessments, such as PISA or TIMSS, also led new research involving the possible didactical interpretations of these results, as relations between epistemological analysis and also new thinking about an effective use of the results for teachers (Chesné 2014; Grugeon-Allys 2016; Roditi and Salles 2015; Martinez and Roditi 2017).

Furthermore, as PISA highlighted for French students, in spite of several efforts, the inequalities between “poor and rich” children increased (not only in mathematics) and a lot of new research has been devoted to tackling this complex issue.

Development of the previous theories

The number of researchers in mathematics didactics is growing, and the research is increasingly organized by some theoretical frameworks. The main ones are still TDS but it evolves, depending on the contexts, and new ones develop too, leading to mixed approaches.

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[3] IUFM: Institut Universitaire de Formation des Maitres -University institute for teacher education – ESPE: Ecole Supérieure du professorat et de l’éducation -Institute for the teaching profession and education.

In TDS, new situations[4] have been designed for secondary or tertiary level, relating to functions, limits, irrational and complex numbers, integrals, linear algebra (in the work of Alson 1989; Bloch and Gibel 2016; Ghedamsi and Tanazefti 2015; Gonzalez-Martin et al. 2014; Haddad 2013; Lalaude 2016). Situations have also been analyzed in the context of students with special needs (see, for example, Bloch 2005; Voisin 2017; Favre 2015).

The notion of "*milieu*", to take another example, was widely developed (Margolinas 1995; Hersant and Perrin 2005), in particular to be adapted to the study of secondary school and university, and to contribute to the design of situations at this level (Bloch and Gibel 2016). For instance, the introduction of new levels to analyze such a situation allows researchers to better understand the emergence of what is expected in terms of proof in the setting up of an adidactical situation and to conceive suitable conditions for it. In Article 6, a precise analysis of this milieu is presented.

Some new didactical engineering (didactical design) has been developed and explored. New themes and methodologies appear and are 'more and more' developed: teaching practices (in regular classes), integration of ICT in education, second generation of didactical engineering, teachers' education, and assessment (recently).

The growing consciousness of the importance of mathematical symbolism leads to the introduction of semiotic components, for instance in a theory such as TDS (Bloch 2005). More researchers with different theoretical frames accord a new place to the study of the formalism in mathematics, their analysis refers to Duval's registers (1993), and even to Peirce's semiotic theory applied to didactic phenomena, extending the reflection on representations.

In the continuation of this work, Yves Chevallard developed another aspect of the study, an aspect which had not been taken into account in TDS: the institutional organization of the school system, related to mathematics teaching, and the way it works. Chevallard named his theory ATD: Anthropological Theory of Didactics, because it was inspired by anthropology, which describes and models the way human beings act in their society (Chevallard 1996).

ATD focused first on didactic transposition – the way mathematics knowledge is converted into different objects within the teaching process, and how teachers cope with this transformation. The mathematical reference analyses are based on the identification of the involved praxeology: this term addresses the classification of

[4] The first research studies on these subjects were developed in 1980-90 but without consequences on university teaching.

human (mathematical) activity into types of tasks, techniques associated with the tasks, technologies (rationales of the techniques) in use and theories on which praxeology is implemented.

The ATD theory is deeply rooted in questions such as: which mathematics for which society, and how it is organized? The theory provides studies of the institutional context, curriculum, and processes of teaching/learning, according to the position of the studied human beings in the institution. It starts from an epistemological ground: mathematical knowledge analyzed in tasks, techniques, technologies (tied to justifications) and theories[5].

ATD has added new dimensions to the theory in the construction of Study and Research Paths (SRP) (see, Chevallard 2009): that is, problems for students joining a dimension of enquiry and, when possible, mathematical modelling of 'reality'[6]. The new context of teaching leads Chevallard to introduce the so-called "dialectic between media and milieu" to take into account both new resources, such as the Internet, and changes in students' scholar expectations. He claims for instance that it is essential to let students take advantage of the technological progress, as new means to question what is true or not in mathematics.

ATD has been used also by Sensevy to develop a theory about the didactic action of the teacher jointly with the students; let us notice that Sensevy, in cooperation with Assude (2009) and Mercier (Sensevy and Mercier 2007), also used TDS, and in particular the notion of the milieu, to analyze the joint work of teachers and students in a situation. Moreover, other 'local' theories have been developed, for instance a theory about different kinds of knowledge: CKC by Balacheff (Balacheff and Margolinas 2005).

2.3. The case of Activity Theory (AT) (and Double Approach - DA) in research in didactics of mathematics[7]

Emergence of Activity Theory in didactics of mathematics

A new focus on teachers' practices emerged, tied to the fact that a lot of researchers have more in mind the training of teachers, if only because of their professional activity, particularly for the secondary level. More precisely, according to their new professional missions, they have in mind the perspectives of teachers'

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[5] From the 2000's years (for instance Florensa, Bosch and Gascon 2015) it has been called a REM: Reference Epistemological Model.

[6] We can find for instance a SRP for the learning of 3D geometry (Petit x, 75), or other examples on: http://educmath.inrp.fr/Educmath/ressources/partenariat-inrp-07-OS/amperes/

[7] All this part 2.3 was partly published in the cahier du LDAR n°18, co-authored by Abboud, Robert, Rogalki and Vandebrouck (2017).

appropriation of some didactical analyses and results for their teaching. It does not mean that they imagine a precise training, but it means that many of them are more aware than previously of the question of transposition. For teachers, transposition implies what is at stake in didactics in teaching to achieve students' learning of the mathematics in focus.

Indeed , according to the well-known difficulty of the teachers to appropriate the results of research in didactics, and according to the French developments of the "professional didactics" (which go beyond the subject discipline), some researchers suggest that studying teachers' practices has to involve not only the aim of students' learning in the discipline but also the professional aims such as having peaceful classes, and so on. The didactic and ergonomic "double approach" of practices is related to this preoccupation, as it emphasizes the complexity of the teachers' practices with its consequences for (future) training. The main new goals concern the contributions to the study of these teaching practices, involving the study of what occurs in the classroom in terms of "possible" students' activities in relation to these teachers' practices (implementation studies). These goals were explicitly coming into the scope of Activity Theory, as already used in professional didactics, with an adaptation tied to the circumstances in practice. And this had another result in terms of AT: these researchers realized that the use of the AT was somehow implicit in the early 2000s for the analyses of the students' learning (Robert and Rogalski 2002, 2005; Robert 2012). Many tools regarding knowledge, teaching and learning, already developed in didactics of mathematics, may be used for AT's analysis. (It had been also applied to teachers (Rogalski 2003).

The inscription into this theory becomes explicit few years later (Robert and Rogalski, cited in Vandebrouck 2008, 2012; Rogalski 2012) for research into the activity of both teachers and students.

A use of the Activity Theory linked to the didactic and ergonomic double approach

These research studies' first focus is on students' learning in relation to the teaching that the teacher is deploying in the mathematics classroom (from primary school to the university).

To address this issue, researchers have chosen to study students' mathematical activities in the classroom: what the students do (or not), say (or not), write (or not). As what students think is not directly observable, researchers work on these activities' observable marks. This theoretical consideration is in line with the Activity Theory approach studying human subjects' activity, in practice, based on the distinction between task and activity. More precisely, it includes the fact that these activities are provoked (to a large part) by the teacher's activities in the working environment of the class.

Therefore, the research objects are the connections from students' activities to their learning (even if in fact it is tied to global hypotheses more than to accurate results) and from the teacher's activities to those of the students. The global aim is indeed relevant to give a diagnosis of what occurs in the class or to suggest some new ways of teaching that have to be explored. Hence, this approach is both experimental and theoretical, in a dialectical way, involving students' and teachers' observations and data collection, and also data analysis, including possibly new methodological developments.

But if the teacher's activity includes what is done before the class (conceiving the scenario and including some anticipation) and during the class (including some improvisation), these elements are not sufficient to understand the teacher's choices and their consequences on the student's activities. They also involve the professional experience, the knowledge and personal conceptions of the craft and of the mathematics to be taught. Researchers have to take into account also the way the teacher logs into institutional and social constraints (such as curricula, school's social environment and so on). This conception of the complexity of the teacher's practices characterizes the didactical and ergonomic double approach. This approach considers 5 components of the practices to interlink: two of them related to the choices of contents and implementation, two other related to the way the teacher takes into account social and institutional constraints and a last personal one, related to knowledge, experience and representations. Three levels to study the organization of the practices are added, which are related to each other. The global level involves the projects, the class designs, and so on, the local one involves what happens in the class (implementation, improvisation), and the third one, micro level, is devoted to the automatisms and routines. It particularly helps researchers to study the practices of beginning teachers, who have not yet global representations nor micro habits in the classroom.

But even if researchers are convinced of their importance, they do not systematically study parameters others than mathematics, tied for instance to affective factors, self-confidence, social and cultural origin. However a lot of research is devoted to the study of disadvantaged classes or schools at the elementary level. For instance, the research on disadvantaged classes went on, including the study of the teachers' practices and leading to descriptions of acute contradictions in those classes, between learning and quick achievement (for instance Peltier (2004), Butlen (2007). This research contributes to highlighting the frequent disequilibria in the classes between devolution (moments where the students work) and institutionalization (moments where the teachers address the knowledge to be learnt); the latter is often reduced, or even missed.

Taking these factors into account would for example involve for the students' activities some levels of organization, such as the global position posture according to the school, including the expectations and the relation to knowledge, the local attitude in class, including the participation to collective activities, and the micro level including some automatisms, for listening for instance.

New developments in the AT theoretical frame are conceiving tools to better target the distance between what students do and / or know and the teacher's actions and mediations according to an adaptation of the notion of a zone of proximal development (ZPD) for mathematics (proximity-in-action and discursive proximities). The aim is to cover the different ways of drawing on what the students already know or have done, more or less close to the general knowledge at stake. Some examples are given in Article 3. But also studying the moments of knowledge exposure through the development of analyses in terms of discursive proximities, in order to appreciate opportunities for possible or even missed proximities between what is general and stated by the teacher and what the students already know or do (cf. Article 3). The specific analyses of activities with technological tools allow access to what is new in terms of working on these instruments, both for the teacher and for the students and to provide the means to take more account of it (cf. Article 5). Unexpected difficulties of students have been brought up to date. Some of them are related for example to knowledge adaptations to be used by students when solving exercises. In some cases these adaptations are not detected by teachers and are left implicit. One could talk about teachers' "naturalization" when it is as if these adaptations are too familiar to teachers to be located. Often some students ask questions about these implicit adaptations, especially since the class is diverse. But if not, they may be overlooked and this likely blocks some students, even for a long time as it is often repeated.

Other developments are about the practices related to assessments, to collaborative research and the clarification of roles, to training and support of school teachers in very disadvantaged classes.

Let us notice that our "appropriation" of the ZPD notion must be specified, insofar as this notion is related to individuals whereas we use it in the context of a class.

3. Noting differences – results and outcomes

3.1. A key difference between the English and the French use of theory

The specificity of mathematics, mathematical knowledge, and mathematical thinking frames the French approach and constitutes the starting point of the French research. The building of mathematics teaching and learning processes and procedures on the basis of research on specific content is the common programme and development in the community. The English approach is perhaps to start from an understanding of teaching and learning in general, moving subsequently, or in parallel, to the specificity of mathematics. What is meant by ‘understanding’ is contested *within* the disciplines of psychology, sociology and philosophy, let alone *between* these disciplines. Hence, the English philosophical tradition of *pragmatism* has been pointed to as the framing of the proliferation of theories, and their relations to the practical traditions discussed earlier. We would argue that it is perhaps differences in how the work of the field is to be orientated between mathematics leading to teaching and learning, and teaching and learning leading to mathematics. More precisely, we now give a contrasting glance at the results, difficulties and perspectives in each tradition.

3.2. Results in English and French research

English research

In the French perspective, as seen above, there are three main headings. As we have set out in the English/UK section, there are many more theoretical orientations and indeed some of them are in opposition to each other. This makes it very difficult, if not impossible to identify ‘results’ that would be accepted across the community, and therefore it makes this an idiosyncratic account, dependent on the two particular authors of this part of our article. In this light we will summarise some ideas that we think most important to highlight, and in doing so we are perhaps looking to areas we consider have and are continuing to produce results, as well as those that are of major interest in the English community. We remind readers, however, that earlier theories are not replaced by new ones but continue their ‘internal’ development. We list five, below.

Strong centralized regulation and policy studies

Where there have been policy studies in mathematics education they have revealed the strong hold on what is taught, how it should be taught, and how learning is measured by the Education Department of Government of both left and right. The reports from the powerful and influential framework of inspection of schools are taken in place of research to inform policy. Performance in mathematics in PISA and TIMSS reports that have shown an apparent slip in UK achievement over the years are another element in what informs policy. The critiques produced by the research community (see e.g. Lerman and Adler 2016), inevitably, do not impact on Government. Nevertheless we consider that such studies by mathematics education researchers, revealing the negative effects of such control, and positive ones where they appear, are needed.

Studies into classroom practice uses of technology and mathematical understanding

The relationships between classroom practices and the theories used to analyze and explain didactical and pedagogical approaches to creation of mathematical understanding are still central to English research and relevant to classroom practice in the UK. Practical traditions are alive and well, pursued by teachers and teacher educators, in relation to political forces and school organizations. Research results inform such practice. A central interest is in developing practices which achieve students' mathematical understanding. As indicated above, a range of theoretical perspectives are used by researchers in these areas. The use of technology is now firmly embedded in the curriculum, but research and associated theory into this use varies with focus. Much research is very small scale with teacher educators and teachers exploring situations at a local level and using theory as it seems to support their own research questions and design. Both constructivist and sociocultural theories, as well as enactivism and instrumentalisation are used. Large scale projects are few, due to limited sources of funding and the requirements of funding bodies.

Informing equity studies

The replication of social class differences in terms of achievement in school mathematics remains an intractable problem. The main factor associated with success and failure in mathematics remains family socio-economic status. In the research field there are insights that offer ways forward, such as gender studies, critiques of setting[8], teacher expectations of who can achieve in mathematics, and working with challenging mathematical problems, but whilst many teachers have adopted and use the materials that have developed from research, these in general have not been taken up in curricula or learning goals as prescribed by Government. We should note here, although the evidence comes from outside of the education research community, that achievement overall, including in mathematics, has improved for all children, though to a lesser extent for children from low socio-economic groups. This has been achieved in some areas of the UK, particularly London. The causes are likely to do with levels of investment, embedding of higher expectations of all children, and other factors outside of the research field of mathematics education.

Meaning and relevance in mathematics

In the mathematics education research community internationally there are growing numbers of studies of the role of everyday reality of students to be brought into the classroom to make mathematics relevant and meaningful. Theoretical perspectives developed include ethnomathematics, critical mathematics and 'funds of knowledge' (see Civil 2016). In the English research, critiques of these approaches have come, in the main, from theories in sociology of education.

[8] The grouping of students according to some concept of 'ability'; thus producing a hierarchy of 'mathematics sets' in a school year group (see Boaler and Wiliam 2001).

Drawing on Bernstein or Bourdieu (e.g. Cooper and Dunne 2000), researchers have identified how turning to the every day to provide meaning that may motivate students better than the decontextualised mathematics that predominates in textbooks and curricula may indeed further disadvantage students who do not succeed in school. These studies have, we believe, pointed to important issues in learning mathematics in general and for students from disadvantaged backgrounds in particular. This is not seen to devalue ethnomathematics, etc., but to indicate the tension between the relevance and applications of mathematics and gaining a grasp of the esoteric symbolization of mathematical knowledge and the need for both.

University mathematics

Finally, to point to the growing body of research on the teaching and learning of mathematics at University level. For rather too long the research community has treated this field as unproblematic. The recent decade or so of the growth of this field has shown that this view is quite wrong. Cognitive studies explore the mathematical learning of large numbers of students, whereas socioculturally rooted studies seek insights into pedagogical practices and their impact on learning outcomes. A main difference with these studies and those conducted at classroom level relates to the number of students in a particular cohort (often between 100 and 300) and an economic need to teach them all together. Thus research results into practices at school level (where class sizes are around 30) are often not applicable at this higher level. The research findings and their implications are informing University mathematics staff, though there are still many who place the whole reason for student failure on the students themselves.

French research

We now summarize some salient features of the main research in the French tradition.

Theory of Didactical Situations

The framework TDS is particularly concerned with the design of learning situations of which the implementation has to be studied. Some evolution occurred in the research so that now the ordinary classes and the resources production are also studied. But the main aim remains to study the cognitive potential of a given situation, that is the study of what the students may learn according to the choices of mathematical content. This leads to the identification of what could be due to the *milieu* (present in the situation independently of the teacher) and to the didactical contract, but it also leads to work on the didactical variables that enable the teacher to play on possible student actions. This induces a conception of the actors as generic subjects, having a specific function (student, teacher) rather than as singular, active subjects as it is for AT and double approach.

However, research studies remain mostly at a local level of analysis, even if the curricula are obviously taken into account; depending on the adopted framework, students and teachers are considered as more or less “generic”.

Anthropological Theory of Didactics

The ATD, concerns more a global vision of the mathematics education system including teachers, emerging from existing constraints and norms. Moreover, the phenomena identified are related to different levels of determination, ranging from class to society. This leads to a conception of the actors as subject to a given institution, and, again, not as singular actors.

ATD allows us to:

- study the modifications of the institutional context, e.g. how it works in a professional environment such as in Engineering schools, and also which mathematics are taught and why;
- take into account the conditions and constraints of teaching, and analyze the teachers’ role, for instance how they introduce and validate tasks, techniques, technologies and theories $\rightarrow$ in algebra, geometry, calculus; what is the relation between these *praxeology* (cf. above) and the school level; how these conditions can appear in didactical studies and how they can be taken into account in teachers’ training;
- build “inquiry-based teaching” as SRP (cf. above).

Activity Theory

By giving *a place to students and teachers in their singularity*, as “human beings”, the AT framework is specifically adapted to study what effectively happens in class, whether practices are ordinary ones or not. Local analyses are more developed than the global ones.

In terms of results, researchers can stress obtaining important results on the teacher's practices and on their stability[9] (shown by several of our research studies), ensuring therefore the validity of the extension of our local outcomes. Taking into account contents and constraints, some “robust” scenarios have been proposed and tested. “Robust” means that whatever the implementation in the classroom is, if not extraordinary, the expected activities are possible for students.

This research also enables researchers to propose a critical view of institutional instructions. Tensions may exist for instance between diverse expected rigour

[9] For experienced in-service teachers, the component tied to the implementation choices seems particularly stable.

requirements needed for the different contents that have to be taught during the year. The didactical contract may be raised to describe those requirements, but AT research addresses the issues on teachers' practices in terms of choices and students' difficulties. Another example is relevant: in order to give sense to mathematics, the institutional injunction makes the students work on complex tasks. But to solve these complex tasks, students use diverse procedures. Then this diversity makes it difficult for teachers to highlight the knowledge aimed for between the various paths that have been used.

Finally, it is important to be aware of the fact that French theories, whatever they may be, are by no means a sort of enclosure but rather guarantors. It is important to use them to guarantee a certain coherence in the division of the observed reality, but also to identify what could be unexpected, and even to know how to transform what first appears as a "disturbing noise" into a new development. Likewise, if data gathering must be adapted to the theoretical frameworks, this, fortunately, may still produce unexpected phenomena; these are opportunities that the research has to grasp!

Conclusion

It is quite impossible to compare general uses of theories since, despite the many differences articulated above, the deeper issues and outcomes of activity in the two domains, such as mathematical understandings, impact and scale of research findings, are not so different. Articles 3 and 5 in this volume, are devoted to research on some key issues and allow us to understand more deeply how the theories are used or developed in each case. We address successively examples of different uses of AT as a lens to study what occurs in a classroom, what occurs with uses of Digital Technology in mathematics teaching, and of how practice and theory are related in the use of video in teacher training contexts. Article 6 provides an insight on the use of TDS in two contexts of training and lets us understand their importance, relative to each outcome. In the concluding article 7, we take up again the ways in which our research approaches are different or comparable and ends with a glance towards the biggest issues that face us all.

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USE OF ACTIVITY THEORY TO MAKE SENSE OF MATHEMATICS TEACHING: A DIALOGUE BETWEEN PERSPECTIVES

Abstract. This paper examines the interactions between teachers' decisions, discourses and acts, and the intended students' learning. The focus is theoretical and methodological as it attempts to exemplify theoretical perspectives in studying mathematics teaching in its complexity. It takes into account, together or separately, the overall setting: sociocultural and institutional and the epistemological point of view on mathematics and its teaching in class. For some of the authors, the study of teacher activity in relation to students' mathematical activity, and affective and social needs has been the focus of their research for many years, using different theoretical constructs and empirical data. As for the others, their research in the same area was focused more on the presumed cognitive needs, in relation to the practices and the mathematics at stake. The article reveals that Activity Theory has been used differently by the two traditions (English and French) as a framework for analyzing and interpreting the relations and interactions between teacher and students' mathematical activity in research studies of the authors. This article exemplifies these different ways of using AT and discusses issues the perspectives raise for interpretation and analysis.

Keywords. Teacher activity, student activity, cognitive aspects, social aspects, affective needs

Résumé. Deux perspectives pour l'utilisation de la théorie de l'activité dans l'étude de l'enseignement des mathématiques. Ce texte est centré sur l'étude des relations entre les activités des enseignants et celles des élèves, les premières étant décrites en matière de décisions, de discours et d'actions. Il s'agit d'adopter un point de vue théorique et méthodologique, en lien avec les perspectives adoptées pour ces analyses complexes ; cela fait intervenir, sans qu'il y ait exclusion d'un des aspects, l'ensemble des déterminants socioculturels et institutionnels, les déroulements en classe et le point de vue épistémologique. Une partie des auteurs fait notamment intervenir dans l'étude des pratiques enseignantes les besoins affectifs et sociaux, l'autre insiste davantage sur les besoins cognitifs présumés et les mathématiques en jeu. Tous les auteurs se réclament de la théorie de l'activité comme cadre théorique pour analyser et interpréter les relations et interactions entre l'activité enseignante et les activités mathématiques des élèves. Nous illustrons chaque point de vue par un exemple en discutant des questions qui se posent à l'autre point de vue.

Mots-clés. Activité de l'enseignant, activité de l'élève, aspects cognitifs, aspects socioculturels

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Introduction

In this paper we, English and French researchers, present briefly the different ways that Activity Theory (AT) has been used in our research and exemplify them through the analysis of two data extracts. The extracts have been chosen to be illustrative of our approaches and provide opportunities for contrasting them. Indeed, our collaboration has demonstrated that the contrasting approaches in using AT results in the need for different qualities and characteristics of data generated for our empirical purposes. Thus, it became clear to us very early in our collaboration that we could not easily share data that had been generated specifically for either the English or French perspectives. The first extract comes from a group tutorial session at a university in the United Kingdom where first-year students work on tasks of finding partial derivatives of a function. The second extract comes from a high school classroom in France where the focus is a lesson (i.e. moment of teacher exposition) on the sign of an inequality of the second degree. Even if the situations are quite different (work on tasks for United Kingdom, a lesson for France) the teachers' goal in both extracts is for the students to make sense of the underlying mathematical ideas, while the students' goal is less visible to the researchers. In both cases the teachers are more or less guided by what students say and do, and act to enable students to achieve the teachers' goals for the students. Research questions are closely related to the theoretical perspective adopted and consequently the English and French groups are concerned to address different research questions. The English group is concerned with the nature of teaching in the tutorial and how this is linked to student mathematical meanings. The French group is concerned with the distance between what students do and/or know and the teacher's goals for the students during a lesson; and how students' responses to the teacher influence the actions and mediations of the teacher in trying to reduce this distance.

For the English group, the analysis is framed by Leont'ev's work on consciousness as the basis of personal knowing and establishing notions of Activity Theory (AT), which is built on Vygotsky's psychological interpretation of Marxist dialectical materialism. This articulation of AT is manifest in categories of actions and goals, division of labour, inner contradictions and mediating tools. These categories are used in the analysis of the first extract looking for relations and tensions (that emerge from the activity's inner contradictions)[1] between the teacher and students' activity and how these tensions were resolved.

Constructs from Vygotsky's work and the French Didactics, such as ZPD (Zone of Proximal Development) and the 'Double Approach' are used for the analysis of the second extract. The approach entails mathematical analysis of 'relief'; that is, the specificities on the learned notion intersecting with curricular and students' difficulties. The approach is also concerned with the dynamics between conceptual and applied aspects and corresponding occasions of proximities (between a student's present and intended knowledge or conceptualization). The French approach thus shows, again the process of bringing closer, the teacher's actions and the students' expectations and needs (Bridoux, Grenier-Boley, Hache, & Robert, 2016).

The two perspectives do not have the same starting point or the same focus when investigating class activity (teacher and students). The French perspective is firstly concerned with students' activities in order to detect what characterizes and what differentiates teachers' practices, according to the adopted hypothesis on students' learning (conceptualizing). Whereas the English perspective begins from the teacher's activity and the mathematics she is dealing with, to study what occurs in the class in terms of students' activity. More globally, a critical analysis of the different ways of using AT will be developed to look for the similarities and complementarities of the different perspectives and on how they contribute to our learning about the complex relation between mathematics teaching and learning. Such a reflection will contribute to possible new ways of theoretical networking.

1. Activity Theory

In Sections 1.1 and 1.2 below we present, first, the English (1.1) and then the French (1.2) perspectives on Activity Theory. We explain briefly in each case the main theoretical constructs that underpin our use of AT to characterize the activity of mathematics teaching-learning. The English and the French perspectives relate approximately to different levels of the general frame of AT as grounded in the work of Vygotsky and Leont'ev, and later developed in some contrasting ways in the English perspective and in the French tradition. Considering Leont'ev's three layers of activity (Activity-Motive; Actions-Goals; Operations-Conditions: Leont'ev, 1978, 1981), the French approach is centred on the actions and operational layers whereas the English one also gives consideration to the motives and goals of activity. Moreover, the French analysis focuses on the teacher/student

[1] Beside this theoretically based concept of tension, the everyday notion of tension is used in this chapter (and in chapter 4) as denoting the idea of divergences between intentions or goals.

relationship within the classroom related to mathematical objects within teaching and learning issues. Tensions are seen to be situated in the gap between what could be initiated from students' mathematical actions and the mathematical aim of the teacher. From the English point of view, the tensions are considered to emerge from contradictions arising within a larger activity system including institutions; they deal with the specific goals of each teaching-learning event within the system and relate to the mathematical objects at stake. Even if the two perspectives refer to the same theoretical source, Vygotsky, they follow different paths.

The English perspective presents itself in line with the evolution of AT as developed at a general theoretical level in section 1.1 below. Nevertheless, this perspective focuses on classroom interactions, seeking to analyze interactions in terms of the more general concepts of AT. The French perspective is presented in section 1.2, it starts from Vygotsky's theory but focuses on his developments on conceptualization and the key notion of ZPD (Vygostky 1986, chapter 6). This theoretical input leads to question precisely the tasks presented to students and their intended mathematical activity. These contextualized tasks and their implementation in class may be considered as tools mediating the teaching-learning activity. The ZPD starting point is developed in an epistemological way that analyses how the teacher makes use (or not) of possible proximities between students' previous knowledge and the mathematical content at stake. These proximities could be considered as didactical devices that the teacher uses to bridge the gap mentioned above.

1.1. Activity theory from an English perspective

Our analysis of mathematical discourse in a university tutorial seeks to explore and explain the exposition and appropriation of mathematical meaning by tutors and students respectively. In doing so, we take a socio-cultural approach, that is cultural-historical activity theory, which emphasizes consciousness as the basis of sense making and hence personal mathematical meaning. Roth and Radford (2011), in their articulation of AT, explain that 'consciousness' in activity is theorized as "the relation of a person to the world" (p. 18). They argue, based on their interpretation of the work of Leont'ev, that consciousness is the basis of personal knowledge, rather the cognitive and constructivist positions that invert the relation by positing knowledge (schema) as the basis of consciousness: "consciousness, …, is not characterized by comprehension, not by the knowledge of the significance of the subject matter, but by the personal sense that the subject matter obtains for the child," (Leontyev, 1982, p. 279, in Roth & Radford 2011, pp. 17 &18). Consciousness emerges within 'activity', which is the sole, indivisible unit of analysis, or in Leont'ev's terms, "the non-additive, molar unit of life" (Leont'ev, 1981, p. 46). Our purpose here is to theorize the university mathematics tutorial within terms of AT; for a deeper discussion about the principles of AT the reader is referred to more comprehensive expositions such Roth and Radford (2011), and Leont'ev (1982).

Activity takes place over time and is pursued to achieve an object that results in an outcome or product in the material world, and its realization is its motive in the psychological consciousness, "an activity's object is its real motive" (Leont'ev, 1981, p. 59). In the present case, we see 'activity' as university education in mathematics, as manifested in the tutorial. The motive here is the education of students in mathematics with the object of their enculturation into the mathematical worlds developed historically and seen through the eyes of the research mathematicians who teach them in the university.

Different actors within the activity may seek different, not necessarily contradictory outcomes, for example: engineers and scientists equipped with the necessary skills to contribute effectively to national and societal development; a deep understanding of mathematics; sufficient mathematical knowledge to achieve a degree result that secures employment or admission to further study. AT is rooted in Vygotsky's psychological interpretation of Marxist dialectical materialism, and points to the division of labour, inner contradictions and tools that mediate between subject and object of activity. These characteristics of activity are fundamental to understanding the educational transactions that occur within a mathematics tutorial.

Mathematical ideas are presented in various representations such as graphs, equations, symbols, and expressions, which are the tools that mediate mathematical meaning. However, embedded in these tools are contradictions rooted in mathematics as well as didactical transactions (teaching actions and operations). The tutor may use mathematical representations to lead the students to a deep understanding of the mathematical ideas. Students may also be expected to communicate their consciousness of the ideas using these same mediating representations. However, the representations are not the mathematical ideas that the tutor wants the students to understand, they need to understand and be able to use the representation at a surface level, they also need to become aware of the mathematical concepts represented at a deep level.

In her attempt to address the inner contradiction of the representation the tutor may use a didactical tool, 'inquiry'. She will pose questions about the representations and mathematics and try to provoke curiosity and inspire the students to ask their own questions. However, the division of labour in the tutorial in which the tutor is cast as the expert who teaches and the students are novices who do the learning creates the context for the contradictions of inquiry. The teacher's questions may be intended to cause students to reflect on their own mathematical meanings, to articulate them, and by bringing them into the open allow them to be examined and give the students an opportunity to review and revise them. The student, however, may confuse the tutor's question as an attempt to evaluate. The student may also be reluctant to share naïve meanings because of the reaction of his/her peers in the tutorial.

In Vygotsky's analysis of activity, the division of labour results in contradictory perceptions of the material product in a material transaction; for the producer, the product has an exchange value, it is worth what the producer can get in exchange for it. For the buyer, the product has a use value. The common category in the contradictory meanings of the material product is the notion of value, the transaction occurs because for both producer and buyer the product has 'value'. At this point the contradictions of the transaction in the mathematics tutorial - between teacher as a producer of mathematical contents and students as buyers - may not share a common category, especially if the tutor and students have different goals. For the tutor, the goal may be that the students develop a deep understanding of mathematics. The tutor is experienced, informed and in possession of her own deep understanding of mathematics. On the other hand, the students' goal may be 'instrumental' in acquiring that consciousness of the representations and relationships that will enable them to be successful in an examination. The different goals imply a different consciousness of mathematics. Is it possible to consider a common category 'value' of mathematical competence if the meanings of competence held by tutors and students are so different?

Returning briefly to the theoretical grounds of AT, it is possible the above discussion could convey a notion of activity being a structure of distinct elements – actions that combine into events, operations such as asking questions, and tools such as mathematical representations. Such a notion would be incorrect. The activity exists as actions and the actions can only be understood within the context of the activity, as activity endures over time the actions take place in time. As the activity is established on achieving some object, the actions are directed to achieving goals. Actions are achieved through carrying out operations which are subject to constraints and mediating categories embedded with the activity – the rules, division of labour, tools and acting people's consciousness. Each of these categories can be understood only in the context of the indivisible unit of analysis – activity, and the analysis of activity entails examination of each of these categories and the dialectical relations that exist between them.

1.2. Activity theory from a French perspective

Hypotheses and theoretical approaches

Framing our research in an AT perspective leads us to firstly study class episodes when students are solving mathematics exercises. Indeed, from the perspective we adopt, this kind of students' activity is what determines, for a great deal, their learning (Vandebrouck 2012; Abboud-Blanchard et al. 2017). The analysis considers both the tasks provided and their implementation in lessons. The latter are studied with reference to the expected students' activities deduced from task analyses and from the observed management of the teacher. The context (programmes, mathematical notions involved, and particularities of the school, the class and students) is also taken into account. But between the planned activities and what the students really do, there exist many differences and diversities. We do not have access to the actual individual activities of the students (of each student) but we try to apprehend their possible activities which are associated with the teacher choices in terms of statements, exercises, discourse (mathematics or not), students' work format and management (including what comes from the students themselves). Moreover, these choices are conditioned both by the desire to make students learn and by constraints related to the teaching approach (see Double Approach Robert & Rogalski 2005). These constraints may lead to choices based on, for example, curricula, class heterogeneity, time constraints, and working in a peaceful atmosphere, choices that are not directly related to students' learning.

Studying episodes of exercise solving, enabled us to have a growing knowledge of both students' and teachers' activities, accomplished within these class moments (Robert 2012; Abboud-Blanchard & Robert 2013; Chappet-Pariès, Robert & Rogalski 2013; Chappet-Pariès, Pilorge & Robert 2017). However, there remain other crucial moments in class learning, those of the exposition (specifically, lectures and lessons) or moments of 'telling' when the teacher is directly presenting some mathematical content. The methodological challenge is to study these moments while simultaneously taking account of the mathematics at stake, teaching and learning, and the broad context within which the lesson occurs. The student activities are often invisible and therefore inaccessible. The usual a priori task analysis does not apply here, and yet it is indeed the organized set of lessons and exercises that contribute, in a long-term process, to the intended conceptualization (learning), which is our actual object of study. Indeed the decontextualization and the general formulation (institutionalization) of the elements of mathematics involved (e.g. definitions, theorems, properties, formulas, methods etc.) are indispensable to this process.

We look to AT to conceive and provide the tools to analyze these moments. We draw inspiration from Vygotsky's theories (1986) and especially from the ZPD model to propose a hypothesis that shapes our study. In order to analyze these class moments, we focus on the teacher's discourse that presents the knowledge to be learned, tracking his/her role as a mediator between the specific (contextualized) and the general, and between the old and the new. Indeed, we admit that the challenge entailed in the exposure of new knowledge is to get students to appropriate and use connections between words, formulas and general statements and particular contextualized mathematical tasks proposed to them. We must consider what may have happened before and what would happen afterwards in the classroom; the connections may emerge at first provisional and partial, during and after the course. In other words, the more the teacher succeeds in bringing together the general elements at stake with what students already know or have already done, including contextualization, the more the conceptualization (learning) aimed at could progress. That could be done by means of comments, of making explicit connections with existing or future knowledge, by explanations of the use of some statements, noting what is invariant or related to historical references, and so on. We call 'meta' all the elements of the teacher's discourse about mathematics and about mathematical work (see Robert & Robinet 1996; Robert & Tenaud 1988). The 'effectiveness' of the lessons, conceived as elements of a long process, then depends on the opportunities, involving the chosen tasks, and the quality of all teacher's mediations.

In order to carry out such a study, it is necessary to provide tools to analyze the content of the lessons (supplementing the tools for analyzing 'exercise-type' tasks) and their implementation.

Methods

The data we collect is mostly a video recorded by the teacher herself with a static camera at the back of the classroom, its transcription and, if possible, a teacher's account of what has preceded the lesson and of the context of the class.

First, we study what we call the *relief* (or landscape) of the mathematical notion to be taught, combining therefore a threefold analysis of this notion: epistemological, curricular, and the already known difficulties that students experience when meeting this notion. This enables us to estimate the distance between what students already potentially know and the new concepts to be introduced, and to reflect on this introduction. It is also important to understand if and how the difficulties the students may experience are taken into account within the lesson. These analyses are subsequently used, on the one hand, to characterize each specific lesson to be studied, with its precise environment, and to have an idea of the possible alternatives. As for the transcriptions, a first examination makes it possible to specify the modalities of the implementation, the moments of exchanges, listening, copying the dialogue or even the repetitions, which makes it possible in particular to track down what comes from the students (answers or questions).

Once these two stages of the analysis are completed, we try to detect the teacher's choices related to the approaches taken in the lesson. We pay particular attention to what can be more or less qualified as attempts of alignment that the teacher operates between what has been done in class and what he/she wants to introduce. We distinguish in particular the connections between general and particular and those which are made at the same level of generality. These are what we call the discursive proximities that we will detail in the following. Notice here that it is the researcher who interprets, on the basis of the relief she/he has already established that there may or may not be such alignment or need for alignment; the search for what is implicit is thus valuable in this respect. What is at stake here may concern: the level of generality of non-contextualized statements, rigour and vocabulary, written versus oral properties, and anything that can illuminate the functioning of the presented knowledge, in particular its status (accepted, demonstrated or presented without comments), and its usefulness for future applications or for consistency throughout the course.

The proximities are hence elements of the teacher's discourse that could influence the students' understanding according to their existing knowledge and their activities, which are in progress. This occurs in the operationalization of the mathematics class within the presumed ZPD. Three types of proximity are to be distinguished in the way the teacher organizes the movements between the general knowledge and its contextualized uses: we call ascending proximities those comments that make explicit the transition from a particular case to a general theorem or property; descending proximities is the other way round; horizontal proximities, however, consist of repeating or illustrating the same idea in another way.

The study of the transcription in a more detailed way gives access to what happens during the lesson. More precisely, we can distinguish between the proximities introduced by the teacher from the outset and the proximities arising from students' answers to the teacher's questions or resulting from students' spontaneous questions. Thus the researcher can have a fairly accurate view of all the proximities, of what motivates them and of what remains implicit in the studied lesson.

This enriches the comparison between different lessons and classes, from the same teacher or between teachers. The developments of these theoretical tools enable us to target the gap between what students do and/or know and the teacher's actions and mediations. The theoretical tools also facilitate the study of the moments of knowledge exposure through the development of analyses in terms of discursive proximities. Moreover they enable us to appreciate opportunities for possible or even missed proximities between what is general and stated by the teacher and what the students already know or do.

2. Examples illustrating the perspectives

2.1. Discussion of analysis with regard to theory in English perspective (cf. 1.1)

The analysis is illustrated through an episode from university mathematics teaching within a tutorial setting with first-year mathematics students in England. The students are expected to attend lectures in calculus and linear algebra and work every week on problem sheets that their lecturers have set. In the tutorial the tutor (third author) works with students (one hour per week) on material related to the lectures, often taking questions from the problem sheets that according to her would reveal key concepts in mathematics and might cause difficulties for her students. The episode comes from the tutorial in Week 6 of Semester 2. Four students and the tutor are present in this tutorial. The tutor has chosen to work with the students on questions from the problem sheet set by the lecturer of the calculus course involving differentiation of functions of two variables. The students work together on the following question:

Question: The three graphs of Figure 1 show a function f and its partial derivatives f_x and f_y Which is which and why?

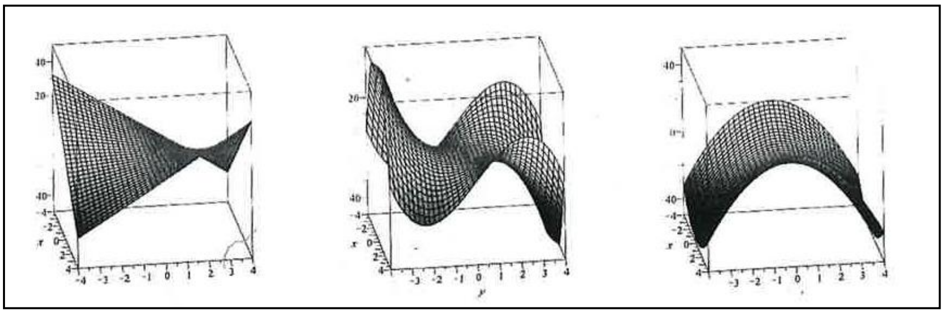

Figure 1: Extract of the problem sheet

A transcript from the first 6 minutes of the tutorial is presented in Appendix A. In this we see a dialogue between a tutor and 4 students in a university small-group tutorial focusing on distinctions between partial derivatives of a function represented graphically. In the analysis, the tutor's knowledge of the mathematics at stake is accepted. The tutor also has knowledge of the students, which developed through engagement with them through the previous semester, and this knowledge guides her engagement through the tutorial. The tutor's goal is that students will develop a deep understanding of the mathematics through engaging in a critical manner with the graphical representations, transformations, mathematical language and expressions that are used in the question (Fig. 1), the students' presumed prior knowledge and the content of the course they are currently studying.

The main research question that is addressed here concerns the nature of teaching in the tutorials (including the characteristics of teaching – what the teacher does, her *actions* and associated *goals*, how mathematics is addressed, what tools she uses to engage students and encourage their understanding) and how this is linked to students' mathematical meanings. Initially, we analyze the episode line by line using a grounded approach to see the actions and goals of the tutor and the students' responses, and to start to interpret them. The approach, which we have used throughout our research over many years, takes the data as a point of departure, and begins with a process of data reduction out of which the main themes emerge and are subsequently categorized using open coding. Essentially the approach does not apply any theoretically rooted categories until after the initial open coding. Then we use constructs discussed in Section 1.1 in the context of the Activity of university mathematics teaching, and of tutoring in particular, and its motive, student learning and understanding of the mathematical concepts; the tools that are used to achieve goals; the emerging contradictions between the tutor's goals and the students' responses. All three stages, data reduction, open coding and application of theoretical constructs, were undertaken independently by three analysts (authors 2, 3 & 4), before meeting to agree the interpretation set out below.

A grounded analysis of the episode – a summary

The following figure presents the first 6 turns of tutorial transcript, the complete 6 minutes transcript is reproduced in Appendix A.

1.	T: [Tutor and 2 students are present] I thought we'd have a look at Q3 first. I've selected all of these questions for a purpose, because each one of them highlights what I would call key concepts. [She refers to question 3 as presented above. Two more students enter the room – tutor greets them and repeats her words above]
2.	T: So, first of all what are these things fx and fy? Alun. What is, what do you mean, if you write fx and fy?
3.	S: (Alun) dee-f-dee-x
4.	T: And how would you write it?
5.	[He indicates with his hand the partial derivative symbol, ∂]
6.	Yes partial df/dx and similarly fy is partial df/dy. When you say df/dx so you want to be clear, we would say here partial df/dx and partial df/dy [She writes on the board $\partial f/\partial x$ and $\partial f/\partial y$]

Figure 2: First 6 turns of tutorial transcript

Turn by turn scrutiny of the transcript reveals the following characteristics of the dialogue:

- Tutor (T) states her goals for her approach in the tutorial (turn 1).

- Tutor questions to students (turns 2, 4, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33)[2].
- Student responses to tutor questions (turns 3, 5, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32).
- Tutor explanation/clarification of concepts (turns 6, 7, 33).
- Tutor focusing on ‘meaning’ explicitly (turns 2, 15, 33) or implicitly (‘why’ questions: turns 13, 21, 25).
- Student responses that (start to) reveal meaning (turns 5, 12, 14, 16, 18, 26, 32).

These details reveal an alternating pattern of tutor questions and student responses; some of the latter not revealing student thinking about the concepts. Those that do reveal some potential insights for the tutor become the focus of further tutor questions.

14.	E: … because it is got the, er, the slants of the first one, and the…
15.	T: so you’re seeing a relationship between the one of the middle and the other two. What do you mean by the slants?
16.	E: er, I don’t know, just the, the gradient there.
17.	T: if you’re right and the function is middle one, erm, before we go any further, Alun, do you think the function is the middle one or would you say one of the others?
18.	S: (Alun) … it looks like the more complex
19.	T: aah..“It looks like the more complex”. So would you expect the function graph look more complex than its two …?

Figure 3: Tutorial, turns 14 to 19

The tutor tries to prompt meaningful student articulations, but this is only partially successful. Student use of language “slants” (turn 14), “gradient” (turn 16) and “complex” (turn 18) suggest meaning to the tutor who probes and prompts with further questions (turns 16-19).

The teaching approach here can be interpreted as a questioning approach that prompts students and probes their meanings (Jaworski & Didis, 2014). It tried to include students by addressing them singly, by name, and as a group. Further interpretation suggests students either do not know the answers to the questions posed, or are not able to articulate their understandings. The tutor mainly avoids providing her own answers to questions posed, seeking rather to draw out the

[2] We have included all instances of each type of turn here to emphasize the frequency with which these occur within a 6-minute episode (see Appendix A).

students' own articulation of meaning. However, in the university culture in which they all participate, it is unusual for students to be asked to articulate their mathematical thinking, so perhaps not surprising if they show inability or unwillingness to do this.

As the tutor is also one of the researchers, she provides information about her goals in the tutorial teaching in general and in the episode in particular. Although the tutor is not 'teaching' the calculus course, she has a responsibility to help the students make meaning of the mathematics. So, her questions, as well as seeking out what the students know (what they can express in words), also have the purpose to assist conceptualization. She works according to a belief that a focus on 'meaning', with direct questions encouraging students to express meaning, will bring meaning into the public domain in the social setting.

Activity - actions and goals - tools

Activity here is the university mathematics teaching and in particular the tutoring. The object of the activity is student enculturation into the professional community of mathematicians; the motive of the Activity is the development of scholarly knowledge of mathematics. The participants/subjects of this activity are the tutor and the students and, following the above analysis, the episode comprises actions directed towards their reciprocal goals of communicating and appropriating understanding of selected key mathematical concepts related to partial derivatives of functions of two variables and their associated graphs.

We perceive an enculturative process to involve development of mathematical meanings as the objective of mathematical activity (rather than perhaps the limited goals of procedural functioning). In this particular episode the tutor's goals are to get students to:

- express what they 'see', their images, their connections, their symbolic awareness, their thinking;
- get used to talking about the mathematical concepts, to express ideas in words;
- link to formal mathematics ideas;
- listen to each other and build on what another person expresses;
- feel comfortable about not knowing, but to recognize that working together can enable more than they could do alone.

These are goals for the students, but the tutor also has goals for herself:

- to phrase questions in ways to which students can respond;
- to listen to the students and discern meaning from what they say;

- to maintain a focus on the mathematics that is important, without telling, guiding, funnelling in ways that will foster a surface recognition without deeper meaning.

In order to gain access to students' meanings and develop further their mathematical meanings she needs some tools. One tool is the question of the problem sheet, part of which is the three graphs as an iconic representation as well as symbols and terms that are used. Her questioning approach is another tool.

The tutor's actions relate to these goals. Her main action is to ask questions, and the different kinds of questions relate to different goals. For example, the prompting and probing questions seek to engage the students in thinking about the mathematical concepts and taking part in the tutorial dialogue. The 'why' questions seek to discern students' mathematics meanings through their articulation of reasons for their answers to her questions. Her use of the lecturer's problem sheet both aligns with the expectations of the university system in mathematics and provides a source of opportunity for students to address the mathematical concepts of the calculus module. The limited offering of her own explanations and exposition is intended to elicit explanations from students rather than providing them herself.

The goals of the students are not made explicit in the episode, and we do not have the relevant data to talk explicitly about them. Nevertheless, as the tutor has observed from tutoring these students for a whole semester and from her other tutoring experiences, the students show more satisfaction when they see how to apply certain procedures and find the solutions of the problems given than to develop deep understanding of the key concepts that the tutor wants them to achieve. Their main goal in participating in the tutorial is to be successful in the class examinations. As we discuss below, these different goals arise from the inner contradictions of the activity and they cause tensions that the tutor needs to handle. Tensions emerging from inner contradictions are also related to the way that the students handle the representations (tools) that the tutor offers to them. Also, the students bring informal tools such as informal language and images in their attempt to make sense of the key concepts that the tutor wants them to understand.

Contradictions, tensions and convergences

There are emerging tensions for the tutor that are of pedagogical and didactical nature. She is familiar with these students and is aware of the factors which influence their participation; the demands on them from their other courses; their difficulties in understanding mathematics, expressing formally and engaging analytically. Her approach has to take into account the wider context. There is no point in manifesting expectations that the students have no chance in meeting. She might be drawn into her own explanations and expositions which the students will not understand any more than they understand the lectures they have attended. Nevertheless, she has to be aware of the key mathematical ideas, and keep the focus on these ideas. Keeping a focus may be in tension with fostering students' own articulations of meaning. Maybe there are other strategies (tools) she could employ, and she does so at other times in this tutorial and in other tutorials. In contrast with her own values in seeking conceptual meaning, the tutor has to be careful to ensure that students see some value in the time spent in the tutorial, otherwise they might not attend on future occasions. Thus, she has to ensure there is some outcome of positive value perceived by the students, even if it is not clearly in line with her main goals. So, for example, students value tutor actions that enable them to answer questions in a test or examination, and they might prefer to gain procedural awareness of how to address a mathematical question without caring for the deeper understanding. So, sometimes it is necessary for the tutor to focus on procedural competency such as how to differentiate a two variable function with respect to one variable. This is something they have done in a previous tutorial.

Another contradiction related to the representations concerns whether the students understand the key concepts that the tutor wants them to articulate or their attention is on the representation itself. The tutor's focus is on symbols – meaning appears to be emphasized with the word 'partial', and later by the idea of imaging (not imagining) planes parallel to x-z and x-y. There is further focus on interpreting graphical representations – features in terms of 'dominant' shape, zeros, stationary points (and types). Distinguishing between the graphical representations of f and its partial derivatives appears to rest on a notion of complexity. It is not possible to grasp or present the key (ideal, generalizable) concepts, it is only the representations that the tutor can express, point to, inspect, etc. Thus the tutor is confronted with the fundamental contradiction in teaching mathematics. What does she do to bring the key concepts to students' consciousness?

We have seen in the transcript above some of what the tutor does and how the students respond. It is hard to judge the outcomes from these actions in terms of the expressed goals. To what extent are students enculturated in mathematics according to the motive of activity? Activity is, of course, ongoing and not limited by the beginning or end of a tutorial. The wider story must deal with actions and goals beyond this tutorial.

2.2. Discussion of analysis with regard to theory in French perspective (cf. 1.2)

We will illustrate the approach we developed for studying moments from mathematics lessons using one example. In such moments the teacher presents to the students general and somehow formal mathematical knowledge. The access to students' and class's activities is more limited than in exercise sessions. Students listen (or not) to the teacher, copy onto their note sheets what is written on the blackboard, perhaps take notes, and think about what the teacher is telling: but these activities escape the classroom video-recording.

Context and content of the recorded lesson

The lesson we use to illustrate is with a 10th-grade class. The declared aim of the teacher is to bring students to use a sign table in order to solve an inequality composed by the product of two factors.

An introductory phase, that was not recorded, took place around the solving of the following problem: A firm wants to make mouse pads consisting of a square image of side 10 cm framed by a strip of colour of constant width. The width of the coloured strip is x cm. For economic reasons, the area of the large square thus formed must not exceed 225 cm². Determine the possible widths of the coloured strip.[3]

The teacher gives the following account of this phase. First, students were given a few minutes to reflect on the problem and then a discussion ensued. A resolution scheme is then sketched, followed by setting the inequality: $4x^2 + 40x < 125$. After having made a value table, students drew the curve of the function $x \rightarrow 4x^2 + 40x$ and tried to solve graphically the inequality by drawing the straight line: $y = 125$. A question of the teacher guides the students' activity: show that the inequality is equivalent to: $(2x - 5)(2x + 25) < 0$. Students are encouraged to solve the case where the product is equal to zero, and then to apply the rule of signs. The teacher draws a sign table by recalling the lesson on the previous chapter about the sign of an affine function and checks that the solution is consistent with the graphic resolution.

In the lesson that follows this activity, first the teacher presents the graphical resolution of general inequalities such as $f(x) > k$ and $f(x) < g(x)$ by the means of curves. Then he writes on the blackboard the next title: algebraic resolution of inequalities. In the first paragraph he presents two tables showing the sign of $ax+b$ according to the sign of a. It is only then that the recorded episode starts; a full transcription is provided in Appendix B.

The teacher recalls, with the students' participation, the rule of signs with numbers, seen in the introductory phase. Then he presents a more general proposition on the rule of signs with a product of two factors A and B (numbers or algebraic expressions) and provides a summary table that the students copy. Then follows the

[3] No student was using a geometrical solution: maximum area is 225 cm^2, hence maximum length is 15 cm, so maximum x is 2,5 cm. It can be inferred that it is an effect of the didactical contract (at this school level) that the approach has to be algebraic.

statement of a method, deduced from this generalized rule of signs, to determine the sign of an "algebraic expression product", which is introduced through an example: find the sign of $(2x + 1)(x - 4)$. After a short discussion about the methods (to develop, to factorize) proposed by the students, which the teacher refutes or comments upon, he returns to the proposal to make a table of signs. He makes precise the nature of the factors involved (as “affine functions”). He recalls, through a series of quick questions to the students, that if the slope is 2, positive, the corresponding affine function is increasing. He then prepares an empty table of signs that the students copy. It is then completed by both teacher and students. After a question from a student who did not understand, everything is repeated once more.

The aim of the analyzed episode is to learn how to design and use a sign table in order to determine the sign of a product of expressions of degree 1 ($ax+b$), the so-called “rule of signs”.

The relief of the mathematical content at stake

Students are supposed to recall what they have learned about linear functions and especially what was done previously for their sign, leading to the algebraic resolution of an inequality as $ax+b > 0$ with the corresponding table.

Students are expected to be able to recognize and use the rule of signs for numbers. In fact, for some students it is probably not “available” knowledge, particularly if numbers are not given as numerical values (such as +3, -7) but expressed as a, b, without any explicit sign. They are also expected to move fluently between three registers: “the number a is positive”, “a is greater than zero”, “$a \geq 0$”, and to associate the signs “+” and “-” as indicating a position with regards to zero (for instance, in +2, the sign + indicate a positive number, greater than zero, such as $+2 > 0$).

In the curriculum and in the textbooks, the “rule of signs” for numbers has already been seen in earlier years (an item of “old” knowledge). As concerning linear functions, they are first introduced at grade 9; their study is developed for 10th-grade students, not only relating to the algebraic formula and the graphical representation, but also introducing the value of the zero of the function as the value where the signs change. A specific aim is to introduce the construction of the sign table for a product of linear functions.

Indeed, what may be difficult here is the difference between the direct algebraic study of an inequality composed of a single linear function and the algebraic study of an inequality composed of a product of such functions, which, moreover, may not be directly visible in the given algebraic form. The impossibility to solve the second type directly, leading to a detour with the use of an extension of the sign rules, remains difficult for a long time. Furthermore the link between the graphical resolution and the algebraic one is not obvious, as the first one does not involve the product of linear functions.

The lesson in progress

The teacher introduces the session with a rule expressed for the "product of positive and negative things" (A and B). Then he points out that "A and B are numbers or algebraic expressions", and announces a "method to determine the sign of an algebraic product of factors [...] something times something, a product". It is done by extending the rule of signs and is based on what is known for the sign of affine functions. The presentation is developed for the specific case of a product of first order simple expressions ($2x + 1$) and ($x - 4$). The teacher quickly draws the table on the blackboard for students to copy it, comments on the number of lines and announces that "the method is to have one line for each factor: a line for $2x + 1$ and another line for $x - 4$", without commenting on the role of the first line (x values) and of the last one (signs of the product), until a student questions the teacher's announcement, "I bring down the zeros on the bottom of the table". Finally, he recapitulates the whole process by answering a student who apparently did not understand anything.

Proximities

We track in the teacher's discourse elements which we presumed were oriented toward making links between previous knowledge and the mathematical content presently at stake. We name them "discursive proximities".

The proximities directly expressed by the teacher were of various types:

- an ascending proximity concerns the rule of signs, when he expresses the similarity between the (yet known) rule for numbers and the new rule for expressions;
- the teacher then announces that the method will be deduced from this rule: another ascending proximity;
- for the table of signs, there is a descending proximity between what students know about the sign of an affine function (recalled just before); a horizontal one - at a general level - is involved when he says, "the method is to put one line for each factor";
- the importance of the values of zeros is commented with a descending proximity, "in order to use the table of signs for the affine function, as we had done (just before)";
- the same proximity is used for fulfilling the line of signs for each factor
- another descending proximity is present for the sign of the product "we apply the rule of signs".

The proximities linked to students' utterances:

- In the case of answers, two descending proximities appear when the teacher interacts with students for studying the sign of each factor and for completing the line of x values with the two zeros in the appropriate order;
- We identify a local horizontal proximity triggered by students' questions, when the teacher relates the term "product expression" to the product known as "something times something, a product"; a descending proximity when the teacher explains why the question "for what value is there a change of sign" was changed into "for what value is it zero?";
- Responding to a student who did not understand, the teacher resumes his explanation, adding several proximities. Two descending proximities are involved in the application of the sign of affine functions previously learned "we wrote just now, and we wrote in the lesson on affine functions, that the sign is ..." and in the generalization of the rule "if I get 15 cases after the zero, I put as many "plus" as there are cases". Two local horizontal proximities were also present: the teacher explicits that before x of x - 4 there is "1" as a coefficient ; he explains that the rule of signs is used along columns as for the null values of a product ("if I take a thing that is zero times another thing that is not zero what does it give?");
- Elsewhere, we observe a refused descending proximity, when a student proposes to use the general form of solution -b/a for the zero of $2x + 4$.

In fact we see that the students have a real influence on the teacher's explanation during the lesson, giving rise to the teacher's descending or local horizontal proximities. However, we notice that there is no questioning related to the students' previous work (possibly giving rise to ascending or general horizontal proximity). This reveals somehow the limits of what could be initiated by the students' questioning. Actually there are notions, properties and notations that remain implicit in the lesson, as it is presented below, what would perhaps involve horizontal general proximity.

***Implicits*[4]**

There is a diversity of implicit use of notions, properties or notations, some of them being evoked later on in the lesson.

- A first implicit concerns the A and B expressions: the reason why it is possible to use the previous knowledge about signs lies in the fact that they are

[4] We use here the substantive "Implicits", it is a neologism – the plural is built on the model of "deficits".

supposed to be expressions with the same variable (x), a notion that does not belong to the students' curriculum.

- A second kind of implicit is related to the use of mathematical registers: “>0”, “greater than zero”, sign +, “positive”; and the notation of the line of x from $-\infty$ to $+\infty$.
- The explanation of the relation between variation of an affine function and graphical representation enabling the visualization of the change of sign is not given. Perhaps it is supposed available as affine functions were introduced in the previous grade and worked on before the session, and also in the first part of the lesson?
- How to use the so-called “method” for solving inequality problems remains implicit, even if the session is just followed (or even preceded) by a specific example.

Some of these implicits could be considered as “missed proximities”, mainly horizontal ones. The appropriate moment for such proximities remains an open question.

If we come back to our “relief” on the algebraic resolution of such inequalities, we may suppose that what some students could miss is more the idea of the necessity of a detour by the study of the appropriate product by the extended sign rule than the technical way (sign table) to do it, which was more developed here by the teacher. It could have given rise to some horizontal general proximity, linked with an appropriate task. We suppose that an appropriate assessment may be used to check this kind of hypothesis.

Conclusion

The theoretical and methodological perspectives presented above and the examples used to illustrate their use shed light on different ways to analyze and interpret the interactions between teacher and students’ mathematical activity. Even though the two perspectives follow different routes, with a shared origin (Vygotsky’s theory), some similarities seem to appear and some questions remain, particularly about the notions of contradiction and tension (without, however, considering the same level of generality).

Through their example, the English group reveals emerging contradictions for the tutor that are of pedagogical and didactical nature. In particular we ask:

1. What do we learn from articulating these contradictions? Why is this of value more generally?
2. What insights does the revealing of contradictions provide with regard to teaching for students’ understanding of mathematics?

The French example pointed out different types of proximities in the relationship between teachers' goals and students' real activity. We can hence add a question:

3. What, if any, are the kinds of proximities that are less likely initiated by students' interventions, and therefore need to be initiated by the teachers?

In relation to Question 1, the fact that there are contradictions in teaching is not new or surprising. We have seen the revealing and naming of them in previous research, particularly at school levels (Brousseau, 1984; Jaworski, 1994; Mason 1988). An example is the so-called "Didactic Tension" deriving from Brousseau's (1984) Topaze Effect as observed by Mason (1988) and used by Jaworski in her analyses of teaching (1994). In this paper we reveal contradictions in *university tutorial* teaching, which is relatively new, and the use of Activity Theory aids this process. Activity Theory, as we have shown above, in its various manifestations, draws attention to contradictions (and resulting tensions) in educational practice (e.g. Roth and Radford 2011).

In Section 2.1 above we see contradictions between teacher actions goals and the responses of students and between teacher actions goals and teachers' interpretation of the meanings behind these responses. We also see inner contradictions in the ways in which mathematics is presented and perceived (that representations are not the mathematics they represent, but that students may see the representation as the mathematics). In starting to generalize, we suggest that the declaring of contradictions is of value more widely, firstly, as the research and teaching community acknowledges the importance of being aware of contradictions and secondly recognizes them in other research or in their own practice. Thus we start to form a classification or knowledge bank relating to contradictions in teaching at a range of levels and opening the debate on how teaching can address such contradictions, whether they are inevitable or whether they can be avoided. In doing so we start to form a theory of teaching in which contradictions are seen as unavoidable, but in which we seek teaching actions that can better address teaching goals.

It seems worth exemplifying these generalities in terms of the examples above. The tutor has certain goals for her work with her students. These include the desire that they develop deep understandings of concepts such as partial differentiation. Her associated actions include the selection of suitable mathematics tasks chosen to reveal the desired concepts; orally delivered questions designed to prompt and probe students' understanding; grasping small clues in their minimal responses (slants; gradient …) in order to judge their understanding and offer further prompts, etc. Whether students develop understandings, deep or otherwise, from this activity is not visible. Hence the teacher cannot decide whether her actions have achieved her goals, or whether some other actions might be needed.

In the tutor's example we can stress the consciousness of the practitioner reflecting on teaching decisions and actions in relation to expressed goals. Here we address Question 2 above. There is considerable debate in university teaching as to whether traditional lecturing achieves learning outcomes that a university desires. The above discussion on actions, goals and associated contradictions offers an important contribution to this debate. From the conceptualization of theory on contradictions and their importance in educational development we envisage a dialogue between practitioners in which the teaching community becomes more aware of the vicissitudes of practice and potentially more critical in their design of teaching to achieve desired learning of mathematics by student cohorts.

In the French analysis we also reveal tensions in teaching lessons, which is relatively new (previous research has been centred on relationship between mathematical tasks and students' activity in classroom exercise sessions). The use of Activity Theory, in relationship with Vygotsky's theorization about conceptualization, aids and supports the analysis. The proposed theorization of proximities would be a model of teachers' mediation aiming at provoking evolution in students' knowledge, from recently acquired mathematical notions ('old' ones) to new ones. It proposes a more fine-grained model than the Vygotskian dyad: spontaneous and scientific concepts. The tensions occur between what is expected or planned by the teacher,[5] what appears to be possible or not according to the students' answers or own questions, what has to be improvised by the teacher to articulate the specific and the general levels of mathematical objects at stake, or between 'old' knowledge and new, through discursive proximities.

Two elements particularly emerge from the analyzed teaching situation. First, there remain some implicit issues in the teacher's discourse, at moments when 'old' knowledge might be mobilized or reinforced; these mainly concern the general level of mathematical objects or activity.

Second - and this is some answer to the third question - students do not appear to make spontaneous connections between existing and new knowledge, or their mathematical actions, and it is up to the teacher to explicitly introduce these connections. In these moments of mathematics lessons, the teacher's activity is neither triggered nor completed by students' initiatives - questions or comments. Establishing proximities appear then, crucially, as the teacher's initiative in articulating knowledge for the (expected) students' benefit.

[5] The data used for presenting the notion of proximities are not analyzed from the point of view of the teacher's expectations and planning, we are referring to our general approach in the studies of teachers' practices.

To conclude, we can say that looking for relations and complementarities between the English and the French approaches to analyzing mathematics teaching through the different uses of AT, led us to recognize connections between proximities and contradictions (and resulting tensions). The notion of proximity is a construct that indicates how the teacher tries to bridge the gap between students' existing mathematical knowledge and the mathematical content that the teacher wishes to communicate, tracked through the teacher's discourse elements. Recognizing different types of proximities, tells us about how the teacher attempts, in different ways, to overcome these tensions and build bridges. The proximities allow us to scrutinize the teacher's actions in relation to his/her attempt to introduce students to new mathematical meanings, taking into account the students' mathematical activity. On the other hand, with the constructs of contradictions, actions, goals and their relationships, the English approach allows us to recognize tensions that are also beyond the classroom interaction and play an important role in the interaction itself and its outcome. Through the different constructs of AT, the analysis contributes to our understanding of the complexity of mathematics teaching. Focusing on critical moments in classroom interaction we identify mathematical, didactical, and institutional factors coming into play that inform teachers' decisions and actions and, as a result offer learning opportunities for the students.

Appendix A

Transcription of an extract of a recorded tutorial in first-year university mathematics

A transcript follows from 6 minutes of classroom dialogue in a university small-group tutorial focusing on partial derivatives.

1. T: [Tutor and 2 students are present] I thought we'd have a look at Q3 first. I've selected all of these questions for a purpose, because each one of them highlights what I would call key concepts. [She refers to question 3 as presented above. Two more students enter the room – tutor greets them and repeats her words above]
2. T: So, first of all, what are these things fx and fy? Alun. What is, what do you mean, if you write fx and fy?
3. S: (Alun) dee-f-dee-x
4. T: And how would you write it?
5. [He indicates with his hand the partial derivative symbol, ∂]
6. Yes partial df/dx and similarly fy is partial df/dy. When you say df/dx so you want to be clear, we would say here partial df/dx and partial df/dy [She writes on the board $\partial f/\partial x$ and $\partial f/\partial y$]
7. So in the question then, we have three graphs; one of them is a function f and the other two are the partial derivatives df/dx and df/dy. Now, which is which?
8. [silence]
9. T: Anybody have a stab at that? What do you say Brian? [He pulls a face and people laugh]
10. [Response unclear]
11. T: No? OK, how about you Erik?
12. E: … not really sure but I guess that, er f will be the middle one.
13. T: OK, why do you think that?
14. E: … because it is got the, er, the slants of the first one, and the…
15. T: so you're seeing a relationship between the one of the middle and the other two. What do you mean by the slants?
16. E: er, I don't know, just the, the gradient there.
17. T: if you're right and the function is middle one, erm, before we go any further, Alun, do you think the function is the middle one or would you say one of the others?
18. S: (Alun) … it looks like the more complex
19. T: aah…"It looks like the more complex". So would you expect the function graph look more complex than its two …?
20. S: I would.

21. T: you would. Why?
22. S: [pause] I don't know.
23. T: do you agree with him, Carol?
24. S: yeah (Carol)
25. T: can you say why?
26. S: erm because it has in this x and y, functions of both x and y.
27. T: well, don't they all?
28. S: more functions, …
29. T: more functions?
30. S: er, I don't know!
31. T: Come on we're getting there. Brian?
32. S: Well, I guess when you differentiate, you're almost simplifying it to your next .[inaudible]
33. T: OK, so if what we have got is, in some sense a polynomial, then when we differentiate a polynomial we get a lower degree, so is that what you meant by 'simplifying'? So is everybody agreed then that the middle one is the function?

    OK. It is!! It is.

    So look to the one on the right, Erik, and tell me how the one on the right fits with what you see in the middle. Is that going to be the partial derivative fx or is it going to be the partial derivative fy?
34. [The dialogue continues in the same style for 4 more minutes]

Appendix B

Transcription of an extract of a recorded course in a 10th-grade class

(Statements of students are in italic – comments of the observer are in italic placed in brackets)

Transcription of an extract of a recorded course in a 10th-grade class

(Statements of students are in italic – Comments of the observer are in italic placed in brackets)

<table>
<tr><th>Time starting from the beginning of the recording</th><th>What the teacher says</th><th>What the teacher writes on the blackboard</th></tr>
<tr><td>4’38

4’46

4’56
Silence 10’’</td><td>So do you remember what we have said earlier about the product of positive and negative things (students give some answers)

We have told negative times negative is positive, negative times positive is negative, positive times positive is positive.
So it is what we call the rule of signs
So we made a small proposal, placed in brackets you can write : rule of signs, not the animals [swans, in French "cygnes" same pronunciation as “signes”] - sign rule (she erases the blackboard)
and we will draw a table
(She draws on the blackboard without saying anything)</td><td>Sign of a product
<table><tr><td>Sign of A</td><td></td><td></td><td></td><td></td></tr><tr><td>Sign of B</td><td></td><td></td><td></td><td></td></tr><tr><td>Sign of A.B</td><td></td><td></td><td></td><td></td></tr></table></td></tr>
<tr><td>5’40

6’15

Silence 25’’
6’47</td><td>So A and B are numbers, or algebraic expressions and the question is about the sign of their product
So you said that if the two are positive the product is positive. If the first one is negative and the second positive it gives negative, if I reverse it, again it gives negative, and if I take two negatives it gives positive. That is what you have just said to me.
A student’s question (inaudible)
Yes. I said if A is positive, B positive, A times B is positive. Minus times plus is minus, plus times minus is minus, and minus</td><td><table><tr><td>Sign of A</td><td>+</td><td>−</td><td>+</td><td>−</td></tr><tr><td>Sign of B</td><td>+</td><td>+</td><td>−</td><td>−</td></tr><tr><td>Sign of A.B</td><td>+</td><td>−</td><td>+</td><td>−</td></tr></table></td></tr>
</table>

Silence 12'' 7'10 7'25 Silence 15'' 7'52	times minus is plus. (*the teacher is silent, the students copy*) And we will deduce a method to determine the sign of an algebraic product of factors. *Student : But Madam, it is normal, in fact it is simple* Yes, I don't disagree. You have known that for a long time, but there are things you do know from a long time, yet you do not know how to use them. So a method, method to determine the sign of an algebraic product of factors (*she dictates*) To do the method we will take a very specific example. We'll take an expression and we will do the algebraic study. (*She repeats*) Method to determine the sign of an algebraic product of factors. Product that is to say something times something, a product. So what example I could give.	
8'30 Silence 15'' 8'33 8'49 9'10 9'23 9'40	Let us find the sign of (2x + 1) times (x -4). I'll wait until everyone has finished writing. *Student: That's in the lessons' part?* It is always in the method, the method, we apply it on an example. *Student: we multiply the factors together?* Chaima, ah, certainly not! *Student : we factorize then* What do you want to factorize? *(Inaudible answer)* We'll make a sign table Actually we use what you know about signs. So the first part: it is a function ..? *Student : affine* Affine. The slope here is equal to… ? (*Student : 2*) 2, is positive so the expression is first negative, then positive, an increasing function. This one is also affine. The slope is equal to…? *(Student :1)* 1, positive, so it is also negative, then positive. At which value the sign changes? *Student : it's –b/a* Yes, there is no need; it is also possible to solve the equation. When does it give zero? *Student: at 4* When x is 4, and this one?	Let us find the sign of : $(2x+1)(x-4)$ We solve $2x+1=0$ $\Leftrightarrow 2x=-1$

<table>
<tr>
<td>Silence 17’’</td>
<td>Student: When x is equal to 0.5.
When x is minus 0.5. We write it
First we solve 2x + 1 = 0 (she writes it) and x -4 = 0 (she writes and leaves a blank). The first one gives 2x = -1;
x = -1/2; -0.5; and that one is much easier, it gives x = 4 so we get both values .</td>
<td>$\Leftrightarrow \quad x = -\frac{1}{2}$

et $x - 4 = 0$
$\Leftrightarrow \quad x = 4$</td>
</tr>
<tr>
<td>10’35

10’51

11’
Silence 25’’
11’52
Silence 12’’
12’16
Silence 17’’</td>
<td>These two values are important.
Student: what is the use of the zero then?
Should first find for what values it is equal to zero, in order to use the sign table of the affine function like we already did.
Student: Why affine ?
Each piece, each factor, we look when it is equal to zero in order to determine the sign and so we deduce the sign table.
(she draws the table and leaves some time to copy)

Then, it is a table that will have 4 lines; however a nice big table. If you still have two lines at the bottom of your page, I do not know if it will hold.
Then the method is to have one line for each factor: a line for 2x + 1 and one line for x -4 (she leaves some time for students to copy) .</td>
<td><table>
<tr><td>x</td><td>$-\infty$ $+\infty$</td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
</table></td>
</tr>
<tr>
<td>12’50

Silence 5’’</td>
<td>Then we write the signs. To fill the lines with signs, we begin by putting the two values; which one first ?
Student : -1/2
Why?
Student : Négative
Especially because it is smaller than the other one. I write the smallest first. -1/2 then 4, with lines below.
Please be careful, you must try to put it just underneath, otherwise the table become unreadable.</td>
<td><table>
<tr><td>x</td><td>$-\infty$</td><td>$-\frac{1}{2}$</td><td>4</td><td>$+\infty$</td></tr>
<tr><td>$2x$+1</td><td></td><td></td><td></td><td></td></tr>
<tr><td>x-4</td><td></td><td></td><td></td><td></td></tr>
<tr><td>(..)(..)</td><td></td><td></td><td></td><td></td></tr>
</table></td>
</tr>
<tr>
<td>13’33</td>
<td>Let us write the signs. We start with 2x + 1, 2x + 1 is equal to zero at which value?
Student: At -1/2
At -1/2, so at -1/2 in the line of 2x + 1 I put a zero. Only at -1/2 eh since it is equal to zero only at -1/2 . Then I fill in with the signs. It's minus, plus, since the slope is positive so here it gives minus, minus, plus.
I repeat, if we take 2, 2 is positive therefore according to the sign table we had earlier on the affine functions it gives minus, plus, plus.</td>
<td><table>
<tr><td>x</td><td>$-\infty$</td><td>$-\frac{1}{2}$</td><td>4</td><td>$+\infty$</td></tr>
<tr><td>$2x$+1</td><td>–</td><td>0</td><td>+</td><td>+</td></tr>
<tr><td>x-4</td><td></td><td></td><td></td><td></td></tr>
<tr><td>(..)(..)</td><td></td><td></td><td></td><td></td></tr>
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</table>

<table>
<tr>
<td>14’17

14’30

14’45

15’09</td>
<td>Now the second one
Student: we put zero at 4.
We put zero at 4, it becomes null at 4 and ...
Student: here it is going to be minus, minus, plus.
Minus, minus, plus (*she is writing*) and the slope is 1. *Student: We do the sign rule.*
And in the third line we put the product, in fact we apply the sign rule.
And the last thing, I bring down the zeros on the bottom of the table.
Student: why do we do it?
Because if this one is equal to zero at -1/2, if I make the product by the other, the product of the two is ..., if this one is equal to zero at -1/2 if I multiply it by (x-4) it will still give ... (*Student: zero)*
And here it is the same for 4, so it is zero at the two values we had found.
Is it okay? No, why? What's wrong? What piece did you not understand? (*Student: Inaudible)*
Then how do we write the plus? Why did you say it's minus, plus?
Student (another one): You put plus when it is greater than zero, minus when it is smaller.</td>
<td>
<table>
<tr><td>x</td><td colspan="4">$-\infty \quad -\frac{1}{2} \quad 4 \quad +\infty$</td></tr>
<tr><td>$2x+1$</td><td>–</td><td>0</td><td>+</td><td>+</td></tr>
<tr><td>x-4</td><td>–</td><td>–</td><td>0</td><td>+</td></tr>
<tr><td>(..)(..)</td><td></td><td></td><td></td><td></td></tr>
</table>

We apply the sign rule
<table>
<tr><td>x</td><td colspan="4">$-\infty \quad -\frac{1}{2} \quad 4 \quad +\infty$</td></tr>
<tr><td>$2x+1$</td><td>–</td><td>0</td><td>+</td><td>+</td></tr>
<tr><td>x-4</td><td>–</td><td>–</td><td>0</td><td>+</td></tr>
<tr><td>(..)(..)</td><td>+</td><td>0 –</td><td>0</td><td>+</td></tr>
</table>
</td>
</tr>
<tr>
<td>15’41

16’

16’11

16’50</td>
<td>The slope here is 2. 2 is positive. We wrote a while ago, and also in the course on affine functions, that the sign is minus than plus. That means minus before zero, after zero it is plus. If I have 15 boxes after the zero, I get 15 plus, I put as many plus as there are boxes after the zero. Basically it's minus, then plus. This one now. Again the slope,1, is positive, so it is again minus then plus. Minus before the zero, plus after the zero.
As for the last line, we applied…what have we applied in the last line? *(Student: the sign rule) (she writes it)* . We apply the rule of signs in columns: minus times minus is plus, plus times minus is minus, plus times plus is plus. An the zeros, we bring down them because if one of the factors is equal to zero then the product is also null. If I consider something equal to zero and something not, then it gives…? (*Student: zero*)
Student : we must systematically bring down the zeros to the bottom of the table.
As for the product, yes!
Student: and if there are more factors?
I can put 15. There are many more values and</td>
<td>She adds one before x in the expression x-4</td>
</tr>
</table>

17’35	the table is much larger. If I take a product with three factors, then I’ll have a third value here and I’ll have a third line here but the rule of signs will work the same, that is if I have plus, minus, minus, minus times plus is minus, these two together give minus, when we multiply by minus it gives plus. The rule of signs functions for more than two factors	
19’26	End of the recording	

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ANALYZING TEACHERS' CLASSROOM EXPERIENCES OF TEACHING WITH DYNAMIC GEOMETRY ENVIRONMENTS: COMPARING AND CONTRASTING TWO APPROACHES

Abstract. The use of digital technologies in mathematics classroom continues to increase. Yet even when well-planned, such use is not unproblematic; indeed, uncertainties are inherent. In this article, we use analyses of teachers' activity in two classrooms, a French one and an English one, when technology in general, and dynamic geometry software in particular, is used. We present two different theoretical frames and show how, in spite of differences related to the context, the object, and the methodological backgrounds, the outcomes in terms of the analysis of teachers' practices turn out to be close. These outcomes provide insights into the complexities of technology integration within mathematics lessons and teachers' decision making both in the moment, and over time.

Keywords. Technologies, geometry, teachers, activity, hiccups, tensions

Résumé. Analyser l'activité instrumentée de l'enseignant en classe dans un environnement de géométrie dynamique : différences et similitudes de deux approches. L'utilisation des technologies numériques en classe de mathématiques continue à se développer. Cependant, cette utilisation reste complexe et demeure régie par des incertitudes lors des mises en place avec les élèves même quand les séances sont bien préparées en amont. Cet article présente les analyses de l'activité de deux enseignants, un français et un anglais, lors de séances intégrant des logiciels de géométrie dynamique. Nous présentons deux cadres théoriques et montrons que malgré les différences liées au contexte, aux notions mathématiques en jeu et à nos choix méthodologiques, les résultats en termes d'analyses des pratiques enseignantes sont très proches. Ces résultats fournissent un éclairage sur la complexité de l'intégration des technologies dans les séances de mathématiques et les décisions que les enseignants sont amenés à prendre in situ et sur le long terme.

Mots-clés. Technologies, géométrie, enseignants, activité, tensions, perturbations

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Introduction

The genesis of this article lay in the authors' mutual interest in each other's work as researchers, work that involved a close look at teachers' uses of, and practices with, digital technologies alongside the more pragmatic need to develop tools that could be used within teacher education programmes. In some sense, our methods

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look at two sides of the same coin, the teachers' classroom practices with digital technology, from our two different cultural perspectives. By working together, our aim is to see whether a knowledge of each side's facets leads to a deeper understanding of the coin as a whole.

The integration of technology into classroom work is known to be a complex process for teachers (Hoyles and Lagrange 2005; Clark-Wilson, Robutti and Sinclair 2014). While some studies have explored the nature of these complexities (Abboud-Blanchard 2013; Clark-Wilson and Hoyles 2017), a key to supporting the development of classroom practice is the availability of methodological tools and framing ideas that enable teachers to both understand the complexities and develop practices as a result.

Teachers of secondary mathematics in England and France have incorporated dynamic geometry software environments (DGE) into their teaching practices, including use by students to support them to engage with, and make sense of, geometric ideas (Laborde, 2001; Ruthven, Hennessy and Deaney 2008). Teachers, often following curriculum guidance, design DGE-based geometrical tasks where students are working in an investigative mode involving conjecturing and generalizing. Teachers support students throughout this investigation in different ways; for example, by introducing new mathematical objects or showing (or possibly proving) a geometric property. However, by opening the mathematics to student exploration, teachers encounter the pedagogic challenge of how to manage multiple student responses to tasks within the technology.

In this paper, we report findings from our analysis of teachers' activity in two classroom video sequences, one from a French classroom and another from an English classroom, using two different theoretical frames.

The first frame (the French context) is informed both by the *Double Approach* (Robert and Rogalski 2005) extended to technology environments (Abboud-Blanchard 2013) and the *Instrumental Approach* (Rabardel 2002). It considers teachers' use of technology as managing 'open' dynamic environments (something that increases uncertainties for the teacher in the classroom) and can be used to analyse teachers' activity in terms of *tensions* and *disturbances* in the planned cognitive route of the class (Abboud and Rogalski 2017).

The second frame (the English context), which is underpinned by Verillon and Rabardel's theory of instrumented activity within technology-mediated environments (1995), introduces the theoretical construct of the *hiccup*[1] to capture

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[1] In English, the word hiccup (or hoquet in French) has the additional meaning: a small problem or difficulty that does not last very long.

the epistemological rupture experienced by a teacher as he/she develops professional knowledge in practice, stimulated by the students' use of mathematical technologies (Clark-Wilson 2010a, 2010b).

The two frames are both affiliated with the theory of instrumented activity (Rabardel 2002), as is detailed in each of the two examples. We explore how the difference between them has a methodological implication as it concerns differences in the relationship between the researchers and teachers they are investigating. In particular, we use the different foci for these two research studies as each 'enters' the mathematics classroom to try to understand aspects of the teachers' (and students') knowledge at stake when technology in general, and dynamic geometry software (DGE) in particular, is used. While the context, the research objectives, and the theoretical and methodological backgrounds differ, the outcomes (in terms of the teachers' practices) could turn out to be close. This raises the prospect of whether the two theoretical perspectives can be connected in some way.

1. Characterizing teachers' classroom experiences with dynamic geometry technology: An example from France

1.1. Theoretical approach

In this example, the theoretical approach, informed by the *Double Approach*, considers the teacher as managing an 'open dynamic environment' (Rogalski 2005). Indeed, the use of technology adds a 'pragmatic' dimension (Abboud-Blanchard 2014) emphasizing the 'open' character of the classroom environment. On the one hand, the approach focuses on the relationship between the lesson preparation (anticipation) and its actual implementation (adaptation). On the other hand, it directs attention to the management of uncertainties inherent to such an environment. These uncertainties are due to the fact that the students' activity cannot be completely predicted, and the teacher is often in an improvisation mode. The conceptual constructs introduced in this frame aim at analyzing the impact of the dynamics of students' interactions with technology tools on the management of the planned (by the teacher) 'cognitive route' (Robert and Rogalski 2005) and the possible divergences from this intended path during the lesson (Abboud-Blanchard and Rogalski 2017).

The teacher's conceptions of the mathematical notion to be taught and of the relation students have to it, are subjective determinants of his/her professional activity. They condition the didactical process that the teacher wants the students to follow, as well as the management of the processes developed during the lesson (Robert and Rogalski 2005). Although the didactic scenario is familiar, the students' diversity and the specific context of the class introduce a factor of uncertainty. In addition, when students are working with a technological tool, the teacher encounters difficulties to control the tool's feedback (which is strongly dependent on students' manipulations) and to identify the interpretations students are making. Teachers have often to deal with tensions due to the presence of the tool and its role in the student's activity but also its interaction with the mathematical knowledge at stake.

Following Rabardel's (2002) *Instrumental Approach*, technological tools could be seen both from the teacher and the students' perspectives. In both cases, the subject-object interactions are mediated by the tool. Nevertheless, the object of teacher's activity is the students' learning, whereas the object of the students' activity is the content of the task given by the teacher; their instruments based on the same tool are thus different. Figure 1 presents how these two instrumented activities are articulated within the dynamics of class preparation.

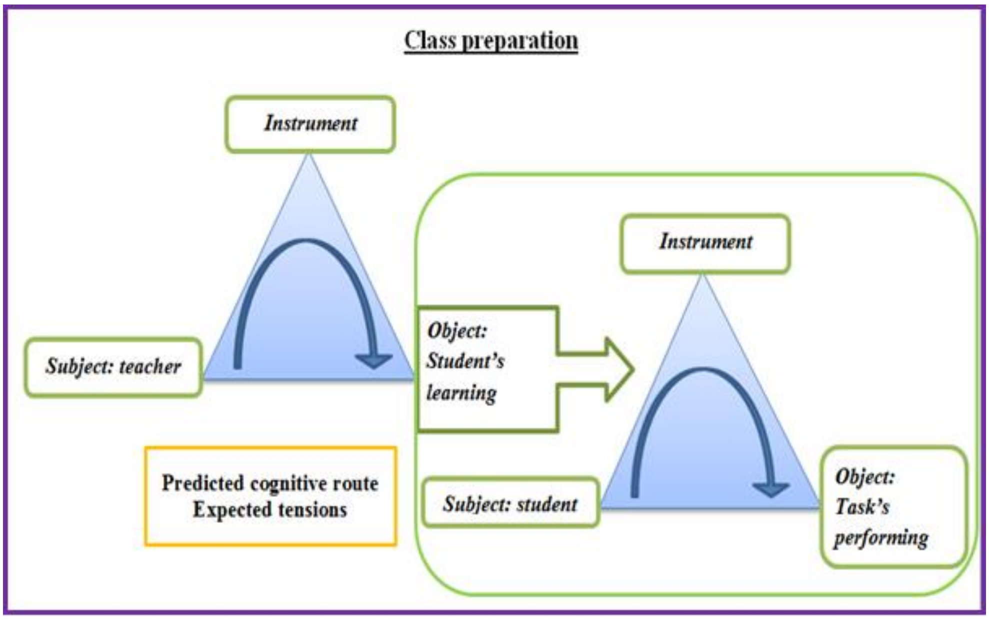


Figure 1: Articulation of the teacher and students' instrumented activities within the preparatory phase

The scene is completed when the two instrumental situations are articulated within the dynamics of class management and indicates possible tensions and disturbances. This is presented in Figure 2.

Tensions and disturbances

In the French approach, there is departure from the way Kaptelin and Nardi (2012) introduced the terms *tension* and *disturbance* when presenting the concept of contradiction, central in Engeström's framework of analysis of how activity systems develop (Engeström 2008). These terms appear in their familiar use; emphasis being put on the analysis of contradictions in activity systems, as key learning sources.

Tensions are not necessarily conflicts or contradictions. In the teacher's activity, tensions are manifestations of 'struggles' between maintaining the intended cognitive route and adapting to phenomena linked to the dynamics of the class situation. Some of these tensions might be predicted by the teacher and so there might be plans of how to manage them. Others are unexpected and constrain the teacher to make decisions, in situ, that direct their actual activity.

Disturbances are consequences of non-managed or ill-managed tensions that lead to an exit out of the intended cognitive route. Disturbances happen when a new issue emerges and is managed while the current issue is not completely treated or when the statement of a new issue is not part of the initial cognitive route.

Here the focus is on tensions and disturbances related to the local level of a class-session; other tensions are or might be managed at a more global level (over several sessions). As indicated in Figure 2, tensions could be related to different poles of the system of teacher-and-student activities; they can be shaped differently along three dimensions (previously introduced by Abboud-Blanchard 2014): temporal, cognitive, and pragmatic.

Tensions related to a cognitive dimension appear in the gap between the mathematical knowledge the teacher anticipated to be used during task performing and those really involved when students identify and interpret instrument feedbacks. Tensions related to both pragmatic and cognitive dimensions are produced by the illusion that mathematical objects and operations implemented in the software are sufficiently close to those in paper-and-pencil context (we refer to Balacheff's (1994) analysis of the *transposition informatique*[2]). Tensions related to a temporal dimension are frequent in ICT environments and linked to the discrepancy between the predicted duration of students' activity and the real time they need to perform the task. Teachers are generally aware of such tensions; they often manage them by taking control of the situation, either by directly giving the expected answer or by manipulating the software themselves.

[2] Balacheff defines this transposition as the process through which the mathematical knowledge to be taught is fundamentally transformed within a computer-based learning environment.

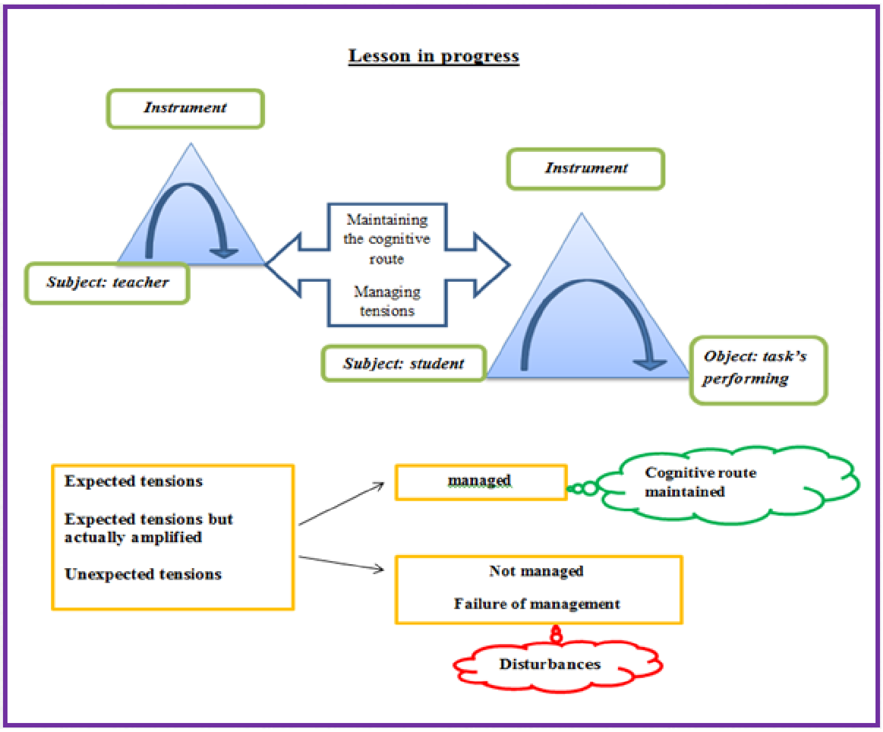


Figure 2: Tensions and disturbances within the dynamics of class management

Finally, a tension non-specific to the ICT environment may concern the didactical contract; students cannot identify the type of answer the teacher is expecting. ICT environments may amplify this type of tension when students are uncertain of the goal of the activity i.e. is the goal about a mathematical object to manipulate with the software or about the use of the software itself?

1.2 Methodological choices

The concern is to analyze the everyday practices of regular teachers who are not involved in research projects and experimental work. The use of technologies these teachers develop and integrate into the day-to-day activity is our actual research object. The choice of data gathering is made to reduce as far as possible the impact of researchers (observers) on the teachers and students' activity in the class. Hence the analyzed sessions are chosen and recorded by the teachers themselves. Deferred interviews and preparation documents are collected in order to identify personal and social determinants of the practices. Comparing the observed succession of episodes with the planned cognitive route, enable detection of tensions and disturbances. The analysis of the practices' determinants makes it possible to shed light on reasons of some of these tensions and on the ways the teacher manages them.

The case study that follows illustrates the variety of tensions, and the management of tensions, which range from routine-based treatments to the non-perception of tensions (Abboud-Blanchard 2015). The latter could entail students getting completely out of the cognitive route without the teacher being aware of this phenomenon. The identification of practices' determinants provides useful information to interpret these outcomes.

1.3 A case study

In this case study, the teacher investigated is Daniel (pseudonym), an experienced (10-year career) secondary mathematics teacher. He was chosen because, on the one hand, he is not involved in any experimental project and is not a technology-expert while, on the other hand, he supports the use of technology in mathematics education. Daniel's interview focused on his teaching experience, the professional context in which he is working, his use of institutional resources (curriculum, textbooks, academic websites, etc.), and on how and under what conditions he integrates technologies into his practices. Daniel chose a geometry session where he uses DGE. In addition of the video-recording, he provided a document explaining the choices made and rationales for the students' task in this lesson.

Summarizing the session

The lesson was an 8th grade (13-14 years) class in a computer room with a data projector screen on which the teacher's computer was displayed. The students were asked to download a file previously prepared by the teacher. When opening the DGE file, students discovered the screen shown in Figure 3.

The teacher then gave a preliminary remark: "please recall that every representation (on paper or computer) of a geometrical figure is inexact; measures given by DGE are approximate values".

Students were first asked to move point M in order to have both triangles AOM and BOM become isosceles at O. Second, they had to find other positions of M satisfying this condition, to observe the AMB triangle and to make a conjecture about the M angle. Last, they had to prove this conjecture, without any further indications of how, and if or not, the computer should still be used.

Approximately midway through the lesson, the students were still trying (or succeeding for some of them) to have angles A, B and the two marked M angles equal to 45°. The teacher made several individual and collective interventions: "You charge yourself with supplementary constraints, so it is difficult to find several positions". Finally, after a 'correct' example was proposed by a student on the projected screen, the teacher showed several cases where the two triangles were isosceles without having OM perpendicular to AB. Over the ten minutes that followed, the teacher moved from the demand of finding more than one (general) configuration to finding “the maximum [number of positions]” for M, and then he asked for “all possible positions”.

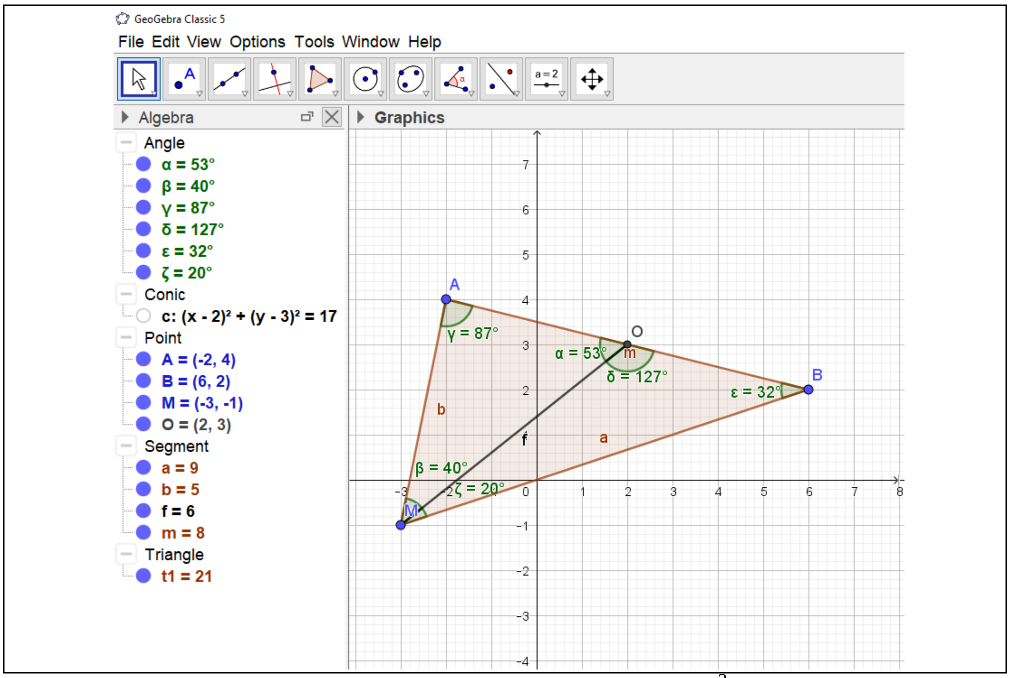


Figure 3: Students’ computer screen[3]

From that moment, the teacher’s goal changed. Rather than discussing students’ responses to this task, he moved to focus exclusively on his overarching goal to establish that the locus of all possible positions was a circle. When a student proposed this idea, he immediately approved and drew the circle and placed M on it. It is only after this episode that he got back to the earlier conjecture and (re)formulated a student’s proposal: “it is always a right angle, yes; that is, the triangle seems to always be a right-angled triangle at M”. He decided then to dictate the present state of shared or, supposedly shared, knowledge; that is to say, the locus of all possible positions of M forms a circle. He postponed the proof of this conjecture because it was already the end of the lesson.

[3] The obvious discrepancy between the displayed angle measurements and the real angle values (for the shapes as defined by the points’ coordinates) is due to the teacher’s choice of rounding for the measured units.

1.4 Analyzing the teacher's choices

What was at stake in this lesson was the theorem related to the circumscribed circle of a right-angled triangle. The presence here of a dynamic geometry software allowed a process of investigation that is difficult to achieve in paper-and-pencil environment. The task was meant to be an introductory task, and not a task requiring a functional use of this theorem in problem solving. It is an unusual task concerning this theorem in the French curriculum and textbooks. Two main choices seem to have been made by the teacher when preparing the task. The first choice was to construct the DGE figure himself and to let students only download the corresponding file. Such a choice limits the instrumented students' activity as concerning the construction of figures through DGE. The second choice was related to what was made visible to students on the screen; in the graphical window, he indicated the angle measures (another choice in the DGE options is to round measures up to units).

Daniel explained these choices by the fact that his aim was to bring students directly to the mathematics exploration of the figure and not to spend time doing it. This was thus meant to restrain the students' instrumented action (limited to handling skills) and to focus their attention on the geometrical exploration and to devote more time to the process of conjecture validation involved in the last question. Unfortunately, (for Daniel), moving M in such a way that the angles become equal is not so straight-forward a task. First, the coordination between observing the angle measurements and moving the point is somewhat complex. Second, the teacher wanted students to focus on angles measures in the graphical window, while several students were focusing on the side lengths in the algebraic window (a cognitive tension that is expanded upon below).

Identifying tensions and analyzing their impact

A pragmatic tension is related to what the teacher expected from the use of DGE and how students actually used it. A part of this tension was indeed predicted by Daniel. That explains the choices he made when preparing the task (see above) in order to minimize the impact of this tension. Yet other parts had not been predicted and these necessitated the teacher's specific interventions. For example, students tried to move points A and B to positions that Daniel did not welcome. DGE allows this manipulation, and students thought that searching for isosceles triangles might be easier if they moved not only M but also A and B. The teacher intervened throughout the session to explicitly forbid many students from moving A and B (given that the teacher did not define them through *Fix object*, within the file initially prepared). When a student was still moving the two points half an hour after the beginning of the lesson, Daniel took control of the computer himself, reset to the initial state and re-explained the task to the student. This is evidence of another pragmatic tension due to students working at different paces through the task, which is often noticed in technology-based lessons and could be highlighted as a characteristic of such a context (Abboud-Blanchard 2014).

A cognitive tension is due to the fact that the teacher considered that the isosceles character of a triangle can be treated by the students both from the angles' property and from the sides' property. Yet, in earlier teaching, the isosceles triangle was defined by the equality of sides – with the equality of angles only having the status of a property. In fact, some students moved the point M by trying to obtain the equality of the sides OM, OA and OB (without controlling the variation of the angle measurements). Daniel was not aware of this and this led to a misunderstanding and even a disturbance for some students. For example, a student encountered the following phenomenon: in the OBM triangle, angle B and M were not equal (45°; 46°), whereas the sides OM and OB were equal (2; 2). The teacher, focusing on the angles (not seeing the sides values) reacted by saying that the equality must be more precise. The student mumbled after the teacher moved away: "I don't understand… it is precise!".

Another cognitive tension linked also to the question of precision goes through the session: Daniel aimed at the continuous objects of (theoretical) geometry, whereas using a software necessarily discretizes them. He also considered that DGE provides approximate mathematical information while students considered DGE information as reliable. This was strengthened by the fact that Daniel rounded all measures to units. This tension provoked several interventions (collective or individual): the teacher reminded students that they must not forget the "approximate character" of what they saw on the computer screen and at the same time he asked them to use what they saw to make conjectures. By rounding to units, there is a finite number of possible positions of M, where there is angles equality. When Daniel changed the initial task by adding a sub-task aiming to find the "set" of all possible positions of point M, he had to state that even if DGE gives a limited number of such positions, there are actually infinitely many such positions. He hastened to bring an end to this contradiction by immediately drawing the circle.

A major temporal tension occurred due to the gap between the planned time for the instrumental task (an average of one third of the total duration of the lesson) and the actual time this task took during the progress of the lesson. Two thirds through the lesson, students were still trying to find several positions of M so as to make a conjecture about the angle AMB. Being aware of the slow progress of the students' activity, Daniel decided to interrupt them and called for a "first assessing" where he gave the correct answers and dictated the conjecture, thereby ending the instrumental task.

The resultant of the set of tensions was thus a major disturbance; in this lesson he had to abandon the aim of engaging the students in an angle-based proof.

Inferring determinants of the teacher's activity

The activity of the teacher was determined by different combined factors. The analysis of the lesson as it progressed enables us to infer the impact and articulation of these factors. The analysis of his post-lesson interview responses indicates particularly personal and institutional determinants.

First, managing conjectures in an investigation process is promoted by the mathematics curriculum for the French lower secondary school (6th to 9th grade). The curriculum also promotes the use of dynamic geometry software for constructing figures and investigating them. Daniel explains his choice of this particular task by referring to these institutional determinants. A plausible inference is that he was expecting (and hoping) that students would engage with a relatively new geometrical topic in an investigative way.

Second, there was evidence of interactions between personal and social/institutional determinants. Daniel chose to present information about the measures of the angles of the 'to-be' isosceles triangles and not about the lengths of their sides; this is unusual. However, starting from the measures of angles allows one to validate the conjecture that the angle AMB can be computed and shown equal to 90°, using the theorem of the sum of the angles of a triangle, something already known by the students, and the fact – implied by the design of AMB – that angle M is composed of two angles, equal to the other angles of AMB.

Third, the use of DGE impinges on Daniel's will to modify and develop his teaching practices (personal determinant). He sees this lesson as an opportunity to introduce a new way for teaching the geometrical chapter devoted to the circumscribed circle and the right-angled triangle by using an innovative task promoted by professional literature (Soury-Lavergne 2011).

Finally, there is the personal determinant of 'being rigorous'. Here, the teacher's choice (about the angles) opens the possibility for a real mathematical proof of the central property about the right-angled triangle and the circumscribed circle - using wide-scope knowledge, instead of referring to figural properties of the rectangle (drawn on AMB by a central symmetry). In fact, during the lesson, Daniel frequently employed logical connectors (so, because, as, then…) in his discourse. An interpretation of this observation, along with considering his will to let students spend more time on the proof process, is that Daniel is strongly oriented toward students developing a logical treatment of mathematical tasks. Such an orientation seems to be a personal determinant of his choice in this particular task.

2. Characterizing teachers' classroom experiences with dynamic geometry technology: An example from England

2.1. Theoretical approach

The complete research study from which this example has been selected aimed to expand knowledge of *how* secondary school mathematics teachers learn through their classrooms experiences to appropriate new technological tools in their teaching (see Clark-Wilson 2010a; 2010b). Whilst the existing research had categorized aspects of teachers' classroom practices (Noss, Sutherland and Hoyles 1991; Ruthven and Hennessey 2002; Drijvers et al. 2010) there was no research that shed light on how these practices had evolved. Initially, Verillon and Rabardel's (1995) theory of *Instrumented Activity* was adopted to gain insights into the nature of the interactions between the *Subject* (here the research lens was firmly trained on the teacher), the *Instrument* (the chosen technological tool) and the *Object* of the activity (the teaching of an aspect of school mathematics to a group of students).

Teachers' professional development is conceptualized as that of 'situated learning' as it is anticipated that the teacher develops their professional knowledge 'in and through' their classroom practice (Lave 1988). This professional knowledge spans the subject at stake (i.e. mathematics), how it is best taught and learnt, which resources might support this alongside institutional knowledge of the curriculum and its assessment.

Following the analyses of sixty-six lessons taught by a cohort of fifteen teachers over a period of a school year, it became apparent that teachers were repeatedly reporting (in their post-lesson reflections) incidents from their classroom that they had not anticipated in their initial lesson design. These *hiccups*, are defined as "the perturbations experienced by the teachers during the lesson, triggered by the use of the technology that seemed to illuminate discontinuities in their knowledge and offer opportunities for the teachers' epistemological development" (Clark-Wilson 2010 p. 138). Key to this definition is that the teacher must have noticed the hiccup. A second phase of research involved the analysis of fourteen lesson observations (of two teachers) that yielded a total of 63 hiccups. The cross-case analysis of these hiccups using a constant comparison methodology led to the classification of seven underlying triggers, as follows.

1. Aspects of the initial task design, such as a poor choice of initial example or subsequent sequencing of examples is unclear/inappropriate.
2. Interpretations of the mathematical generality under scrutiny. Difficulties encountered when relating specific cases to the wider generality (or students failing to notice the generality).

3. Unanticipated student responses as a result of using the technology. Students develop their own instrument utilization schemes for the activity.
4. Perturbations experienced by students as a result of the representational outputs of the technology. Students doubt the ‘authority’ of the technology.
5. Instrumentation issues experienced *by students* whilst actively engaging with the technology. Students are unclear exactly how to grab and drag dynamic objects.
6. Instrumentation issues experienced *by teachers* whilst actively engaging with the technology. Teacher is unsure how to display a particular representation, i.e. displaying the function table for a given function.
7. Unavoidable technical issues[4]. Displaying the teacher’s software or handheld screen to the class.

If lesson hiccups are to be interpreted as a vital contributory element of teachers’ situated professional learning as they appropriate mathematical technologies, it is necessary to describe how the hiccup prompted specific aspects of this learning. What follows is a case study of a particular classroom teaching sequence from the research data to justify the *hiccup* as an epistemological construct.

2.2. Methodological choices

Central to the research methodology was the need to observe closely the teachers’ development, enactment and reflections on their lesson tasks and approaches in their classrooms through an ethnographic approach. Consequently, a close professional relationship was developed with the teachers such that they felt sufficiently confident for the researcher (Alison) to observe and video-record their teaching, participate in interviews and post-lesson exchanges. The teachers shared their lesson design artefacts (software files, presentation slides, written plan, students’ work) in advance of the lesson and, following teaching, produced a written reflection of their teaching, which often included a redesigned task.

2.3. A case study

This example is taken from an English classroom and features an experienced teacher, Tim (pseudonym) and his class of fifteen 14-15 year olds who were being introduced to Pythagoras Theorem through a dynamic task mediated by the

[4] During the study, the teachers were using prototype classroom network technology. This did result in some equipment failures during some lessons. Although these occurrences were definitely classed as hiccups, they were considered to be outside of the domain of the research study as they were not related to the mathematics being taught and learnt.

geometry environment of their handheld devices that were wirelessly connected using classroom management software. Tim had written the following mathematical objective for the lesson, “to appreciate Pythagoras’ Theorem, in particular recognizing that the sum of the areas of the squares on the two smaller sides will equal the area on the longer side *if and only if*[5] the triangle is right-angled”. Furthermore, Tim added a specific intention for the use of the wireless classroom network (that connected all of the students’ devices to the teacher’s computer/projector), “Each individual student will explore the triangles on their own handheld – we will use the shared space of screen capture to come to a shared agreement about the necessity for the triangle to be right-angled”.

The task, which was wholly conceived and designed by Tim based on his a priori analysis of what the students were required to understand, included a software file that was transferred in advance to the students’ handheld devices. The task is shown in Table 1.

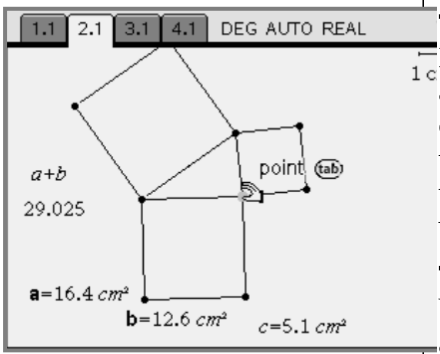
Opening screen on students’ handheld devices	**Description of the construction of the environment**
 	The task was constructed in the TI-Nspire ‘graphs and geometry’ application. A triangle had been constructed onto the sides of which three squares had been defined. The triangle was not constrained in any way. The areas of the squares *a*, *b* and *c* had been measured. *a* and *b* were defined as variables so that the value of *a*+*b* could be calculated and displayed on the screen.

Table 1: Tim’s task ‘Pythagoras exploration’.

Summarizing the session

Tim displayed the opening screen of the task and, following a brief introduction to connect the image to some work pupils had encountered previously, Tim then moved the triangles around by dragging different vertices, highlighting which area measurement related to which square. He then stated the aim for the task, which was to move the vertices of the triangle until the area measurements that had been

[5] Tim’s emphasis.

labelled *a* and *b*, when summed, equaled the area measurement that had been labelled *c* saying,

> "So, you need to think about which square is which and move them around a bit and I want *a* and *b* to add up to make *c*. Do you kind of get what we have to do? You're trying to change the sides so that *a* and *b* adds to make *c*."

At this stage Tim gave the students five minutes to respond to this challenge, during which time he moved around the room supporting students and monitoring their work. Simultaneously, the students' handheld screens were on public display to the class, refreshing automatically every thirty seconds. In this period Tim chose to send one student's work (Student A) to the teacher's computer, which captured the student's response to the task at that point in the lesson (see Figure 4). Tim concluded this phase of the lesson by alerting the students that they were going to be stopping and reviewing the class display of the individual handheld screens in a few minutes and that they would, 'scroll down and have a little chat about them [the screens] and see how we're getting on'. With the students' attentions back on the screen capture view of their work, Tim began to pick out screens and check that the numbers displayed satisfied the desired condition by talking out loud. For example, he focused on Student A's screen, saying: "Okay I'll go through these and we'll have a look at them so... *a* add *b* is twenty-eight point six-ish, and there's *a* and *b* is two point five, add them together that's kind of alright – that's really good."

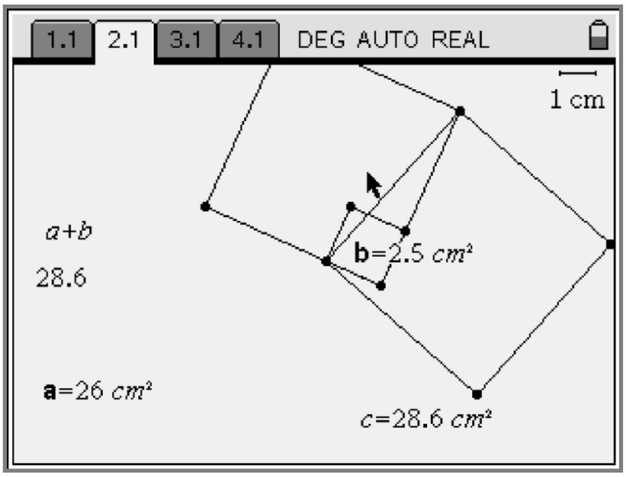


Figure 4: Student A's handheld screen.

He then moved on to Student B's screen, shown in Figure 5, saying: "We've got this one here and we've got three and twelve, that's fifteen and that's nineteen so that's close but a little bit off but it's close".

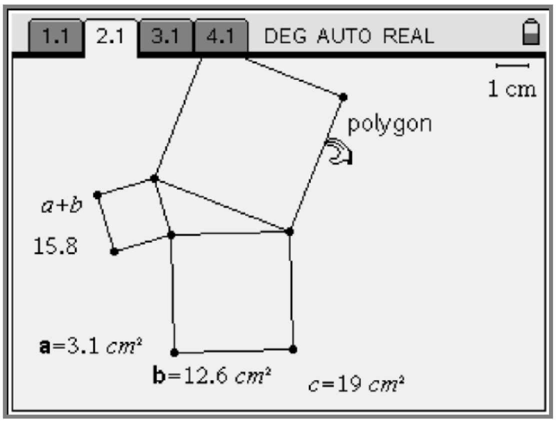


Figure 5: Student B's handheld screen.

At this point he reminded the students that "...we're kind of looking at the ones that do work and the ones that don't..." and he invited the students to volunteer their screen number if they thought that their screen 'worked'. At this point, there was a noticeable increase in students' participation and involvement as a number of students were heard to call out 'mine works', '22 works' and 'mine's 12' and Tim tried to locate these screens and move them so that they were visible to the class.

Tim then directed the students by saying,

> "Okay I'd like you to look at the ones that work that we've identified and compare them with the ones that don't work and I want you to look at the shape of the triangle... ...in the middle. This is what I am asking you to look at now. Look at the shape of the triangle. Look at the ones that work, look at the ones that don't work and my question to you and you've thirty seconds to discuss this now, my question to you is: is there anything different about the shape of that triangle in the ones that work compared to the ones that don't quite work? You've got 30 seconds to talk about it."

After a short period of pupil discussion Tim asked if anyone had noticed anything and a student volunteered the response, 'Is it right-angled?'.

Tim responded by displaying the student C's handheld screen shown in Figure 6 and making the following comment, directed towards the author of the screen:

> "Yours is quite easy to see isn't it? - that this is a right-angled triangle because you've actually got a square and you can see it's a corner of a square in there – yes it is a right-angled triangle".

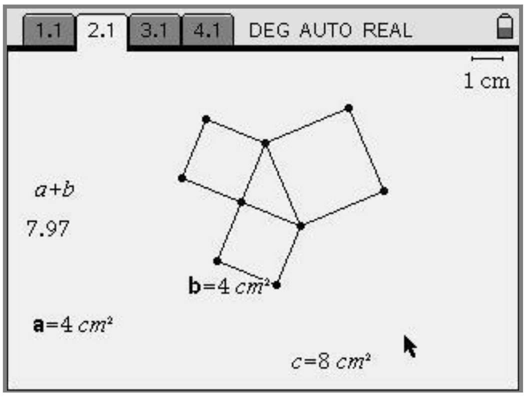


Figure 6: Student C's handheld screen.

Tim then selected a student's screen that did not appear to satisfy the initial task instruction that the value of a and the value of *b* should sum to give the area measurement of *c*, but did appear to work visually in that it appeared that the central triangle was right-angled (see student D's handheld screen in Figure 7).

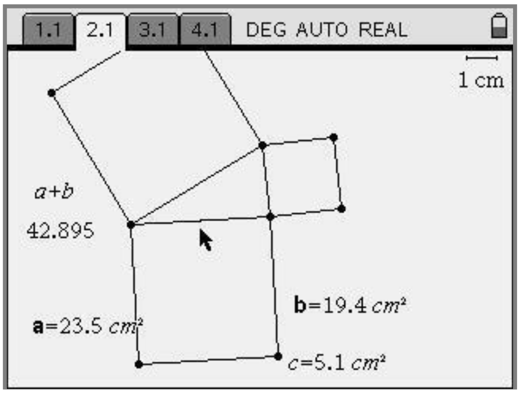


Figure 7: Student D's handheld screen.

Tim said:

> "This one, we've got a add b... doesn't quite make c either, but yours kind of works the other way round, if we look at this square here, that's five, and this square here is about nineteen and five and nineteen is about twenty four-ish and that's twenty four – so yours works a different way around".

Tim selected two more examples and spoke his thoughts out loud to reason through the calculation of the sum of the measured areas to verify whether they did or did not meet the initial task constraint He then asked the students to make a conjecture by saying, ‘So what do you think we are learning from this then? What do you think we are noticing about the ones that work and about the ones that don’t work?’.

One student responded, ‘The more the equal they get then... you know...’ to which Tim requested the student to ‘say that mathematically?’. The student added, ‘They’ve all got a right angle in them’. Tim then prompted the students by saying ‘So if the two small areas make the bigger area...’, leading to the same student’s response ‘it makes a right angle’.

Tim concludes this teaching sequence by consolidating his key learning objective thus, ‘Okay, so that’s what we’re learning here if the two smaller areas of our squares make the bigger area then we… it’s a right-angled triangle. If it’s a right-angled triangle, then the two smaller areas - of the squares - make… the biggest area’.

2.4. Analyzing the teacher’s choices

Here, the focus is the choices made by the teacher in planning and implementing the lesson. Central to Tim’s design was the intention to explore the regularity and generality of the mathematical context provided by a dynamic construction of squares on each of the sides of a triangle. In this task, he had interpreted the notion of the variables a, b and c as the registers of memory of the measured values of the areas of the squares. Tim was explicit in directing the students to change the various parameters within each of the environments, by the dragging of free vertices, with a view to students arriving at their own example that satisfied the constraint that the sum of the two areas labelled a and b should equal the measured value of the area labelled c.

From planning through to classroom enactment, it was clear that Tim set an expectation that the students would arrive at their own interpretations of the generality under exploration, although Tim did take the lead in the selection of the screens that would be discussed. A discourse analysis of the lesson transcript evidenced that, on five separate occasions during the whole class discourse, he was encouraging the students to focus on aspects of the similarity and difference between the properties of the central triangle when the areas of the two smaller squares did, or did not, sum to equal the area of the third square. Early on in this discourse, Tim introduced the notion of ‘it not quite working yet’ to describe a student’s screen where the condition was not met and later on in the discourse, Tim explicitly asks the students to focus on ‘the ones that work’.

2.5. Evidence of a hiccup

What follows is a detailed analysis of one particular hiccup that took place during the lesson in order to show how this event *may* have contributed towards Tim's situated learning during and soon after the lesson.

The hiccup was observed during a point in the lesson when Tim was clearly reflecting deeply on the students' contributions to the shared learning space and 'thinking on his feet' with respect to responding to these. It coincided with his observation of an unanticipated student response. The particular hiccup occurred when a student had found a correct situation for the task; that is, the two smaller squares' areas summed to give the area of the larger square, but the situation did not meet Tim's activity constraint of a + b = c. This hiccup can be classified as Type 1 as Tim's initial task design made it difficult for pupils to identify which of the measured areas (a, b and c) referred to which of the three squares on the screen.

Tim commented about this in his personal written reflection after the lesson:

> "One student had created a triangle for which a+b did not equal c, but (I think) a+c=b. This was also right-angled. This was an interesting case because it demonstrated that the 'order' did not matter... when the sum of the smaller squares equaled that of the larger square, then the triangle became right-angled".

Tim revised the TI-Nspire file after the lesson, providing some convincing evidence of his learning as a result of the use of the technology in that he intended to do something different next time. Tim gives an insight into his learning through his suggestions as to how he thought that some of these perceived difficulties might be overcome by some amendments to the original file.

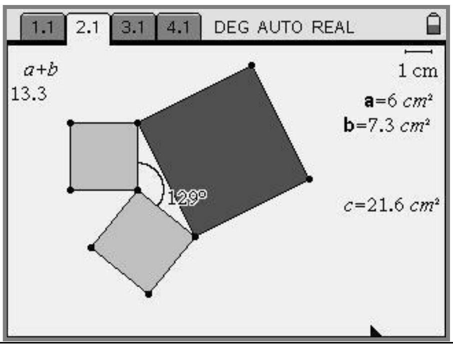
Opening screen	Revisions to the construction of the environment
	Task 2 (revised): The squares whose areas were previously represented by 'a' and 'b' have been lightly shaded and the square represented by the area measurement 'c' has been darkly shaded. Tim also added an angle measurement for the angle that is opposite the side that was intended to represent the hypotenuse.

Table 2: Tim's revisions to the TI-Nspire file for the 'Pythagoras exploration'.

Both of these amendments to the original file suggest that Tim wanted to direct the students' attention more explicitly to the important representational features. He wanted to enable the students to connect the relevant squares to their area measurements and 'notice' more explicitly the condition that when the condition for the areas was met, the angle opposite the hypotenuse would be (close to) a right angle. Here Tim was still trying to overcome the inherent difficulty when using mathematical software concerning the display of measured and calculated values in the hope that students would achieve an example where the areas were equal, and the measured angle showed ninety degrees. This conflicted with his earlier willingness to try to encourage his students to accept an element of mathematical tolerance when working with technology with respect to the concept of equality.

3. Comparing and contrasting the analyses

In the research in the English classroom, the aim was to articulate more deeply the nature of, and processes involved in, teachers' learning as they introduced a multi-representational technology (MRT) into their classroom practices. The identification and analysis of one classroom hiccup, and the identification of Tim's subsequent associated actions, provided evidence of his possible situated learning in relation to the use of the technological tool to privilege students' explorations of variance and invariance. This learning was related to a number of factors.

First, the decision to use the technological environment for this activity, and display the students' results publicly, resulted in an unanticipated student's responses becoming the focus within the classroom discourse. Consequently, Tim was prompted to develop a new repertoire of dialogue in response to this classroom experience that acknowledged the student's correct response within a wider mathematical sense.

A second factor was the design of the task in the technological environment and the way in which its appearance (on the computer screen) would support, or not, students to notice the variant and invariant features relevant to this task. This was achieved by modifying the objects' labelling and introducing a new piece of information (angle measurement) in order to focus students' attention toward the property at the core of the mathematical theorem at stake (in Tim's intention).

Overall, in his original design for this activity, Tim had not envisaged the scenario of the student response that led to this lesson hiccup. The analysis presented of this one lesson hiccup provides an insight into the relationship between Tim's situated learning in the classroom and the potentially more epistemic learning as evidenced by his direct actions in redesigning the activity.

The French analysis focused on the relationship between the lesson preparation and its actual implementation. The focus was on the teacher's management of uncertainties inherent to students' activity in a technological environment; the teacher needs often to be ready to react immediately to students' feedback when working with computers. In Daniel's activity, several types of tensions were observed, manifestations of 'struggles' between maintaining the teaching goals and adapting to the classroom current situation.

Some tensions were anticipated by Daniel, and he planned how he might manage them. Other ones were unexpected or even not consciously noticed by him and not managed within the lesson time: these led to disturbances in the management of the mathematical activity of the students.

The tensions were analyzed along several axes. A pragmatic tension was related to what the teacher, Daniel, expected from the use of DGE and how students actually used it. In addition, cognitive tensions were identified related to definitions of mathematical object or to the intrinsic discontinuity and approximation of measures in the technical environment. Temporal tensions (almost a constant feature of lesson management) were exacerbated by unexpected difficulties, some of which could be due to students' lack of experience in using 'basic' commands of the software.

In this analysis of Daniel's activity, only tensions and disturbances related to the local level of a class-session are considered; some tensions were, or might have been, managed at a more global level (perhaps over several sessions).

Contrasting the English and French studies, a major difference is the positioning of the researchers. In the English study, a close 'insider' relationship was established between the researcher and participating teachers, which required a "theoretically-based, innovative, iterative design process - for reliable developmental outcomes" (Jaworski 2004, p.3). In the French study, the researchers worked on videos of the lesson chosen by the teacher and identified tensions and disturbances from an 'outsider' point of view. While interactions between the researchers and the teacher occurred later, the teacher was not directly involved in the research process and is considered as an 'ordinary' teacher – with the research process aiming, in a way, at some generalization (for activity analysis and for teacher training).

A second contrast relates to the way in which classroom incidents were both identified and theorized. The notion of such 'contingent moments' in mathematics lessons is currently receiving increasing attention in research literature (for example, see the special issue of *Research in Mathematics Education*, Vol. 17, Issue 2, entitled *Tales of the unexpected*). Within the English research, such contingent moments - the hiccups - were conceived as an epistemological construct through which to identify aspects of the teacher's (mathematical) professional learning.

In comparison, the French study conceived the existence of tensions (and the possible disturbances) as inherent to the characteristics of the teaching situation, particularly when involving technological environments - as tools both for the teacher and the students. The research focus was not on evolution in the teacher professional knowledge but on the dynamics of managing tensions, and on the factors that influence this management: on the one hand, it depends on the ‘contingencies’ in classroom mathematical life and, on the other hand, it is oriented by several forms of determinants of the teacher’s activity (from institutional to personal ones).

Conclusion

In comparing and contrasting the analyses, several themes emerged relating to the respective theoretical perspectives, the methodological approaches, the relevant unit of analysis, the research outcomes, and the long-term intentions. In this section, our discussion addresses each of these themes.

In terms of theoretical perspectives, in the English research the notion of the hiccup was employed to articulate teachers’ professional learning over time as they integrate dynamic mathematics technology in their lower secondary school mathematics lessons. In the French research, the idea of ‘tensions and disturbances’ aimed at a better understanding of the issues involved in the integration of dynamic mathematics technology into lower secondary school mathematics lessons by ‘ordinary’ teachers. Researchers envisage to investigate again (the years to come) the same teacher if the opportunity arises, in order to see the evolution of his practices.

The methodology of the French research entailed lesson analysis based on video recording of the lesson, together with post-lesson interview with the teacher (providing insights into his practices’ determinants), analysis of the tasks proposed to the students and how the teacher implemented the tasks in the classroom. In the English research, the methodology entailed pre- and post- lesson interviews, lesson observation (with the lesson audio and video recorded), plus analysis of the lesson artefacts such as the teacher’s plan, the software files, the student productions, and so on.

Given the theoretical perspective of the English research, the unit of analysis was the individual teacher’s professional learning. Here, the ‘grain size’ was both ‘micro’, in terms of detailed analysis of individual hiccups, and ‘macro’ in terms of identifying teachers’ learning trajectories over time with respect to their mathematical, technological and pedagogical knowledge. In the French research, given the theoretical perspective of the research, the unit of analysis was the individual teacher’s anticipation and adaptation in implementing their lesson. Here the ‘grain size’ was ‘micro’ in terms of detailed analysis of tensions and disturbances, ‘meso’ in terms of analysis of the teacher’s adaptations during the session itself and in a later session, and 'macro' in terms of inferences about the determinants in the teacher's activity.

The longer-term intention of both the English and the French projects was to provide deeper insight into the ways that teachers use mathematical technological tools in their classroom practice so as to inform the design and implementation of professional development activities to this effect. On the English side, it was anticipated that it might be possible to address common types of hiccups within professional tasks for trainee and practicing teachers to promote and encourage reflection in, and through, classroom practice. On the French side, an additional aim was to provide theoretical and methodological tools that can be used in teacher educator courses in order to improve their understanding of the complexity of ordinary teachers' practices related to technology and to adapt, accordingly, their training actions. Both of these research endeavors contribute to the call made by Sinclair and colleagues (2016, p. 704) for "further research on the preparation of teachers [in the use of technology] to help them ensure that students gain deeper understanding of geometrical concepts and theory".

In conclusion, as evidenced by the discussion thoughts we have presented immediately above, the French and English studies provide insights into both sides of the same coin, that of teachers' classroom practices with digital technology in the classroom. In fact, on the one side, the French analysis is particularly oriented toward the reasons producing *tensions* and *disturbances*, on the other, the English analysis emphasizes the consequences of *hiccups* for teacher learning.

Whilst the context ('ordinary classrooms') and overarching longer-term intentions (theorizing about aspects of technology integration) of the two studies are closely aligned, the complexities of technology integration in mathematics lessons are illuminated in ways that explain teachers' decision making both in the moment, and over time. By contrasting the two studies, we have shed light on the many and varied considerations that mathematics teachers face with integrating technology.

Here we have illustrated how the hiccups, tensions and disturbances when integrating digital technology in mathematics classrooms leads to teacher learning. Yet the occurrence of such hiccups, tensions and disturbances can, potentially, put teachers off using digital technology for ever or mean that they do no more than the minimum. Jones (2011, p. 44) has suggested the notion of *canalization*, a term usually used to indicate that there is a 'normal' pathway of development, to capture the idea that when more is known about the complexities of digital technology integration in school mathematics, then technology use "may be more likely to reach a 'tipping point' and move the pathway of education to a radically new route". Our research contributes to a deeper understanding of the complexities of such technology integration in school mathematics towards when there may be such a 'tipping point'.

The theorizing evident in the French and English studies emerged from the analysis of digital technology-rich mathematics classrooms. Nevertheless, the theoretical constructs (*hiccups*, *tensions* and *disturbances*) may well be useful when analyzing lessons where there is no use of digital technology. This needs to be validated by further studies.

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THEORY AND THE ROLE OF THE MATHEMATICS TEACHER EDUCATOR: COMPARING THE USE OF VIDEO IN TEACHER EDUCATION SESSIONS IN FRANCE AND ENGLAND

Abstract. In this article, we compare and contrast practice-based approaches to using video in the context of mathematics primary and secondary teacher education. We look across country boundaries, with a focus on theory, in relation to the role of the mathematics teacher educator. We place the article in the context of developing interest in the facilitation of professional learning of mathematics teachers using video. In contrasting our different practices, we ask: what guides the planning of video sessions? what guides the action of facilitators during sessions? and, what are the intentions, in terms of teacher learning? We uncover similarities and differences in our practices which we theorise in terms of our espoused, enacted and intended theories, as mathematics teacher educators.

Keywords. Using video, teacher educator knowledge, role of theory, mathematics teacher training

Résumé. Une comparaison de l'utilisation de vidéos en France et au Royaume-Uni pour la formation des enseignants de mathématiques : théories et rôle du formateur. Dans cet article, nous comparons des pratiques effectives d'utilisation de la vidéo en formation d'enseignants du 1er et 2nd degré, pour l'enseignement des mathématiques. Au-delà des différences culturelles, nous nous interrogeons sur le rôle joué par la théorie dans ces approches, et nous nous inscrivons dans le courant de recherche actuel sur le rôle du formateur d'enseignants de mathématiques. En comparant nos pratiques, nous nous demandons ce qui guide l'organisation de la formation et l'action du formateur pendant ces séances utilisant des vidéos : quels sont les enjeux, en termes de développement professionnel, pour les enseignants ? Nous mettons en lumière les similarités et les différences dans nos pratiques, que nous analysons à travers l'idée de « théories du formateur », explicites ou non, transmises ou non aux enseignants formés.

Mots-clés. Vidéo, connaissances professionnelles pour la formation, rôle de la théorie, formation des enseignants en mathématiques

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Introduction

We came to write this article through our participation in two Symposia focused on making connections between English and French approaches to mathematics teacher education, from a theoretical perspective. We place our writing within the field of growing attention paid to the role of the didactician or facilitator of teacher education (e.g. Jaworski and Huang 2014). In this article, we illuminate similarities

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and differences in the three authors' (all teacher educators) uses of video when working with mathematics teachers. We discuss some cultural differences between our English and French perspectives with respect to teacher education. In particular, we focus on the way that theory informs what we do: what guides the planning of video sessions? what guides the action of facilitators during sessions? and, what are the intentions, in terms of teacher learning?

Jaworski and Huang (2014) proposed the word 'didacticians' as a label for the specific group of mathematics educators involved in the training of pre-service teachers, or in the professional development of in-service teachers. From a French perspective, 'didactician' singles out those educators who are also researchers in mathematics education in distinction to those 'formateurs' who are involved in the education of pre-service and in-service teachers. We use the phrase 'teacher educator' to describe what we do as we are reporting here on our work with mathematics teachers – all three of us are didacticians in the French sense of the word as well.

1. The role of the teacher educator

In a commentary on a journal special issue with the theme of the practices and professional development of teacher educators, Even (2014) suggests that although there is growing interest internationally in the education of teacher educators, there is currently little research addressing this area. Even calls for 'a more comprehensive research effort on the education and professional development' of teacher educators (p.331). We view this article as a contribution to such an effort, as we share what we have learnt from each other's practices and the influences that have led to us acting in the way we do. We are in a peculiar role as teacher educators, in that, through teaching teachers we are enacting and exemplifying in practice what it is to teach, as well as, perhaps, discussing and espousing theoretical perspectives or beliefs about what it is to teach. We recognise that our actions may not always match our expressed beliefs and stances. We also recognise that the learners in our sessions (pre-service or in-service teachers) may pick up more from what we do (in the role of teacher) than from any explicit content. Our concern in this article is with the role of theory in our work as teacher educators. In particular, how do theories of learning and teaching, guide or influence the actions of teacher educators?

We begin by setting out our methodology for comparison of practices (section 2) and then offer a description of theory and practice in one English context (section 3) and theory and practice in two French settings (section 4), before analysing similarities and differences (section 5).

2. A methodology for comparison of practices

The collaboration on which we report in this article is in the tradition of teacher-educator self-study (Loughran 2002; Tzur 2001). We examine our own practices in an effort to understand better what we do and what each other does with a focus, as stated above, on the influence of theory. We are teacher educators and researchers at the same time, although the roles are different, and the decisions we make, knowingly or more implicitly, may depend on which role we assume.

It is one thing to analyze what takes place in a training session or program (with theories as tools for analysis), and another thing to actually teach a session (where we act in-the-moment to make decisions and perhaps only later analyze those decisions in relation to theory). Being an educator as well as a researcher doing research on teacher education, it can be hard to separate our different expertise and practices. Here we analyze sessions that were meant for training and not for an experiment on training, but our two roles are intertwined.

As researchers, we are involved in problematizing the teacher education system, leading us to look at different levels of theory in our field. We identify the following uses of theory in our own work:

- the theories (or elements of theories) that guide the choices made when designing training programs;
- the theories that inspire the effective implementation and actions of teacher education;
- theories that we use to analyze classroom and teacher education activities;
- the elements of theory that might be among the knowing that we intend on offering to teachers.

In essence, we take ‘theory’ to mean any set of distinctions relevant to one of the purposes above. We conceptualize these different uses of theory under the following headings: espoused, enacted and intended. Our espoused theories are the ones we perhaps write about and use to justify our research and that may inform our planning. We distinguish, however, espoused theories from theories that are enacted. In the performing of a teacher education session there may be a more or less close match between the theories being espoused, those being enacted (and, indeed, those intended for the students to learn). A caricature of a mismatch between espoused/intended and enacted theories would be a lecture given on the importance of learner-centred education. However, some differences between the espoused and enacted theories might be inevitable, since teachers’ education and mathematics teaching have their own specificities. We identify the intentions of the teacher educator in relation to the intended theory that will be taken on by the student teachers in a session. There may be an intention for students to espouse the same theories as the teacher educator, or enact the same theories, or something different. Part of the contribution of this article to the field, is to offer the framework of espoused, enacted and intended theories, and to exemplify its use for the comparison of teacher educator practices, when using video with mathematics teachers.

2.1. Comparing use of video[1]

In the last decade, there has been increasing use of video for teacher education and professional development, across the world and across subjects with, broadly, one of three intentions: linking theory and practice of teaching; analyzing professional practice; and, implementing institutional reforms (Gaudin and Chalies 2015). Our own work is part of this growth and crosses the aims of linking theory to practice and the analysis of practice. A similarity across our three contexts is the use of video as a mechanism to provoke teachers to reflect on practice and a conviction about the need, for effective working with video, to support teachers in moving beyond generalized and evaluative descriptions of what is seen and into a space where it is possible to dwell in the detail of events and allow new perceptions and connections to arise (Coles 2013), even though we do not deal with this need in the same way. We all tend to use local video recordings of teacher practices rather than, for example, internet-available videos that may have been edited for particular purposes.

Despite these similarities, we had the experience of co-running a workshop at a conference in 2016 (ICME 13 in Hamburg) where we tried to find one common lesson video clip we could use with participants, to exemplify our different ways of working. We could not find a suitable one, in terms of mathematical content, of length (a short clip or an entire session), of exhaustiveness (with or without editing), of available information on the context of the video (experience of the teacher, moment in the planning of the year, curriculum) – demonstrating that despite many similarities of aims, we have different expectations when choosing a video for teacher education. We needed an English language video (due to the language of the conference) and the examples Alf had available were not suitable for Aurélie and Julie’s purposes. For example, one video Alf has used frequently involves a clip of a class working on the question: “How many numbers are there between zero and one?”. Without more elements about the context (what had been studied before, intentions of the teachers…), this question was too broad for the others, and, as will become evident later in the chapter, would not have been a good choice to illustrate the French way of working, which requires an anticipation of possible student answers and how they are put into use in the mathematical

[1] In the present article, we will use this term to mention any video clip extracted from a videotaped session of mathematics teaching.

content at stake. Aurélie and Julie offered a related question, which would have worked if we had a video of a class working on it: “Can you give me a fraction between a third and a half?”. We note, in passing, that we learnt more about each other’s practices when faced with the practical need to choose a common video than we had in many hours of discussion of our practices prior to that. Part of the problem, we recognized, is that we can use the same word to mean different things and hence interpret what each other is saying through the prism of our own practice. In writing this article, for example, we are aware that we have different connotations for the word ‘theory’, hence in part wanting to look at theory use as it espoused or enacted and at scales from local theories informing practice to overarching orientations.

In the next two sections, we offer exemplifications of our use of video. These necessarily involve a degree of description, in order to give access to the context of our work. We have chosen to structure what we present in the same way – initially offering a theoretical perspective and some institutional background, then going into the detail of our ways of working, either drawing on a specific example or generalizing across sessions and ending with an articulation of intended outcomes. We then look across these examples in order to highlight similarities and differences in relation to the use of theory. We conclude by returning to three questions that have motivated this article: what guides the planning of video sessions? what guides the action of facilitators during sessions? and, what are the intentions, in terms of teacher learning? We have structured our exemplifications of practices in order to make our comparison, in relation to these questions. There are differing amounts of detail offered about specific sessions and this simply relates to the available data we had in each context.

3. An example of video use in an English setting (Coles)

The practices of English teacher educators have no central or shared theoretical basis. University teacher education courses sometimes have a strong theoretical background, but this tends to be due to the presence of a researcher with a particular perspective (for example the University of Cambridge primary education course makes use of the Knowledge Quartet, which is a set of distinctions derived from that institution, Rowland, Huckstep and Thwaites 2005). So, in this section, I reflect on the use of theory in my own context with no claim to wider generality.

3.1. Overarching theoretical background

Through the influence of Laurinda Brown, the University of Bristol’s mathematics education courses are designed from an enactivist perspective (Varela, Thompson, and Rosch 1991). The courses are designed to get novices learning about teaching (or, in master’s and PhD work, learning about researching) in the same manner that experts learn, but from the very beginning (see Brown and Coles 2011). The enactivist thinker, Francisco Varela, recognized a characteristic of expert performance and learning (in any sphere), which is that experts tend to act spontaneously – and in most cases their automatic responses are effective. However, on occasion, their expert functioning breaks down and, in these cases, they are able to reflect on what occurred in a manner that brings to light the ‘intelligent awareness’ (Varela 1999, p.31) that led to the behavior (that was not effective). Having identified an awareness that in the past has led to effective behavior but now does not, the expert is able to identify what they need to do differently in future in the same circumstances. A clear example of the breakdown in effective behavior occurs when an experienced teacher moves to work in a different school. Years of developed behaviors that are effective in one context may no longer ‘work’ in the new scenario. Student teachers on English teacher training courses can experience such a change when they change placement schools (at Bristol, teachers have extended placements at two different schools generally, over their training year). For example, in one school, waiting in silence for a class to quieten down may be effective. Changing to a new school, the same strategy may lead to more and more disruption. The expert is characterized not so much by being able to second guess what will be effective in a situation, but by being able to change and adapt quickly.

Varela’s insight (1999, p.30) is that even novices can learn like experts. What is needed is to support novices to steer a path between unconscious behaviors and over-deliberate actions. In other words, they need to act spontaneously and then be supported to reconstruct the awareness that led to any ineffective behaviors. It is this insight that informs the teacher training courses at the University of Bristol. From the very start of the year, the student teachers are placed in the role of having to act as ‘teacher’. In the first week of the course, they have to teach each other (in groups of around fifteen) something non-mathematical (that they have prepared) for ten minutes. The rest of the group then reflects on what they learnt, or anything that hindered their learning and the student teacher giving the lesson will begin the process of reconstructing what led to ineffective behaviors and therefore developing ‘action targets’ for the next time they teach.

The overall aim is to induct student teachers into a cycle of reflection which begins with describing the detail of experience; moving to identifying issues arising from their experience; and then, committing to new actions, linked to the issues identified. This cycle is set up in the first week of the course and it informs: all sessions at University; the writing tasks for students; and, the ways in which, as tutors, we run feedback sessions in school after observing them teach.

3.2. Ways of working with video

I was involved in a research project in 1999, as a teacher-researcher that made use of video recordings of lessons both for professional development of teachers and as a tool for analysis. Arising from these experiences, when I became head of a school mathematics department (with responsibility for the professional development of mathematics teachers) I was keen to use video. I encouraged staff to take video recordings of lessons (with a fixed camera on a tripod at the back of the room) and I would use small clips of these recordings to discuss at departmental meeting (see Coles 2013). I moved to a role at the University of Bristol in teacher training in 2010 and have subsequently made use of video in a number of scenarios, for example: on the training course for secondary mathematics teachers; in Master's level sessions on observational methods; at one-off invited sessions with groups of teachers.

I have come to have conviction in the importance of starting work on video with a reconstruction of events. By this, I mean that the initial discussion needs to focus simply on describing what took place. However, focusing on the detail of our observations is, for most people, an unusual experience and can be hard to do, for participants. There is an ambiguity also, in that descriptions can potentially be at any level of detail. It is probably not going to be helpful, if the focus is on learning about teaching, to go into the minute detail of ergonomic movements. The intention, in starting with reconstruction, is to focus discussion on agreeing the words that were said and possibly some basic description of movements.

The practice of starting with a reconstruction of events is a strategy taken directly from the practice of working with teachers on video developed by the Open University in the UK (see Jaworski 1990). There will often be a need, as the teacher educator (in this context, the facilitator of discussion), to impose the discipline of only offering descriptions of events. Particularly if groups are not used to working in this manner, it is common for initial comments to tend to the general and the evaluative (e.g. 'the class seemed bored', 'the teacher had a lovely rapport with the class'). The teacher educator, in these instances, needs to intervene – cut the contributions short if necessary – and re-inforce the discipline of only describing events that can be observed. You cannot 'see' a pupil being 'bored', for example, but only interpret this. The aim of this section of the way of working is to get teachers to put descriptions like 'bored' to one side and focus on what they actually saw (e.g. 'two pupils at the front were looking out of the window').

What Jaworski (1990) reports from her work with teachers on video, is that typically, we respond to video clips of lessons with evaluative comments (e.g. 'my pupils could never do that') and, as a result, little of value comes from discussion. Combining this insight with my enactivist convictions, what is needed is a mechanism to try and get teachers talking (about video) in a non-evaluative manner so that there is the possibility of ‘seeing’ what is on the video differently and therefore allowing discussion to throw up the possibility of acting differently. The discipline of starting with a reconstruction of events is one way of cutting out evaluative comments to allow the possibility of new insight.

At some point, I will always re-play the clip or a section of the clip for teachers to watch again. I try to look out for points of difference amongst the group, in terms of what they saw on the video clips, as these points of disagreement (e.g., about what was said, or the order in which it was said) can provide a motivation to watch again. There is always a delicate decision about when to replay a clip. Leave it too long and the reconstruction task turns in to one of memory, and teachers may lose engagement. Replay too soon and teachers may not have an experience of doubt or questioning about their own recollection of events.

When the replaying works well, teachers in the discussion often comment with amazement at how much more they hear in the clip the second or third time around compared to the first time. Particularly if the focus for re-playing is on a small section, it can become clear that whole sections of dialogue were not heard the first time around. This realization in itself can be a powerful learning point from working with video, with the obvious question it raises of how we cope with this complexity in the real time of classroom decision-making.

Having reconstructed the video, with the aid of re-watching, the final part of this method of using video is to move to an interpretation of the events on the clip. It is necessary for the teacher educator to mark this shift in the discussion, i.e., that it is now moving to a discussion of teaching strategies linked to the particular focus of the group. No matter what the focus is, given the work we are discussing here is with teachers, a focus on teaching strategies is relevant. I take a teaching strategy to be anything that a teacher does, and that can be described in a manner that makes it repeatable.

3.3. Institutional background

In England, there are routes into teaching via under-graduate degrees or one-year post-graduate degrees. Post-graduate provision is split between degrees offered by Universities and ones offered by schools with a University partner who accredits the qualification. The context, in 2018, is that there are not enough mathematics teachers at secondary level, partly as a result of high numbers leaving the profession after a few years (Des Clayes 2017). When teachers are in school, continuing professional development opportunities are available from a range of sources, including Universities, government funded « hubs » and private providers. These opportunities could be: ongoing Master’s degree courses; other courses that run over time; or one-off conferences or seminars.

3.4. Example of a session

In the session I describe here, I draw on data from a « video club » that I ran for primary mathematics teachers. The session chosen typifies my use of video and was not exceptional but does illustrate the way of working. A group of in-service primary teachers volunteered to join the club, which committed them to attending six meetings (roughly one per fortnight) after the end of the school day. The volunteers knew that they were committing themselves to taking some video recordings of their teaching and sharing these within the group. An open call had been advertised to teachers in the Bristol area and no one who applied was excluded. Eight teachers joined the group and I analyze here audio recordings from the first session with the group.

What is the video and why was it chosen?

The first session of a video club that involves teachers who do not know each other, is the one instance when I will use a video clip that is not from one of their classrooms. In this case I chose one from the Video Mosaic collection (https://videomosaic.org). The way of working necessitates a clip of 3 to 4 minutes. I have always worked with video clips that show a phase of whole class discussion, i.e., where there is one conversation happening in the room (or at least one predominant one) and also where something unexpected (Rowland and Zazkis 2013) occurs, to which the teacher has to adapt.

In the clip I chose ('Alan's Infinity'), the teacher (who is in fact a researcher) is working with a class of 4-grade students, and the clip starts with the teacher asking the class "How many numbers are there between zero and one?". What happens next on the clip is a discussion amongst students in the class, with two boys doing most of the talking, one of whom thinks there will be infinitely many numbers and the other who disagrees. Three other students are also seen to contribute. The teacher makes some prompts and on occasion directs who will speak next.

The activities during the session using video

Before watching the video: in the first group meeting, before watching any video, I invited each teacher to say something about why they had joined the group and what they were wanting to develop in their own teaching. For example, one teacher (J) described wanting to develop his teaching so his students became more independent in their learning. In setting up the first video watching, I explained the question the class had been offered (how many numbers are there, between zero and one?) but we did not work on, or discuss, the question ourselves. I then said to the group:

> 'Don't worry about taking any notes. We're going to watch a short clip. And the first thing we're going to do as a group is to literally try to reconstruct

> what happened, what was said … and then given all the things you're thinking about we then might do some thinking about what the teacher's doing or what reasoning or what teaching strategies; things that might be more of an analysis. But the first bit is going to be literally what was said. So, the children are thinking the problem, how many numbers are there between zero and one?

After this, I played the video and sat down.

During watching the video: I consider the whole process of 'reconstruction' as taking place 'during' watching – in fact, there is a movement between watching and discussing, re-watching and discussing. The dialogue, straight after the video clip ended, was as follows:

> P: I can't stop watching thinking about your [looking at Teacher J] independent children and unfortunately the children that weren't paying attention.
> J: yeah, yeah, yeah
> Alf: So, that's an interpretation and at this stage the invitation is to say what happened, what you saw
> N: She invited them to as what's inside that line
> Alf: Anyone remember anything before that, so say that again, so he puts his hand up
> N: It was about splitting the line into zillionths.

As the facilitator, my role during the reconstruction phase is to direct conversation back into the detail of events and to offer a re-watching of the video, when the group has arrived at conflicting memories of what took place.

> P: Someone talked about atoms didn't they?
> J: That was when he said about a really long number line.
> J: I thought that was interesting because
> Alf: That sounds like an interpretation
> J: Interpretation, yeah, yeah, yeah.
> Alf: Try and stay with detail, we'll go on to that in a second. Let's try and see if we can get the chronology … and we can go back and look, but we got something from the teacher, a possible question, we think
> C: How many numbers
> Alf: okay
> J: How many numbers do you think?

In this transcript, as well as re-emphasizing the need to avoid interpretation, I articulate where, as a group, we have some questions about what took place on the clip – in this case, what the teacher actually said at the start of the clip. Just before I do re-play, I comment:

> Alf: okay, so we can quickly watch it again. There are some questions about this dust particle and what the dust is all about, something about what's said at the very beginning. Okay so let's try it again.

I then offer one further re-watching:

> Alf: So I might stop it after the first break and we can see if anything else has emerged or if we have any answers to those questions.

At each re-watching, we look at a smaller and smaller section – as we focus on specific questions of what took place.

After watching the video: we spent twenty minutes working to reconstruct the clip, re-watching sections of it three times, before we move to the analysis phase. I provoke this new phase as follows, and Teacher P is the first to respond:

> Alf: Any reflections on what the teacher was doing then or what the students were doing, or any teaching strategies?
> P: I thought she was very controlled and very restrained. I talk far too much in my maths lessons I think. She just let them get on with it.
> Alf: Okay [AC writes 'controlled/restrained/let them get on with it']

My aim in this phase is to support teachers to generalize from what they observed, identify issues relevant to their teaching and, if possible, share strategies related to these issues. For example, the issue raised here by Teacher P, I interpret as 'letting [the pupils] get on with it' and we would then share strategies, i.e. things as teachers we can do, which relate to this aim.

Outcomes from the session

The initial focus on a pure reconstruction of events tends to mean the interpretation of events is rich in detail and noticing. As a teacher educator, my aim is to support the articulation of new ways of seeing in the classroom. The move here is away from the fine detail of classroom events, but not to become so abstracted from the context that the link to direct actions is lost. If discussion moves into the realm of philosophy, for example, whether the class acted in an 'autonomous' manner or not, then my sense is that this is unlikely to be of benefit to teaching. There needs to be some abstraction from the detail, but the link to future action is vital.

To sum up, my aim as a teacher educator working with teachers on video of lessons, is to support new ways of seeing what is there on the clip and get to novel (for those teachers involved in discussion) articulations of features of the video clip. I do not use video with the aim of directing discussion onto particular and pre-identified aspects of pedagogy, beyond having in mind the overall focus for meeting, which is usually some aspect of teaching and learning. My belief, born out of the enactivist world-view, is that learning for teachers will be most effective if what arises out of discussion for them to work on has come from their own awareness. The shift in perspective of Teacher P, from initially seeing children who 'weren't paying attention' to later on getting to a realization that 'I talk far too much in my maths lessons I think', is an example of the potential power of the way of working in terms of shifting participants' attention away from their own immediate reactions (that are often emotional) and towards potential learning points.

3.5. Analysis in relation to theory use

The way of working on video is theory-driven (Jaworski 1990) and although the origins of the method are not enactivist, the principles behind what I do fit well with my espoused enactivist principles. These espoused principles and theories are not part of the training sessions using video. What is made explicit is the distinction between observation and interpretation, which is an important element in the discipline of noticing (Mason 2002) and features in enactivism (Maturana and Varela 1987). The intended theories, in relation to the teachers, are two-fold. There is an intention that teachers will become conscious of the observation/interpretation distinction; secondly, the hope is that teachers will find 'issues' (for example, for Teacher P, perhaps 'letting the pupils get on with it') that will inform new actions in the classroom. I might describe such issues as local 'conjectures' about practice, or local 'theorising', mindful that from a French perspective 'theory' denotes sets of ideas that are far more developed and established.

4. Two examples of video use from a French perspective (Chesnais and Horoks)

First, this is our personal French perspective, rather than one that could represent every French teacher educator's view on teacher education, and it is mainly inspired by the frameworks we use while doing research about teachers' practices. If we tried to analyze education programs for teachers in other universities in France or observe and analyze what educators do, we would probably see that there is a wide variety of practices, some inspired by other theories, even outside of the mathematics education field, but also from former experience as teachers, as in Sayac (2013), who explores the practices of teacher educators from different backgrounds.

4.1. Overarching theoretical background

To analyze and interpret teachers' practices, but also to consider our own practices as educators, we use the Theory of the didactic and ergonomic Double Approach (Robert and Hache 2013), which combines didactic analyses of pupils' mathematical activities with ergonomic analyses inspired by the analysis of the practices of a professional activity. The fact that the Double Approach was inspired by activity theory plays an important role in our choices. The main postulate of these frameworks is that teaching practices (teachers' activity) influences pupils' activity, which is responsible for pupils' learning. It allows us to take into account some constraints of the profession, which can explain some of the decisions made by a teacher (or a teacher educator) when teaching (or training), by defining five components of teachers' practices (see chapter 3). The first two concern what happens in the classroom:

1. the cognitive component "corresponds to a teacher's decisions regarding content and tasks, including their organization, their quantity, their order, their inclusion within a curriculum beyond the class period, and plans for managing the class period". (Robert and Hache 2013, p.51);
2. the mediatory component describes choices regarding class events, and the effective implementation in class of the content and tasks (teacher's speech, pupils' participation, assistance to pupils, validations and explanations of knowledge).

The other three components might have an influence on what happens in the classroom, but depend on factors outside of the classroom, such as the professional environment:

3. the personal component (including representations, knowledge, experience of the teacher);
4. the institutional constraints (related to the nature of the mathematics to be taught, curricula, the schedules, the resources available, the administration and inspections);
5. the social constraints (resulting from the various groups formed by pupils, parents, colleagues...).

We divide and analyse the complex system of a teacher's practices into these five deeply intertwined components, which allows us to try to understand the rationale behind a teacher's actions, regularity and coherence relating to his/her decisions for a class. Some of our hypotheses about teacher training come from the Double Approach: taking into account the constraints of training and teaching (for example the fact that not everything is possible for any teacher in any classroom, and also that the teacher is not alone in his or her classroom or in the institution) and, taking into consideration the actual practices and needs of the teachers during training. This is why we believe in the use of videotapes and "the collective discussions about practices, using a professional vocabulary which will help the participants with the necessary ''depersonalization'' in order to achieve a scientific debate, rather than an ideological one" (Horoks and Robert 2007). Videos seem to be a good tool to get an insight into teachers' and pupils' activities and access the complexity of the teaching-learning process, without having to take the risks inherent in actually running the class. For example, it allows us to work on two components simultaneously: the cognitive and mediative ones, since the video shows both lesson content and ways of managing the class. It also allows us to approach the multidimensionality of teachers' activity and the experienced needs of teachers. Videos also inform about pupils' activity, since we have access to some of their questions, comments or discussions, and to what the teacher says during the pupils' activities (potentially influencing them) and about them. Hence, analysing videos can contribute to a better comprehension of the links between teaching practices and pupils' activities.

Our theoretical approach on practices and their development, encourages us to try to foster their evolution in a bottom-up process. Instead of studying the content first with student-teachers, then elaborating tasks and reflecting on how to implement them in classrooms (as some training programs, inspired by other theoretical frameworks, would recommend), we choose to start from actual practices, inside the classroom and to face directly some aspects of their complexity. From this starting point, the teacher educator tries to make the student-teachers trace back to the generalization of some questions or problems experienced by all teachers (about content or pupils' activity or ways to deal with pupils' activity etc.). The role of the teacher educator is fundamental here in order to allow this movement towards a more general point of view.

4.2. Ways of working with video

From our points of view, teachers' choices in their practices are essentially considered as the result of multiple constraints (complexity). The teachers' decisions are also supposed to be driven by underlying logics for action, at least partially explicable. Therefore, the aim with future teachers is to allow them to be aware of the constraints, the tensions that may appear between preoccupations, choices that are made (including adaptations to contingency) and of other possible choices ("marges de manoeuvre"). In order to complete this objective, teacher education is organized to make teachers develop a reflexive posture on the activity of teaching mathematics, oriented towards didactical concerns. In particular, it includes the ability to manage and evaluate pupils' activity as the result of the teacher's choices (even if pupils' activities, of course, do not depend only on the teacher's choices). It is here that the use of video can play a powerful role in training. Related to this objective, we try to equip teachers with didactic tools which can help them analyze what happens in the classroom and make choices as teachers and evaluate their effects on pupils' learning.

One of the main tools is the *a priori* analysis of pupils' tasks, through the identification of the *adaptations* of pieces of mathematical knowledge (Robert and Hache 2013) inside those tasks. Examples of adaptations of knowledge could be: having to use a basic geometrical relationship within a more complex diagram;

having to use insights about adding fractions to dealing with algebraic fractions; having to move from using Pythagoras' Theorem in 2D to 3D. There is a need to predict what students can currently do and then what extension, or adaptation, of this existing knowledge is needed in the novel context of the task being analyzed. The a priori analysis of mathematical tasks is used both when working with video and more generally, and it is often the first tool that is used when working with teachers on tasks. To analyze mathematical tasks, Robert distinguishes simple and isolated tasks (SIT), defined as tasks where "a single piece of knowledge is used, potentially with simple replacement of general inputs by the given information in the context of the exercise" (Robert and Hache 2013), from tasks where pupils need to adapt the relevant piece of knowledge, "in relation to the required recognitions, initiatives, additions and combinations" (ibid.). Robert developed a list of seven types of adaptations of pieces of knowledge in mathematical tasks. Adaptations are considered by Robert both as means and criteria for learning: being able to adapt a piece of knowledge in a suitable way to solve a task is the sign of a certain level of conceptualization (Vergnaud 1991; Robert and Hache 2013) of it and becoming able to do so is related to the fact of having encountered various tasks in which adaptations of this piece of knowledge were to be made. Moreover, some research results have shown that the way teachers deal with adaptations (choices in scenarios or the way they handle them in classrooms) is variable and these differences have potential effects on pupils' learning (see, for example Chesnais (2013) or Horoks (2013)). The notion of adaptations is one of the tools we intend on offering to pre-service teachers and explains our use of videos. After completing the a priori analysis of a task, we would have students compare it to what happens in the video where a teacher is using this task with his/her students. The a priori analysis allows participants to apprehend the complexity of a mathematical task, and the way the teacher handles this complexity in the classroom.

We also rely on 'The Theory of Didactical Situations' (Brousseau 1997). This theory is "shaped by Piaget's theorization of cognitive development as a process of constructive adaptation and … refined in the light of Bachelard's theorization of knowledge growth as encountering intrinsic obstacles" (Ruthven et al. 2009, p.330). The concept of "situation" refers to the system formed by a problem-solving task and its environment that are especially designed to help the pupils construct some specific new knowledge. We present some of the concepts of this theory to the student teachers, to allow them to analyse and design tasks, but also, when these tasks are implemented in class, to analyse mathematics sessions in terms of phases within a situation (devolution, research, comparison of pupil's procedures, institutionalisation) in a video. Many of these concepts (such as didactical contract, didactical variables, a priori analysis, etc.) are relevant to build situations for the classroom, and to teach or to experiment in class with a research question. Some of the important elements of this theory are explained in Article 6 and not repeated here.

4.3. Institutional background

Since 2013 in France, future teachers are trained in University structures, called "Ecoles supérieures du professorat et de l'éducation" (ESPE), higher teaching and education schools. The training for primary and secondary school teachers varies with the University where it takes place but always includes, during the second year, both an internship (consisting of taking charge of one or two classes for half the time a tenure teacher usually does) and following courses at the university (in order to validate a master's degree). These courses include ones on didactics and epistemology (about all the subjects for primary school teachers, and about mathematics for secondary school), and aim at helping the student teachers for the classes they have (internship support). Student teachers are also offered general courses about pedagogy (somehow related to the internship) and an initiation to research (in a didactic or educational field) for which they are supposed to produce an essay (a classroom-based action research project, relying on a review of the research literature of the field). Each ESPE is free to decide and organize the content of these courses, so the examples developed below cannot be considered as representative of all ESPEs.

4.4. First example of a session (Chesnais)

I describe in this section a specific teacher training session that I have been implementing for five years in the ESPE of Montpellier in the south of France. Being both a teacher educator and a researcher, the choices made for the internship support course are based on a point of view of teaching practices inspired by my own use of the Double Approach as a researcher. The organization of the internship support course is highly influenced by examples of teacher training sessions and teacher educators' training sessions based on the Double Approach described in Robert and Vivier (2013), Chesné et al. (2009), Chappet-Pariès and Robert (2011).

After a first session (3 hours) where teachers work on the *a priori* analysis of several tasks including several adaptations and discussions about these adaptations, the second session aims at showing them how adaptations might help analyze what happens in a classroom in relation to teachers' decisions, i.e. a means to apprehend the complexity of teaching and learning, in accordance with our main goals.

What is the video and why was it chosen?

The two videos used in this session are part of data collected for my PhD thesis (Chesnais 2009). They show two different teachers in first year of secondary school classrooms (11-12-year-old children). Both videos last for about ten minutes each and both teachers (T1 and T2) use the same task where students are supposed to construct the mirror image of a given point with respect to a given line, using a set square and a compass.

Activities during the session using video

Before watching the video: student teachers are given the text of the task, on which the students are working in the videos, and are asked to make an *a priori* analysis of the possible activities pupils might develop in response to it. This includes trying to anticipate possible answers and procedures, pieces of knowledge that are necessary to implement these procedures with the potentially needed adaptations, pieces of knowledge that are available at this stage of learning for pupils (in relation to the curriculum). Student teachers are also asked to anticipate how a teacher can orchestrate the implementation of the task in class (organization of pupils' work, material that is needed, timing, interventions of the teacher etc.) and in particular the means that the teacher has to respond to these adaptations and the difficulties that may consequently arise. I focus on the second question of the exercise (figure 1) with the student teachers. They are aware that the method that first year secondary school pupils (age 11-12) could use to complete the task is the one using set square, to draw a line, perpendicular to (AC) through point B, and a compass, to duplicate the distance between B and (AC), and thus find the position of E on the line.

1. Copy the given figure [just the triangle ABC, shown in figure 2].
2. Construct the point E, symmetric to point B with respect to the line (AC).
3. Give, without measuring it, the length of the segment line [AE].

Figure 1: Text of the exercise.

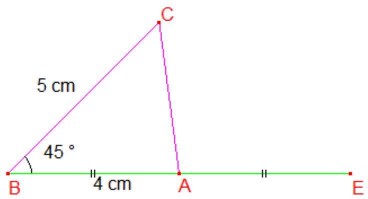

Figure 2. Mistake in the construction, encouraged by the horizontality of segment line [BA].

Nevertheless, the task contains adaptations: one of the main ones is to be able to recognize how to use the method that they know. For example, the line is not drawn, but only the segment line [AC]; there are other elements than just a point and a line; the horizontal segment line encourages pupils to construct the mirror image of point B on the same line (see figure 2).

During watching the video: after identifying and discussing adaptations, student teachers are shown the videos and are asked to take notes on what pupils and teacher do, or do not do, the pieces of knowledge that appear, and to concentrate especially on what happens about the previously identified adaptations. They are encouraged to try to identify common points and differences between the two videos and especially between the choices made by the two teachers.

After watching the video: student teachers have a couple of minutes to discuss their subjective impressions on, for example, the pupils' levels of concentration, and I then direct the discussion to the choices made about the adaptations. What emerges from discussion is the fact that T1 takes charge of the adaptations (for example by indicating to draw the line before the pupils even start to work on the task); whereas T2 leaves the pupils some time to try to figure out by themselves how to identify the configuration and use their knowledge.

The question then arises of the frequency and time of these kinds of choices in the process of teaching and the possible effects on pupils' learning (about geometry - for example the ability to recognize a given figure in a complex one - and in general), and finally the question about the reasons that may explain these choices (for example the need to control what students are doing in a difficult class…).

Outcomes from the session

The conclusion of the discussion emphasizes the necessity of the *a priori* analysis of the tasks (especially the identification of adaptations that are necessary to complete the task successfully) in order to: (a) choose tasks while understanding what is really at stake in them and organize the teaching of a particular piece of knowledge; (b) anticipate pupils' difficulties and be able to identify them and their origin when they occur; (c) anticipate (different) ways of dealing with them, contemplating possible choices for the teacher.

4.5. Second example of sessions (Horoks)

In the University of Créteil near Paris, I run a course dedicated to the initiation to research, which forms a significant part of students' training (pre-service primary school teachers). The student teachers have to choose from several fields of research to do this initiation, and the following example concerns the course in the research field of maths education. The purpose of this initiation is to give student teachers some objective means to take a step back and reflect upon their own practices. We are also trying to give the student teachers some tools to be able to undertake some actual research (even though they will likely not become researchers in the end), therefore transmitting some selected parts of our theories, that we consider useful to analyse what happens in a classroom. The student teachers are required to write a fifty-page essay throughout their training, making hypotheses about teaching and learning mathematics, and testing these through an experiment in one of their classes.

We make the hypothesis that we tested in Horoks and Grugeon-Allys (2015), that training teachers through an initiation to research, introducing research tools and methods, might facilitate the development of a more objective stance, in order to reflect on their own practices when teaching mathematics.

What are the videos and why are they chosen?

Using video allows us to look at the practices of another teacher from a researcher's point of view: showing the need for theoretical frameworks to inform how we analyse what happens in the classroom, based upon the analysis of the situation through its mathematical content and the way it is put in use.

The videos that I choose have usually been recorded for research purposes, or within the training program, and usually show a teacher (experienced or not) giving the pupils a task to be solved. As in the previous example, we focus here on a session where we compare two videos that feature pupils in the first year of elementary school (5-6-year-old children), where the task (counting a "big" quantity of objects) leads, in both cases, to grouping the objects by sets of 10, to introduce the decimal structure of whole numbers.

Activities during a session

Before watching the video: I give the task to the student teachers to be analysed first before viewing the video, which raises research questions about the choice of certain didactical variables, and the effects of these choices on the pupils' mathematical activities and learning. In order to inform the a priori analysis, I offer student teachers some ideas about learning the decimal system, inspired by epistemology and research results in the mathematics education field, and let them reflect on the task given to the pupils in each video.

During watching the video: after working on the task and anticipating the students' possible answers and difficulties, students are shown the videos, to analyze the potential gaps between the a priori analysis and what actually happens in the classroom.

After watching the video: the student teachers are invited to comment on the pupils' procedures, when they are visible, compare them to the ones we anticipated, and analyse the role of the teacher in the different phases of the session. The choices of the teacher can then be interpreted, using the Double

Approach, to take into account different constraints that are not directly linked to the pupils' learning. It generally raises more questions, as we usually do not have enough data within the video clip alone to corroborate our hypotheses about learning.

In these two videos, for example, the different choices related to the various phases of the situation (with more or less initiatives for the pupils, more or less time for the pupils' independent exploration and for the comparison of their procedures) might have an influence on the pupils' learning, but it needs to be investigated further, in order to be asserted, in relation to the students' mathematical activity. This is the kind of experiment that could be undertaken by the student teachers for their research essay.

Outcomes from sessions

The goals here are to show researchers' methods - analysing a video as a researcher would do while doing research - and provide the student teachers with necessary tools to begin to interpret teacher practices (and balance the analysis) in a given context (with particular constraints). These are the tools that we hope will help the teachers to reflect on their own practices, but we ask them to take the researcher's posture for now (which can be unsettling for the students) and adopt these methods to enable them to complete the writing of their research assignment. The students are also working with videos sometimes in other parts of their training, but not with a research question in mind (and with or without an a priori analysis of the task or situation, depending on the teacher educator's status, practices and goals).

4.6. Analysis of theory use

Our ways of working with and on videos are driven by the Double Approach and its hypotheses about teachers' practices and their development. The theoretical tools from this theory (to analyze teachers' practices) are both at the origin of our decisions for teacher education (espoused and enacted theory), and at the centre of the video analyses conducted with the student teachers. Therefore, they are relatively explicit in our practices as educators, as intended theoretical and/or professional tools. These are tools that can also be used to analyse our own practices as educators and, explain our choices in a given context.

5. Analysis of use of video in relation to the use of theory

From the French perspective, Julie and Aurélie would say they do not espouse theory on mathematics teacher education, but rather have theories about teaching that inspire hypotheses about working with teachers: Activity Theory informs their way of thinking about how teaching practices work and how they develop (in a comparable manner to Robert and Vivier (2013), Chesné, Pariès and Robert (2009), Chappet-Pariès and Robert (2011), but neither this theory nor the Double

Approach are theories that are meant to be teacher education tools. Their work in teacher education is more a ‘logic of action’. For Alf, enactivism, being in part a theory of cognition, can be put to use in thinking about teacher education as well as mathematics teaching. It is from an enactivist perspective that the way of working on video is conceptualized, in which participants begin with a description of the detail of events.

Enactivism is committed to the non-separation of knowing and acting (‘all doing is knowing, all knowing is doing’, Maturana and Varela 1987, p.27). One of the insights of enactivism is that we live most of our lives in ‘readiness for action’ (Varela 1999, p.10) responding immediately, and effectively, with those around us. Alf’s aim in using video is to support the development of ‘readiness for action’ in the classroom. The ‘espoused’ theory of enactivism suggests ways this can be enacted (focusing on the detail of events and initially avoiding evaluation before moving to an interpretation and labelling new distinctions).

From the French perspective, the influence of the TDS can be seen in the way that Aurélie and Julie use problem situations to trigger the identification of needs and questions, and the need for tools (such as categories, established by researchers, to classify mathematical tasks or identify moments of the sessions) to analyze practices and what happens in the classroom, and also the organization of the training sessions into different phases. The influence of TDS can also be seen in the way Julie and Aurélie both start by getting student teachers to do the task that features within the video recording. From a researcher’s point of view on the training session, one could mainly use the Double Approach to interpret the teacher educators’ choices (a priori or a posteriori) taking into account their professional constraints (type of audience, personal background, research, etc.). This theory prompts the use of two contrasting videos, in both scenarios.

It is apparent, looking across the descriptions of teacher training, that there are differences in relation to the role that theory plays. From the French perspective, in the case of the sessions described in this chapter, not everything comes from the pre-service teachers’ practices, Aurélie and Julie bring something new, that comes from their research background: elements of theory, more or less transposed to be used in teacher education, as ‘expert didactical tools’, which is a different way of building a teacher education program, giving a more or less important role to the students’ practices.

For Alf, the only explicit use made of theoretical constructs to guide teacher talk, is the distinction between description and interpretation. This distinction is enforced if needed, by Alf, so that it is enacted in the discussion of video. Aurélie adapts tools from the Double Approach that student teachers use to inform discussion of video and their subsequent planning of activities for the classroom. For Julie, the work on video entails a deliberate use of the Double Approach and TDS which student teachers use in analysis and, subsequently in writing an assignment.

We can also see the more or less important role of mathematics in these sessions (the videotaped and training sessions) in terms of what is explicit. In the French perspective, the theories used give significant importance to mathematics and to the specificities of the mathematical objects that are studied, which leads to focus also on mathematics during the training sessions. In Alf's description, although the context is mathematics teaching, the way of working is potentially more general. From an enactivist perspective, if teachers are supported to make new (to them) distinctions about what they see on a video, then they are developing theory and hence discussion is about their own theorizing. What is made explicit is this theorizing, and not Alf's espoused theory. We referred, at the start of this article, to a workshop we co-ran at a conference, exemplifying our uses of video. It was apparent from discussion that the detail of the mathematics was more present in the talk during Julie and Aurélie's way of working than Alf's, although such detail would not be precluded from Alf's methods.

There is a difference in the amount of theory that we are trying to get student teachers to engage with and understand, by communicating elements of it, or not, more or less transposed, during sessions. While theory informs Alf's actions, the intention is not for teachers to become committed to enactivism – indeed, as suggested above, it would be unlikely that the term 'enactivism' is used at all during a training session. The intention is to support teachers to develop their own teaching and theorize their own practice. However, the observation/interpretation distinction is important for teachers to use. Aurélie teaches chosen elements of a theory, as tools to teach and analyze teaching practices. Julie teaches theories as tools to conduct some research as a detour to help the student teachers to reflect on their practice. These differences can be related to the goals of the training sessions in each example, and in particular in the last one, where research tools are among the content to be taught to help the student teachers achieve the writing of a research essay. But the fact that we, as educators, make choices about the extent to which theories are visible or not to the student teachers during training raises important issues. We summarize these similarities and differences in table 1.

	Alf	**Aurélie**	**Julie**
Espoused theory (informing session planning)	Enactivism	Activity Theory Double Approach TDS	Activity Theory
Enacted theory (used in training sessions)	Interpretation / observation distinction	Double Approach	TDS and Double Approach
Intended theory	Teachers' own	Tools to analyze	TDS to engage in

(for teachers to use/adopt)	theorizing to support new action	tasks inspired by Activity Theory and Double Approach	classroom research and also to be used as a tool to organize teaching
Role of mathematics in the sessions	Dependent on observations of teachers	A priori analysis of task on video	A priori analysis of task on video

Table 1: The roles of theories and mathematics in our work as educators.

Conclusion

We began with the questions: what guides the planning of video sessions? what guides the action of facilitators during sessions? and, what are the intentions, in terms of teacher learning? The table above summarizes what we found. One of the main similarities in the three examples presented is the way we all start from the actual needs of the student teachers, videos are then an artefact that allows these needs to arise: being close enough from what teachers do in the classroom but with sufficient distance to make them able to reflect on it, especially because they are not directly involved in the situation. They then have access to the complexity but without being responsible for dealing with it (cf. Gaudin and Chaliès 2012). The idea here is close to one developed by Robert when she suggests that there is something like a Zone of Proximal Development for teaching practices: she calls it the Professional ZPD (PZPD): training programmes would allow teachers to take advantage from them if they reach this PZPD. The idea of PZPD is in both perspectives as we all start from student teachers' actual practices and representations about teaching.

Questions remain for us, to research the effects of our choices for training, and especially the place and impact of theory, which is more or less present in our three examples (from making one theoretical distinction explicit (Alf), to making use of a transposition of theory (Aurélie) to inducting to the use of a theoretical framework (Julie)). As stated, our choices can be linked to the differences in our goals and audience when running the sessions described here, but we need to ask ourselves what kind of tools we want to offer to (student) teachers, and what such tools might occasion in relation to their professional development?

We do not consider that the ways we use videos are the only or best ways for teacher education. However, Gaudin and Chaliès (2012) suggest that there is not a lot of reflection in teacher training programmes about the various ways videos are used. The comparison of our practices in mathematics teacher education, around the use of video is helping us to understand our ways of working, as researchers as well as educators, which is in itself a significant step, but also leads us to clarify for ourselves the ways that the theories we use as researchers can influence our work as educators. We continue to share cultural differences and keep reflecting more about our role, as educators as well as researchers, and the way they both influence each other. The framework of espoused, enacted and intended theories helps us to reflect on our practices and become aware of choices that we may not have questioned, 'expanding the space of the possible' (Davis 2004, p.184) for us as teacher educators.

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THEORY OF DIDACTICAL SITUATIONS AS A TOOL TO UNDERSTAND AND DEVELOP MATHEMATICS TEACHING PRACTICES

Abstract. This article aims to discuss how the theory of didactical situations in mathematics (TDS[1]) can be used to answer research questions concerning regular teaching practices, production of resources for regular teaching, and teacher development. In the first part we focus on TDS and the way it may be a tool for the researcher to understand teaching practices and the way it may contribute to develop teaching practices, helping teachers identify questions useful for their practice. In the second part, we present analyses using TDS in two contexts in which researchers worked with teachers, making explicit or not the concepts they used. The third part approaches, from these two contexts, the way TDS may help the collaboration between researchers and teachers (or teacher educators), in research on teacher development, in particular in the case of producing resources helping teachers to prepare their class. The comparison of the two contexts informs on the specific contribution of TDS in understanding and developing mathematics teaching practices.

Keywords. Theory of didactical situations, mathematics teachers' practices, teachers' development, resources for mathematics teachers, multiplication, geometry.

Résumé. La théorie des situations didactiques comme outil pour comprendre et développer les pratiques professionnelles des enseignants en mathématiques. Le but de cet article est de discuter l'utilisation de la théorie des situations didactiques en mathématiques (TSD[2]) pour répondre à des questions de recherche concernant les pratiques ordinaires d'enseignement, la production de ressources pour l'enseignement ordinaire et le développement professionnel des enseignants. Nous centrons la première partie sur la manière dont la TSD peut être utilisée par le chercheur comme outil pour comprendre les pratiques des professeurs et comment elle peut contribuer au développement de ces pratiques en aidant les professeurs à identifier des questions utiles pour leur pratique. Dans la deuxième partie, nous présentons des analyses appuyées sur la TSD dans deux contextes dans lesquels les chercheurs ont travaillé avec des enseignants en utilisant la TSD, en explicitant ou non les concepts utilisés. La troisème partie aborde dans ces deux contextes la manière dont la TSD peut aider la collaboration entre chercheurs et enseignants (ou formateurs) dans les recherches sur le développement des pratiques enseignantes, notamment dans le cas de la production de ressources pour aider les enseignants à préparer la classe. La comparaison des deux contextes permet d'éclairer

[1] In the more recent texts, Brousseau specifies "in mathematics", speaking of the theory of didactical situations. Nevertheless, for short, we use TDS, which is more usual.
[2] Dans les textes récents, Brousseau spécifie « en mathématiques » quand il parle de la théorie des situations didactiques. Nous utilisons néanmoins l'abréviation courante TSD.

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l'apport de la TSD dans la compréhension et le développement des pratiques des enseignants en mathématiques.

Mots-clés. Théorie des situations didactiques, pratiques des enseignants de mathématiques, développement professionnel des enseignants, ressources pour les enseignants de mathématiques, multiplication, géométrie.

Introduction

Unlike the other articles of the special issue, this article refers mainly to one theoretical frame. Indeed, our purpose is to discuss how the theory of didactical situations in mathematics (TDS for short) can be used as a tool to understand and develop teachers' mathematics practices, that is to say all that teachers have to do in order to carry out the mathematics teaching in class in all its complexity: planning, designing, implementing, analyzing and validating teaching units.

TDS emerged in strong interaction with a methodology of didactical engineering and developed concepts and models helping conceptualize the evolution of mathematical knowledge (from informal mathematical knowledge to formal, mathematical knowledge), and identify the teacher's roles in different phases of this evolution. Later, some researchers (e.g. Hersant and Perrin-Glorian, 2005; Margolinas, Coulange and Bessot, 2005) used it to study regular teaching with a nearly naturalistic observation. We discuss here its relevance in the development of teaching practices and in research on this development in two different contexts, the first (case 1) on multiplication with Grade 3 students in Norway, and the second (case 2) on geometry with Grades 3 to 5 students in France.

The contexts differ not only by the mathematical content at stake but also by the purpose of the research in which each of them takes place, by the way researchers and teachers collaborate and by differences concerning teacher education and teacher recruitment in the two countries.

In case 1, the data come from a four-year intervention project in Norway, LaUDiM (Language Use and Development in the Mathematics Classroom) (Rønning and Strømskag, 2017) in a context of pre-service teacher education. The teacher training goal is to help teachers design, implement and analyze a teaching situation where there is an intention of teaching primary students some particular mathematical knowledge (here, multiplication) that could be perceived as meaningful for the students. The research goal was threefold: to design a teaching situation for third graders' first encounter of multiplication based on *a priori* (epistemological and didactical) analyses; to observe the situation implemented in class; and to validate the situation in terms of comparison of *a priori* and *a posteriori* analyses. One teacher, with certain awareness of some concepts of TDS is involved in direct collaboration with researchers who are at the same time teacher educators[3] for pre-service teachers, with whom they will use the results of the analyses.

In case 2, the data come from a research project in France gathering two researchers and five teacher advisors[4] from one educational district for primary schools (about 200 classes). The teacher training goal is to help teachers think about geometry teaching in Grades 3 to 5 (8-11 years old) and to produce reflection and resources to help practicing teachers in this teaching. The research goal was threefold: to elaborate an organization of the teaching of geometry coherent from 6 to 15 years; to work out with teachers advisors a resource for teachers coherent with our assumptions about geometry teaching; and to investigate the way teachers of primary school, not specialists of mathematics, may develop their geometry teaching using this resource. Twelve regular teachers are associated to the project: they implement in their class the situations first designed by the researchers and the teacher advisors. Neither the teachers nor the teacher advisors are aware of concepts of TDS, except perhaps the one of didactical variable.

In Section 1 we focus on the way TDS may be a tool for the researcher to understand teaching practices and to help teacher development. Section 2 presents the two contexts and the analyses with TDS. Section 3 approaches, from these two contexts, the way TDS may help the collaboration between researchers and teachers (or teacher educators), in research on teacher development. Then, we come back to the comparison of the two cases, in relation with the use of TDS to clarify how this theoretical frame can enlighten teaching practices and we draw out some questions for more investigation and articulation of TDS with other theoretical frames related to Vygotsky's work or Activity Theory.

[3] In Norway, most mathematics teacher educators are researchers. The ones participating in the reported research are all researchers. In schools where pre-service teachers have their field practice, there are mentors supervising pre-service teachers' practice in their own class. The mentors are teachers who contribute to teacher education, but they are not researchers, and they have no teaching duties at campus.

[4] In France, most mathematics teacher educators are not researchers. Teacher advisors are teachers of primary school partly or totally without a class. They contribute to teacher education (as mentors or for in-service teacher training). Three among those who participate in the research reported here have a class during two thirds of their work time and contribute to pre-service teacher education; the two others are advisors without a class of their own and contribute mainly to in-service teacher education.

1. How TDS may help the researcher to understand teaching practices?

The theory of didactical situations in mathematics provides scientific concepts that allow one, researcher or teacher, to understand or predict certain didactical phenomena in any situation in which there is an intention of teaching someone a particular piece of mathematical knowledge, whether they succeed in it or not. In regular teaching, TDS allows the analysis of an actual opportunity for a student to learn and gives means to provide such an opportunity. It was elaborated by Brousseau mainly during the 1970s and 1980s, with a methodology of didactical engineering (Brousseau, 2006). During the 1990s, Brousseau stressed the importance of the notion of *milieu* in the theory (Brousseau, 1997b, 2000) and he developed the notion of *didactical contract* (Brousseau, 1997a, 1997b) and insisted, on many occasions, on the fact that TDS is able to represent any situation in which there is an intention of teaching someone some specific mathematical knowledge. More recently (Brousseau, 2000; Perrin-Glorian, 2008) students' learning is seen in TDS as a combination of two processes (see Figure 1).

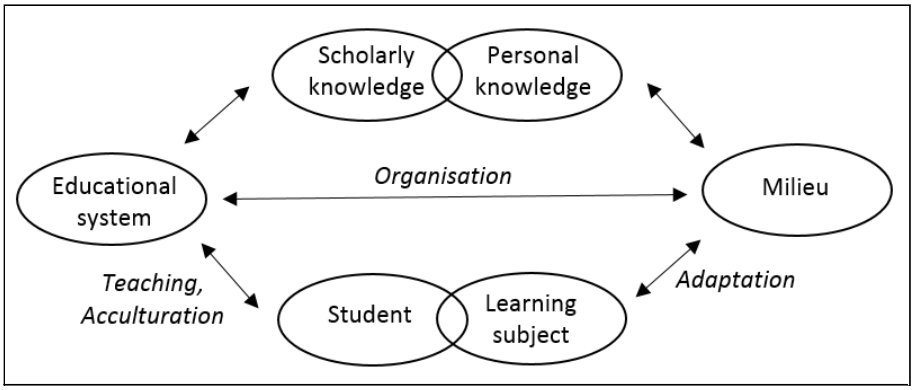

Figure 1. The didactical situation in TDS (translated from Brousseau 2000)

On the one hand, independent adaptation to a milieu (conceptualised through an *adidactical* situation) and on the other hand, acculturation into an educational system (through didactical situations and contract). In this model, the devolution ensures the conditions for adaptation, and the institutionalization ensures the conditions for acculturation. At the same time, TDS became to be used to study regular teaching with a methodology of class observations, with as few interventions of the teacher as possible, in the preparation of the class (Hersant and Perrin-Glorian, 2005; Margolinas, Coulange and Bessot, 2005). TDS was then a tool for the researcher to understand teaching practices by posing questions for observation and analysis of these practices. Answering these questions makes it possible to understand how knowledge can progress in class and who contributes to this progress.

1.1. A brief presentation of TDS

The methodological principle of TDS involves implementing target knowledge in a situation that preserves meaning; that is, the target knowledge appears in some sense as an optimal solution to the given problem. If the teacher succeeds in making a devolution of this problem, that is the problem is taken over by the students as their own, it provides a *purpose* for the students to engage in the situation, and the target knowledge appears as meaningful and useful (what it can be used *for*) because it solves the problem in the situation. The following diagram (Figure 2) recalls the main issues of TDS to represent a didactical situation, focusing on the teacher with the perspective of understanding how the students learn and how the teacher helps them learn some mathematical content with the help of this situation.

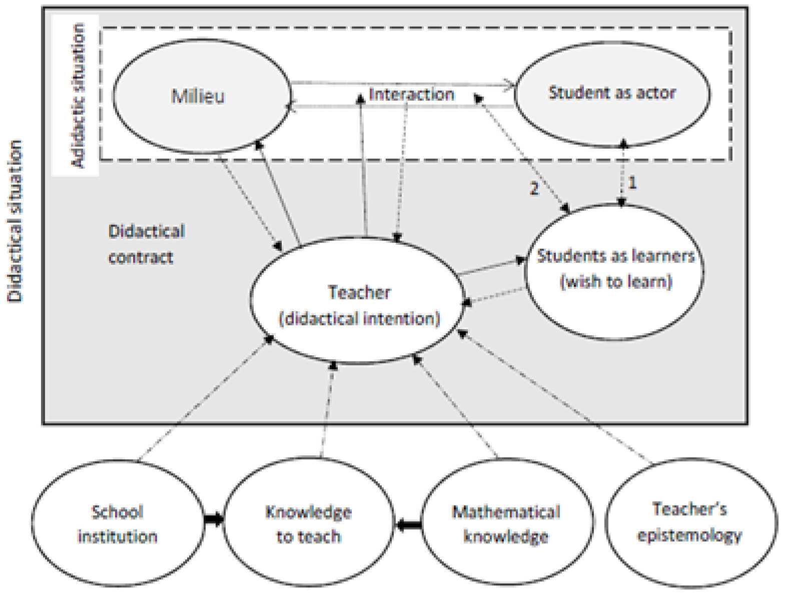

Figure 2: Interactions in a didactical situation (in the sense of TDS)

The didactical situation is represented by the grey rectangle. In this situation, there are two kinds of actors: the teacher with an intention to teach some mathematical object and the students; they are linked by the didactical contract. The white rectangle inside the grey one (with a dotted edge) represents the adidactic situation we can identify inside the didactical situation, as a way to learn a new piece of mathematical knowledge: a generic student, representing any student, acts on a milieu[5] that is able to give feedback on those actions. The adidactic situation may be considered as a game[6] defined by this milieu, rules to interact with it and an aim to reach: how to win. It is constructed or chosen by the teacher such that the knowledge to win will be the knowledge to be learnt and the prior knowledge of students may help them to play the game and interpret the feedback of the *milieu*.

These conditions can be expressed by three constraints on the *milieu* (Salin, 2002): (1) to provoke contradictions, difficulties for the students so that they have to adapt their knowledge; (2) to allow them to work autonomously; (3) to help them to learn some specific mathematical content (by learning to win the game). Thus, to learn, the student has to play the game (acting him/herself or in interaction with others), following the rules (and his/her own idea) and reflect on this action taking into account the feedback of the milieu, whether s/he won or lost.

Black arrows: The teacher interacts with the milieu (to construct it before the class or to modify it during the class), eventually with the relation between the actor and the milieu to change the game (with an aim of devolution[7] for instance) or on the students' knowledge (institutionalization for instance).

Dotted arrows: The teacher takes information on the relationship between the student and the (adidactic) milieu, on the students' knowledge (in act or expressed). S/he will be able to use this information to modify the milieu or to give some help to some students. The students as learners consider the action on the milieu (arrows 1 and 2) and reflect on it as a way to produce new knowledge. These actions may be indirect or implicit (not easy to observe).

Arrows with short lines and dots (at the bottom of Figure 2) represent constraints and objectives of the teacher, coming from the school institution or her/himself. Knowledge to teach is interpreted by the teacher from the curriculum and her/his

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[5] The *milieu* represents the elements of the material and intellectual reality with which the students interact when solving a task. These elements may comprise: material or symbolic tools provided (artefacts, informative texts, data, etc.); students' prior knowledge; other students; and arrangement of the classroom and rules for operating in the situation. For a very short presentation of the notion of milieu, see (Perrin-Glorian, 2008). Examples will be found in the second part of this article.

[6] *Game* is a metaphor – it has to be understood in a theoretical sense, as a model of the problem to be solved with related conditions.

[7] *Devolution* and *institutionalization* are two components of the game that the teacher has to play so that the student learns from the situation. In devolution, the teacher acts so that the student plays the game to win and not to please him/her. In institutionalization, the teacher's aim is to help the students recognise the knowledge gained in the game and to transform it into knowledge usable to solve other problems.

own mathematical knowledge. We do not represent constraints on students, though they exist, coming for instance from their parents or from other students.

1.2. TDS to analyze regular mathematics teaching

The description above, of a didactical situation in TDS, gives a researcher means to observe and analyze a regular teaching class session constructed by a teacher without the help of the researcher because it gives questions to pose, in order to define these elements from the class session observed: the adidactical part of a didactical situation in the sense of TDS (problem and milieu), as well as the didactical contract; to carry out the a priori analysis (i.e. analysis of what was possible) of this situation and to compare it with the a posteriori analysis (i.e. analysis of what actually happened). Of course, to answer these questions we need previous analyses involving the knowledge to teach (e.g. epistemological analysis and analysis of the curriculum) and the previous knowledge of the students. For example and details, see (Hersant and Perrin-Glorian, 2005)).

We can summarize some of these questions as follows :

1. What is the didactical intention of the teacher (the mathematics knowledge s/he wants the students to learn)?
2. Can we identify the objective milieu provided for the students? By objective milieu we mean here all the data independent of the teacher's interventions and from the students' knowledge afforded for the action or reflection of the students.
3. Is there something problematic for the students in this milieu? How may they solve this problem? What knowledge is at stake for the students? What use of knowledge is necessary to interact with the milieu and solve the problem? (Is it needed in order to: Progress in finding a solution to the problem? To formulate the solution in such a way that somebody else be able to solve the problem? To prove that this solution is a good way to solve the problem?)
4. What is the status of this knowledge for the students (quite new knowledge, knowledge in the course of learning, knowledge supposed known)? In this question we include the relations between knowledge at stake (new or old) and the didactical contract (what is expected from the teacher, from the students) in the domain.
5. What are the choices in the milieu that the teacher can change so that the knowledge at stake for the students changes (i.e. didactical variables)?

These questions may be posed with different scales: at the meso-scale of a sequence of classroom sessions or of one lesson; at a macro-scale of the insertion of this sequence (lesson) in the teaching of a mathematical domain; at the micro-scale of interactions between the teacher and the students. At the micro-level, the milieu evolves in the course of the lesson after some actions of the students or of the teacher. Thus we use the notion of ‘situation’ at different scales too. Usually, we begin with the meso-level of the class session including it in a more macro-level of analysis for the knowledge at stake and we consider the micro-level only on some parts where we find something happening in the perspective of the progression of knowledge for the students (progress or difficulty).

Answering these questions helps define a situation in the sense of TDS and provides an understanding of how the knowledge can progress in class. Moreover, to understand who contributes to this progress, we add some other questions concerning the relationships between what the students do and what the teacher does.

1. *Devolution*: what does the teacher do so that the problem becomes each student’s problem all along the session?
2. *Regulation*: what does the teacher do so that the students work really on the content at stake? How does s/he help them?
3. *Institutionalization*: what does the teacher do so that the knowledge used to solve the problem becomes a piece of knowledge to know and to use in other situations?

Clearly, answering these questions depends strongly on the knowledge to be learnt. We are particularly attentive to the different meanings likely to be attributed to the word “knowledge” even if we consider a specific item knowledge in mathematics. From the knowledge, as s/he knows it, and from its definition in the curriculum (knowledge to be taught), the teacher has to choose problems where this knowledge is useful (as knowledge to act in the problem) and to define what s/he wants the students to be able to do with this specific knowledge (knowledge to learn for the students), and then what they actually learnt and are able to do with it (knowledge actually learnt).

To specify some of these questions and answer them, it may be useful to connect TDS with other theoretical frames, on the one hand to analyze the knowledge at stake, on the other hand to analyze the teacher’s action as we shall see with the two examples in the next parts of this article.

1.3. How TDS may help teacher development?

From the point of view of the teacher development, the concepts of TDS may help identify questions useful for the teacher in three moments: in the preparation of the class; during the lesson; in analysing what happened.

The concepts of TDS, mainly those of milieu, didactical variable, action, formulation, validation, devolution, regulation, didactical contract, and institutionalization are quite important for the action of the teacher but it is not really necessary that s/he knows them in a theoretical way (as concepts of a theory) to be able to use them in practice. S/he can access these concepts to analyze and improve her/his practice for instance by a collaboration with a researcher in observations and analyses of situations in her/his classroom or in other classrooms. The teacher needs to relate these concepts to her/his concrete practice, what s/he usually does to prepare or analyze the lesson.

2. Using TDS to help teacher development in two different contexts

Our intention in this section is to present the use of the underlying concepts of TDS through two case studies in primary school: teaching of multiplication in Norway; and, teaching of geometry in France. In the two contexts research questions concern teacher education. TDS intervenes at two levels: 1) How can it help to enlight teachers’ practice and be useful in teacher training? 2) How does it contribute to the researchers’ methodology and analyses? In this section, we give first a description of the class sessions in the two contexts and then some examples of the use of TDS to analyze the teachers’ practices. Questions linked to this use, according to research questions and the different ways teachers and researchers interact in the two cases, will be discussed in Section 3.

2.1. Presentation of the data in the two contexts

The case of multiplication

This section is a description of a teaching sequence on multiplication in a Norwegian Grade 3 classroom (18 students, 8 years old). Records were gathered of: pre-analysis and planning (in a team of a class teacher and five university researchers, one of whom is one of the authors of this paper); two classroom sessions; and a reflective meeting (in the team) after the first session. The researchers who took part in the planning were teacher educators of mathematics. In the project team there were also two pedagogues (general educators) who were researchers, and another teacher. The observations were video-recorded, and the reflective meeting was audio-recorded. TDS was used implicitly to design the sequence on multiplication.

Pre-analysis and planning

The described teaching sequence was the students’ first encounter with multiplicative structures. In preparation for the pre-analysis, all in the team had read an article by Greer (1992), where he proposes that the most important types of situations where multiplication of integers is involved, are: equivalent groups (including rate); rectangular arrays; multiplicative comparison; and Cartesian products. The team agreed that the focus should be on situations with equivalent groups (i.e. of the same size) and rectangular arrays. Researchers suggested that the target knowledge was understanding situations with equivalent groups in terms of multiplication, and being able to write the result as a product, where for example 5·3 would be explained as "five threes", or "five times three", or "five groups with three (objects) in each group". The teacher said that a goal for her was that the students should *write an arithmetic problem*[8] that fitted with the task. "For instance, if Pauline[9] has five bags with three apples in each bag, how many apples does she have all together?" (quoting the teacher). Here the teacher would like students to write 3+3+3+3+3=15 (not 5+5+5=15) to say 5 sets of 3 apples gives 15 apples, which she would subsequently institutionalize as 5·3=15.

It is relevant to notice that in Norwegian schools, multiplication is usually introduced through situations with equivalent groups, where conventionally, 3·5 means 5+5+5, while 5·3 means 3+3+3+3+3 (i.e. a model of repeated addition). It was pointed out that multiplication as an operation is commutative, whereas situations to be modelled by the operation can be either commutative or non-commutative. Both types of situations were exemplified.

Based on the pre-analysis and planning, the teacher made a set of three tasks in the form of word problems. Classroom work on Tasks 1 and 2 (presented below) will be described and analyzed in this paper.[10]

Task 1. Class 3c plans to arrange a class party in the Café. The day before the party, they will bake muffins for the party at school. Pauline has to go the grocery store to buy eggs for the muffins. The recipe says there should be four eggs in one portion. The students have decided that they will bake twelve portions of muffins. How many eggs should Pauline buy?

Task 2. The muffins are placed on baking trays to be baked in the oven. On a tray there is space for five rows of muffins, and there is space for seven muffins in each row. How many muffins can be placed on one tray?

The teacher's didactical intention was: (1) equivalent groups put together should be interpreted in terms of multiplication as repeated addition; and (2) the problem in the task should be written as a product, where the first factor in the product signifies the number of groups (multiplier) and the second factor signifies the size of the groups (multiplicand). The situations in Tasks 1 and 2 are multiplicative

[8] 'Arithmetic problem' is translated from Norwegian 'regnestykke'; it means to write what calculations are needed to solve the problem.
[9] All names used in the paper are pseudonyms. Pauline is the teacher.
[10] Task 3 (on multiplicative comparison) was not reviewed in the analyzed sessions.

structures that consist of a simple direct proportion between two measure spaces, a structure referred to by Vergnaud (1983) as isomorphism of measures. The situations in the two tasks are however different in nature: The first situation (Task 1) is non-commutative, where one factor measures a number of iterations and the other measures a magnitude; this type of situation is understood as equivalent groups. The second situation (Task 2) is commutative, where the two ways of making iterations for counting are equivalently natural; this type of situation is understood as a rectangular array.

Implementation of the tasks

The teacher explained to the students that they would work in pairs on three tasks about an imagined class party at school. She said that she wanted them to draw on sheets how they would solve each task, and that, later, two pairs would be put together to explain how they had solved the tasks. After the students had made drawings and found the answers by counting, the teacher asked them to write “arithmetic problems” that showed the calculations. Later, she initiated a transition to the phase where two pairs explained their solution to either Task 2 or Task 3 (Task 1 was not part of this sharing). At the end of Session 1, the teacher gathered the students at the interactive white board to enable sharing of how they had solved Task 1. She invited them to the board (one at a time) to write and explain their methods. Below, two solutions to Task 1 are shown (Figures 3 and 4).

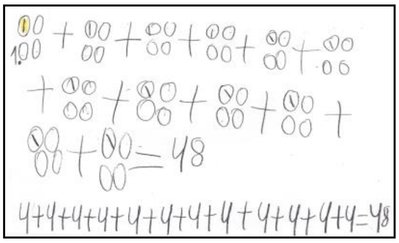


Figure 2. Lucas and Nadia’s solution

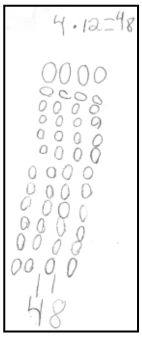


Figure 3. Filipa and George’s solution

The teacher concluded Session 1 by referring to the product 12·4 (which was the solution by only one pair, Filipa and George)[11], and said that they would look closer at 12·4 in the next session. That is, her goal for Session 2 was introducing product notation.

[11] At first George had written 4·12 (see Figure 4) , but changed it into 12·4 after some input from one of the researchers. George was the one who had written 12·4 on the board.

Right after Session 1, the team had a short meeting to reflect and possibly make adjustments for Session 2. The teacher referred to the situation with portions and eggs, and said that it was challenging to sum up at the end, the matter with the order of the factors in a product, and what the factors mean. She commented that it was not possible to swap the factors in Task 1, without losing the meaning of the situation. The team discussed how the situation might be reinterpreted[12]. The teacher described how Task 2 was different from Task 1: For muffins on a baking tray, rows and columns can be interchanged. She decided to use Task 1 to establish the *convention* of the order of the factors, and Task 2 to establish *commutativity*.

Two days after Session 1, the students were gathered at the board, where the teacher reminded them about Task 1, using the image in Figure 5.

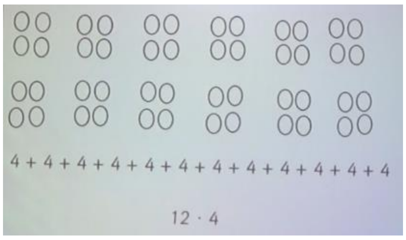


Figure 4. Multiplication as a model of an equivalent-groups situation (Task 1)

It addressed primarily those who had made distinct groupings in fours; there was no mention of the sequence 4, 8, 12,…, 48 (given by two pairs). The teacher's goal was that the students should learn to write 4+4+4+4+4+4+4+4+4+4+4+4 as 12·4. She asked why the repeated addition fitted with the situation. Nora answered that it was because it should be four eggs 12 times. In the discussion that followed, there was some back and forth about the interpretation of 12·4: "twelve times four", "twelve four times", "four twelve times". The teacher wrote on the board 12+12+12+12, and 4·12 under it. Then she asked about the distinction between 12·4 and 4·12. Tanya answered "If four is written first, then you shall take 12 four times, and if 12 is written first, then you shall take um… four 12 times." Some reinforcement of Tanya's interpretation followed.

The discussion continued as the teacher asked why writing 12·4 is "smarter" than writing 4+4+4+4+4+4+4+4+4+4+4+4. Responses suggested it is faster than writing all the fours. However, one student pointed out "We wrote it fast too, with plus." The teacher responded by supposing that they were making a thousand portions of muffins—what would this be? Students replied "a thousand fours", and

[12] 4·12 could be interpreted as 4 groups of 12, where the first group consists of the first egg from the 12 portions, the second group consists of the second egg from the 12 portions, and likewise for the third and fourth groups. This was not meant to be presented to the students.

that it is “a thousand times four”. But Lucas argued “Now you take a thousand four times”. He explained that he just “turned it” and took 1000 plus 1000 plus 1000 plus 1000, and got 4000. The teacher said that this was right, and that there are some smart ways of calculating this, without explaining this further at that time.

Afterwards the teacher turned to a review of Task 2, using Figure 6 as an illustration.

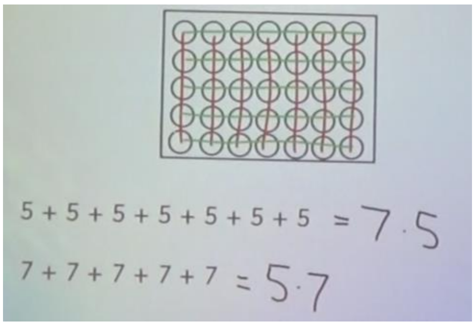


Figure 5. Review of Task 2

The horizontal and vertical lines and the products were inserted during discussion. She showed (what several pairs had pointed out in their solution) that they would get the same number of muffins, whether they counted 5+5+5+5+5+5+5=7·5 (vertical lines in Figure 6), or 7+7+7+7+7=5·7 (horizontal lines).

In the above, two different situations were aiming at multiplication as a model: first, an equivalent-groups situation (portions and eggs), then a rectangular-array situation (rows/columns and muffins). There was no discussion of any connection between the situations.

In Section 2.2 we present a TDS analysis of the sequence (done by the author involved in the project), the aim of which is to identify issues for development of teaching practices.

The case of geometry

A collaboration between researchers (two of the authors of this paper[13]) and teacher advisors was carried out for several years in order to reflect on geometry teaching in grades 3 to 5 (8-11 years old) in France and produce reflections and resources to help teachers in this teaching (Mangiante-Orsola and Perrin-Glorian, 2017). With this aim, we designed situations that were implemented first in the classes of the teacher advisors who had one, discussed, and then proposed to a group of about twelve teachers who implemented them in their classes. The sequences were

[13] They are (or were) at the same time teacher educators.

observed by the teacher advisors eventually accompanied by one researcher; some of them were video-recorded. The data were discussed first in the small group of researchers and advisors, then in the large group with all the teachers. Our approach to geometry rests on the the work of a research team in the North of France from 2000 to 2010 (Duval, 2005; Perrin-Glorian and Godin, 2014, 2017). A main construct is the vision of figures: the natural vision of figures is a vision of juxtaposed surfaces; in mathematics, geometrical figures are defined by relations linking lines and points so that you have to focus your gaze on these components of the figure instead of viewing the figure as a combination of surfaces, as comes naturally to the eyes.

The main idea to build problems for the students is to make them reproduce figures under certain conditions, what we called *"restoring a figure"*. To restore a figure, students have a model figure (always available) and a *beginning* of the figure to reproduce (small part already reproduced, the same size as the model or a different size). They may use tools (usual geometrical tools except tools for measurement[14], but also non conventional tools, such as templates) to take information from the model (for this, they are also allowed to trace on the model figure) or to draw the new figure. When they have achieved their reproduction, they may check it with the figure to be drawn, on tracing paper. Roughly speaking, the *milieu* is constituted by the model figure, the beginning of the reproduction, and the tools available. The game consists in reproducing the model with the tools. You win the game if the figure on the tracing paper exactly fits with your reproduction. The choices of the model, and the beginning and the tools are *didactical variables* because the knowledge necessary to achieve the figure strongly depends on them.

In this paper, we focus on one crucial situation of the sequence. The objective was to help teachers think in a different way about geometry teaching while proposing to them a situation for the class to exercise the way of looking at a figure and to work on the notions of alignment, line and point. The researchers, with help of the advisors, have designed this crucial situation in four phases. Each of them aims at restoring the same figure (Figure 7), but the beginning and the tools are different for each phase. As tools, students always have a non-graduated ruler and an eraser, but the available templates change. The choice of the beginning and of the available templates is of course a didactical variable on which the teacher can act. In the proposed situation, from one phase to the other, the degree of freedom in positioning the templates to draw the figure increases and the perception of alignments is proving more and more critical for the success of the expected

[14] For instance, the ruler is a non-graduated ruler. To move lengths, students may use a compass or other informal instruments allowing to compare lengths without measuring them, like a paper strip with a straight edge or parts of the figure (here templates).

tracings. Figures 8 to 11 present four phases in tables, each of which has two sides where the left side of the arrow shows what is given to students and the right side shows the solution to complete the figure.

At a first glance, on the figure to restore, we can see two or three triangles with a common side lying on a quadrilateral, but to complete the figure, the students will have to see also two large overlapping triangles and certain relationships between segments and points in the model figure: for instance, some sides of the triangles and some vertices of the outer quadrilateral are aligned on the diagonals of this quadrilateral.

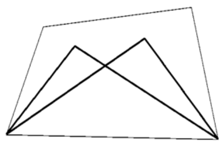
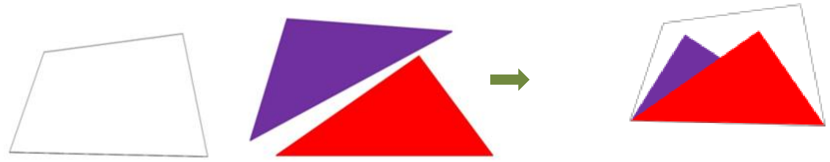


Figure 6

In Phase 1 (Figure 8), the beginning is the quadrilateral and the instruments are the (non graduated) ruler and two large triangles as templates. To restore the figure, the students must recognize them in the model (covering two triangles of the figure to reproduce) and place them on the beginning (the quadrilateral) to draw. The alignment of the sides of the two small triangles is given by the *milieu*: it is a consequence of the use of the templates since a side of the big triangle is the reunion of the two sides of the little triangles.



the beginning and the templates *how to restore*

Figure 7. Phase 1

In phase 2 (Figure 9), students have a “nibbled”[15] template T2, two corners of which are missing. To complete the figure (where the beginning includes a triangle T1), they have to know how to place the nibbled template: as it has no vertex, it is necessary to extend two sides of T1 with the ruler before placing T2 with two sides lying on the extension of those of T1. Thus the students have to use explicitly the alignments of the sides of T1 and T2.

[15] The template is “nibbled” to oblige students to extend the sides and not just make the summits coincide.

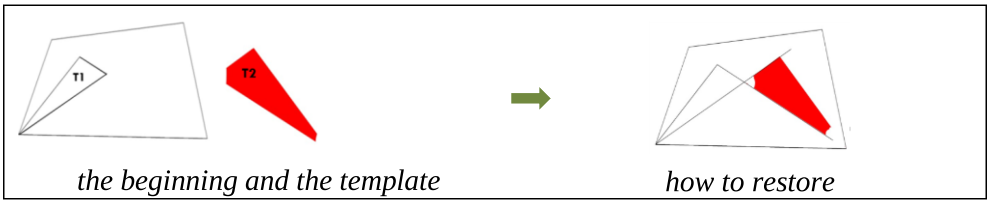


Figure 8. Phase 2

In phase 3 (Figure 10), the problem is to restore the quadrilateral from the two triangles and the beginning of two sides of the quadrilateral. There is no template, only a (non graduated) ruler. The sides of the quadrilateral and of the triangles have to be extended until the lines intersect to find the two missing vertices. It is necessary for the students to use "in action" the fact that we can get a point by intersection of lines.

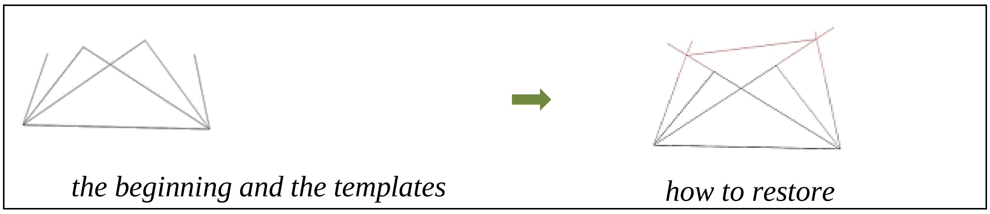


Figure 9. Phase 3

In Phase 4 (Figure 11), the beginning is the quadrilateral, the tools are two "nibbled" templates of T1 and T2. The problem requires students to see and to use the diagonals of the quadrilateral to place the templates before tracing. The templates were "nibbled" to entail the necessity to use the diagonals to place them. This phase may be seen as a reinvestment of the previous ones.

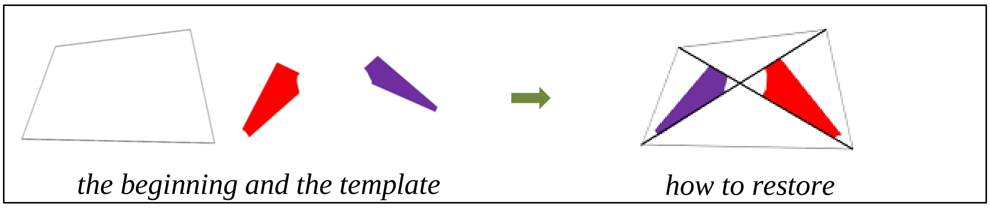


Figure 10. Phase 4

The concepts of TDS were used to elaborate the situation: the knowledge at stake was the notion of alignment (of points or segments) and intersection (of lines); the choice of didactical variables makes them necessary to solve the problems. We shall see in Section 2.2 how they can help the teacher to develop her/his practice.

2.2. Using TDS to develop teaching practices in the two contexts

In this part, we present our analyses using TDS concepts in relation with teaching practices and the way they can fit with certain professional reflections of teachers.

The case of multiplication

Devolution, informational jump and didactical contract

The possibility to draw ensured the devolution of the problem: this implicit model was available to all students. Task 1 did not explicitly need an arithmetic expression (eggs could be counted on the drawings). Nevertheless, the teacher aimed at such a representation for the students, and for that reason she changed the problem during the students' engagement with the task. She asked them to write an "arithmetic problem", referring by this question to the *didactical contract* associated with elementary word problems, which (for the students) involved translating them into "arithmetic problems". Another way to proceed might have been to make an *informational jump* by asking for, say, 150 portions.

Milieu, didactical variables

The objective situation (in Task 1) consisted of a person buying eggs for 12 portions of muffins, when each portion contains four eggs. The *material milieu* consisted of the eggs. The variables in the milieu that could be changed by the teacher are the numbers of portions and eggs. The knowledge supposed known was how to write an "arithmetic problem" representing a word problem.

Action, formulation, validation

Task 1 worked as an adidactical *situation of action* because the milieu was familiar enough for the students so that they could make an implicit model, in terms of drawings. After this, followed exchange of ideas in pairs, the purpose of which was sharing solutions and challenging each other when solutions were different. This did not work as intended. To a varying degree the students listened to each other, and there was no discussion when they had solved the task differently. Since there was no necessity to communicate to solve a task, this was not an adidactical *situation of formulation*. It would have been possible to have one by getting another student to use the explained method with a different number of portions, or with another recipe (with a different number of eggs). Another way to have a situation of formulation would have been to ask the students to agree on a method to apply it to a new question to come (before knowing the numbers).

Recapitulation of solutions at the end of Session 1 was focused on justification of students' methods, and hence it was a *situation of validation*. Because the necessity of validation came from the teacher, it was not an adidactical situation.

Institutionalization

During discussion of Task 1, a conflict occurred between the desired convention about the meaning of the factors in a product (understood as repeated addition), and the commutative property of multiplication as a mathematical operation. The teacher wanted the students to learn the convention that the first factor in a product signifies the number of equivalent groups and the second factor signifies the size of the groups. She used a large multiplier (1000) to motivate for the efficiency of product notation. But this created a conflict since it is easier to calculate $4 \cdot 1000$ than $1000 \cdot 4$ (two products with different senses but with the same reference).

The conflict created by the different commutative properties of the model and the situation in Task 1 was not resolved. The teacher continued on Task 2, where the commutativity of the situation (a rectangular array) was justified. After this, it would have been feasible to come back to Task 1 and say that it can be seen from Filipa and George's solution (Figure 4) that the eggs may be placed in a rectangular array and, as well, be calculated as 12+12+12+12. In this way, a connection might have been created between the two types of situations. This, further, would illuminate the relationship between the situation and the model—that the strength of numbers is to forget about the situation to do the calculations, then get back to the situation.

In summary, we used the concept of formulation to explain why the exchange in pairs was unsuccessful, and to suggest how this phase might be changed. Searching for a *purpose* of students' activity in adidactical situations may help the teacher to develop the (adidactical potential of the) milieu with which students interact. Further, the concept of institutionalization may help the teacher to understand the necessity of connecting students' productions (iconic representations of a non-commutative situation) with scholarly knowledge (commutativity of multiplication).

The case of geometry

We accompanied teachers in the preparation and implementation in class of the situation of geometry. Then we (researchers or advisors) conducted interviews with them. We use one of these interviews to illustrate the way in which some concepts of TDS may explain some difficulties encountered in class and be useful in the communication between teacher advisors and teachers and help the development of the observed teachers' practices.

Didactical contract, devolution

During the interview, one of the teacher advisors drew the attention of the teacher to the difficulties encountered during Phase 1 in the overlay of templates (she said:

"there was another problem, the superimposition, they [the students] refrained from superimposing… They prevent themselves from superimposing").

This problem is not due to a learning difficulty but to the constraints that the pupils give themselves. The concept of *didactical contract* can explain this difficulty and help the teacher to overcome it: the usual contract in geometry makes the students interpret the rules of the game that it is forbidden to overlap the templates. It was not cited by the advisor but it helps her advise the teacher: just allow students to overlap. Here the teacher must understand that this difficulty is not linked to a lack of mathematical knowledge from the student and that s/he needs clarify the rules of the game: tell the children that the templates may overlap; and that this clarification does not change the problem and the knowledge necessary to solve it: it relates to *devolution*. Taking support related to the concept of didactical contract, in this case, helps enrich the analysis of the teacher.

Milieu

In the designing of the situation, the evolution of the milieu (beginning and tools change among the phases) helps the students to change the way they look at the figure: from a vision of surfaces juxtaposed or overlapping to an analysis in terms of lines and points to construct (students' analysis is enriched through the tasks). During the interview, the teacher, in his commentaries about Phase 2, shows that he has understood that the changes in the milieu and the tasks asked of the students help them enrich their analysis or question their first analysis:

> "But me in the reflection of the kid, my interest precisely, it is that! We saw some things and when they get the templates, it exactly allows them to see what they have not seen! See, when B. tells me 'sir, the templates, they are not good', I answer 'ah yes, they are not good?!' So, once the kid knows where we could place the template, I can say 'well you see there is a triangle'."

During Phase 3, this teacher gives the students a string to help them locate the alignments, thus enriching the *milieu*. It is important for a teacher to understand that s/he can help students differently from intervening directly in the students' work. To provide the students with another instrument (templates, string) is a change in the milieu; it is another way to help the students without saying anything. TDS gives means to control the milieu in such a way that the students may learn, as much as possible, interacting with this milieu: TDS aims at characterizing situations (i.e. milieus) allowing students to learn some piece of knowledge by solving a problem, without significant help from the teacher.

Devolution, institutionalization

The teacher has to act so that the students solve the problem as their own, engaging their present knowledge and ready to acquire new knowledge. It is the *devolution* of the problem. In this case, the teacher, as well as some other teachers in our observations, chooses to begin with a phase of analysis of the figure. The difficulty in such a phase for the teacher is to let the students raise questions necessary to make precise the rules of the game, to postpone questions revealing in advance some crucial components of the figure or implying some construction. The interview shows that the observed teacher wants to give the students "good habits" and that, by "good habits", he refers to his own habits: "Myself, I begin like that: when I have a figure to reproduce, I look at it, I try to identify forms that I recognize, to find the links between them, to trace things that are not seen ...".

In fact, the proposed situation confronts the students with the resolution of a problem that makes the need for these "good habits" emerge from the students' reflections instead of being imposed or suggested from the beginning. These "habits" as well as some geometric knowledge linking the use of geometrical tools and geometrical concepts—such as "to set my ruler to draw a new line, I need two points or a segment already traced on the figure"—have to be formulated and pointed out for the students as something to know and use to construct geometrical figures. This corresponds to *institutionalization* in TDS. Clarifying the distinction between devolution and institutionalization helps the teacher develop her/his practice.

3. How TDS may help collaborations between researchers, teacher educators, and teachers in research on teacher development

In this part, we discuss how TDS intervene in the methodology of our researches, in particular we use the two contexts to examine how the collaboration between teachers, teacher educators and researchers might develop, focusing on the crucial question of links between the choice of the situation in relation to the knowledge at stake, devolution and institutionalization. The comparison of the two contexts informs on the specific contribution of TDS in understanding and developing mathematics teaching practices.

3.1. The case of multiplication

In the case of multiplication, questions in two arenas were identified: first, how to integrate a purpose—in the situation of formulation—so that the students would *need* the knowledge aimed at; second, how to solve a conflict—in institutionalization—between the situation to be modelled and a property of the mathematical model used to represent the situation. In collaboration between teachers and researchers, cases like the one analyzed here (with material from students' solutions and responses) may be used to discuss conditions and constraints (using TDS concepts) that enable or hinder students' opportunities to learn the knowledge at stake. This may then be used to modify and enrich the sequence for implementation in other classes.

TDS has been introduced to the LaUDiM project team by one of the researchers as a framework for investigating teaching and learning processes and for supporting didactical design in mathematics, where the particularity of the knowledge taught plays a significant role. After the project had been running for one year, the teacher (who has a Master's degree in mathematics education) was interviewed by one of the pedagogues about the significance of the project for her as a teacher of mathematics. The teacher expressed:

> "That is perhaps what I have learnt most from, I think, getting input from a somewhat different theory [TDS], a kind of model for teaching on the basis of which you can plan, which I had never heard of before".

From how TDS concepts have been used in the project, we understand that by *model for teaching* she means situations of action, formulation, validation, and institutionalization. Later, she said that defining the target knowledge was important: "[…] to choose exactly what [knowledge] we will work on is decisive for being able to design tasks that hit the goal". Further, the teacher commented on sequencing, that she had experienced how important it is to plan what (and why something) should come first in a teaching sequence. This was related to the importance of the pre-analysis, where the mathematical knowledge is analyzed by the team. The teacher claimed that being part of the project had clearly changed the way she thought about how a teacher should start a session on a mathematical topic. She is here seen to talk implicitly about *devolution*.

As part of data collection in the Norwegian project, researchers were asked to provide a written statement on potential impact of TDS on collaboration between researchers and teachers, regarding development of mathematics teaching. Two of the researchers focused on *institutionalization*, and this is what one of them wrote:

> "There is currently much focus on students' presentations of the work they have done in mathematics lessons. Very often this becomes show-and-tell, and some of the reason for this may be that teachers consider this part of the lesson mostly as a summary of what the students have been doing in the actual lesson. The concept of *institutionalization* may be useful to introduce to these teachers, so they can get a better understanding of what the teacher's role might (and should) be in this phase. [Institutionalization] to convey that the teacher has an important role in decontextualizing and helping students to put into words what kind of mathematics that has been worked on."

Even if knowledge of TDS concepts and models is shared among a group of researchers and teachers, there is a need for discussion of what the *target knowledge is* (or should be) in each case of designing a teaching sequence. This was pointed out directly by the teacher in the above extract from the interview, and indirectly by the researchers in their emphasis on the concept of institutionalization. Identifying the target knowledge requires pre-analysis and planning, preferably in a team of researcher(s) and teacher(s). This might not be realistic to carry out with teachers who are not part of a research project (i.e. if they have no reduction of teaching duties). However, analyzed teaching sequences (as the one on multiplication) can be adapted and implemented in other classes, for subsequent analysis. Even if TDS was pointed out as helpful by participants involved in the research reported here, more research is needed to know to what extent it is effective more broadly, for other teachers and researchers.

3.2. The case of geometry

In the case of geometry, the production of resources for regular teaching and teacher development proved to be a way to extend the collaboration between teachers, teacher advisors and researchers giving them a common aim. Our intention in this section is to present how this collaboration makes it possible to focus on the crucial question of links between the choice of the situation and the knowledge at stake, and explain how TDS concepts can be operationalized. We presented in Part 2 some examples showing how these concepts may help teachers interpret the choices made by the small group of researchers and advisors and develop their practices. Thus, on one side, notions arising from TDS can be mobilized by the teachers in action. On the other side, the concepts of TDS are explicitly present for the researchers at each stage of the process and help them interpret the teachers' questions and thus enrich their propositions.

The way this collaboration works is explained in Figure 12. In a first step (arrows n°1), researchers develop a situation based on research questions and hypotheses on the teaching and learning of geometry. TDS is the theoretical reference for the researchers exercising theoretical control on the analysis of knowledge, the definition of the situation, the milieu, the students' knowledge and the role of the teacher. But the theoretical control on the role of the teacher is to be tested and clarified especially in our case since we address all regular teachers. Therefore, in a second step (arrows n°2), this situation is discussed within the small group represented by the inner rectangle and a first document is written. At this stage, not everything can be anticipated by the small group who knows that difficulties will be brought to light during the implementation of the situation in class.

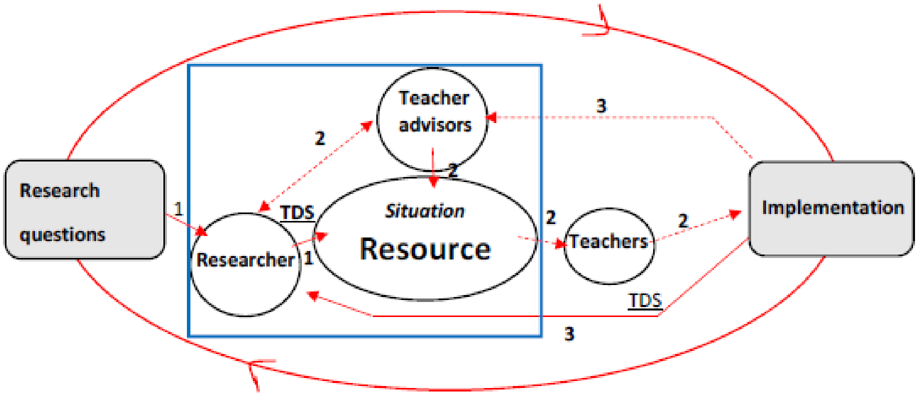


Figure 11. Use of TDS and collaboration between teachers, teacher advisors and researchers in case 2.

The situation is then presented to the teachers of the large group during a three-hour training session and a document (description of the situation and short guidelines for its implementation) is given to them. Implementation in class is accompanied by the advisors, observed (some of them with videos) by researchers or advisors, and followed by an interview. In a third step (arrow n°3), the small group analyzes the observations made and new questions emerge. Some of them give rise to pedagogical treatment but some of them require focusing on the crucial question of links between the choice of the situation and the knowledge at stake. These new questions enrich the work of the small group and the resource is modified. At each step, during the action itself or after the action, the researcher also takes information on the whole design process of the resource and analyzes how the different actors interact. The arrows are dashed when TDS is most often used implicitly (here, during the training) and the arrows are in solid lines when TDS is most often used explicitly (research). The outer arrows indicate the dialectic between research questions and observations.

We now give an example. In the initial document given to the teachers, there was no indication about the way to present the figure in class. When analyzing the class observations within the small group, we decided to take this issue into account and to give indications to the teachers (indeed, if the teacher develops a too precise analysis of the figure with the students, we see a risk of denaturalization of the situation). In a first time, the small group planned to draw up in the resource general advice essentially based on the question of devolution (explaining to teachers that students must understand what they have to do but should not be helped on how to do it before they try to reproduce the figure). Then, an advisor who is at the same time a teacher, while implementing the situation in her class, chose to write on the blackboard the first observations made by the students ("in this figure, I see ... a quadrilateral, two small triangles…”). Then, she hid this list and told the students that they would come back to it later. The other teachers observed did not write anything. This teacher advisor kept a track of the students' analysis in order to be able to complete it gradually with them. Giving a status to this writing, she initiated the process of institutionalization from the presentation of the figure.

This observation led the researchers to propose to the teachers to conduct a first analysis of the figure with students to complete it as the students’ research progress and return at the end. Thus, this observation helped the researchers to see how a more precise control of the role of the teacher could be implemented in the specific context of this situation. This example helped the explication in the small group of the way devolution and institutionalization are differently linked to the knowledge at stake and how this question might be taken into account in the resource for regular teachers. It is an example of the ways the collaboration between teachers, teacher educators and researchers is helpful: it helps researchers to see how concepts of TDS can be operationalized; it helps teachers or teacher educators working with researchers (in the small group) to explicitly approach the concepts; and it helps other teachers (using the resource) to gain some access to these concepts in the course of teaching.

3.3. Discussion

Comparison of the use of TDS in the two contexts

In both contexts, through the study of teaching, we have in perspective the study of the students’ learning and the teachers’ professional development—and our use of TDS is close one to the other. In both cases, the focus was on the design of the situation itself and its study. There are differences, however, in the objectives and research questions in the two contexts.

In the case of multiplication, the objective was to test the theoretical validity of the situation in relation to the essential elements about the target knowledge, whether the didactical intention was achieved or not, and why (i.e. to compare the *a priori* and the *a posteriori* analyses of the situation). Concepts of TDS have been made available to the teachers in order to give them tools for design and analysis of situations (arrows n°1 in Figure 13). As in the case of geometry, teachers implement situations (arrow n°2) and take part in the a posteriori analysis (arrows n°3). This explicit use of TDS concepts (arrows in solid lines) follows from the hypothesis that development of the teacher’s teaching practice is done through the implementation and analysis of a situation designed mainly by the teacher, based on *a priori* (epistemological and didactical) analyses done by the researchers and teacher in collaboration.

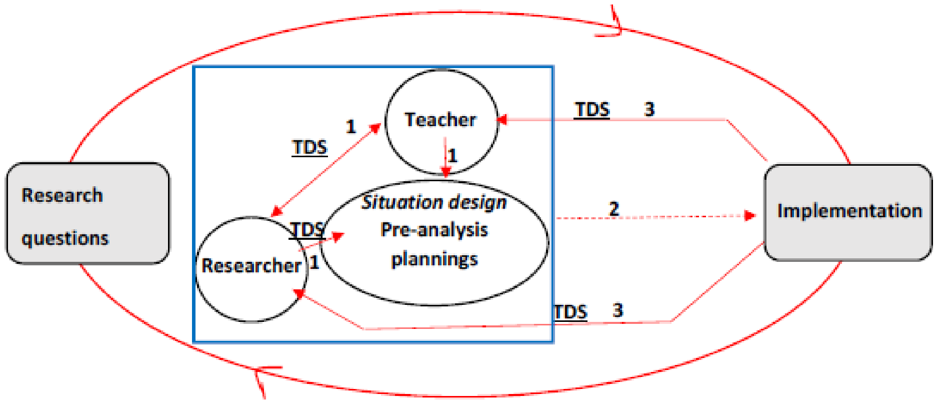


Figure 12. Use of TSD and collaboration between researchers and teachers in case 1

In the case of geometry too, the research questions comprise testing the theoretical validity of the situation in relation to the essential elements about the target knowledge, but they include also the study of the adaptability of this situation in regular education, taking into account the contributions of the teachers and the prospects of evolution of their practices. The objective was, after a first validation in the classes of teachers collaborating with researchers, to describe the situation in a resource with the perspective that regular teachers can use it without any direct interaction with the researchers or teachers collaborating with them. Therefore, the design process of the resource is at the center of the device (Figure 12) and not the situation itself as in Figure 13, and the TDS concepts were used only implicitly with the teachers (dotted arrows). This follows from the hypothesis that development of teachers' teaching practices is done through the implementation, analysis and adaptation of a situation first designed by the researchers.

Thus, the hypotheses and collaboration between teachers and researchers are different in the two cases. In the case of multiplication, the observations concern classes in which the teacher completed the design of the situation. In the case of geometry, except in one case, the observations concern classes in which the teacher did not take part in the design. The use of concepts of TDS is more explicit for the teacher in the case of multiplication than in the case of geometry. In the case of geometry, there is a big difference between the small group and the large group: in the small group, gradually, there is a certain familiarization, at least a use "in action" of the concepts of TDS, without expressing them, in the exchanges during the design of the situations and the analyses of class observations; in the large group the focus remains on decisions focused on practice.

Complementarity between TDS and other theoretical frames

In the two contexts, our research questions concern the teaching of a specific mathematical subject (multiplication or geometry) and the way to design situations acceptable by the teachers to improve their practice. The aim of a teaching situation designed according to TDS principles is students' development of meaningful, scholarly mathematical knowledge. Vygotsky's theory of concept formation is also about students' development of scholarly knowledge. Vygotsky (1934/1987) proposes that concept formation is the outcome of an interplay between spontaneous concepts and scientific concepts. However, as commented by Wertsch (1984), Vygotsky never specifies the *nature of instruction* of scientific concepts beyond general characteristics, in terms of teacher-student cooperation and assistance by the teacher, determined by the student's zone of proximal development (ZPD). On this point, TDS can be seen to complement Vygotskian theory in the way TDS provides tools for a fine-grained analysis of the progress of pieces of mathematical knowledge (from informal to formal mathematical knowledge), and what it takes for the teacher, in terms of designing a milieu and managing its evolution. For a discussion of compatibility of TDS and Vygotskian theory, see (Strømskag Måsøval, 2011, Chapter 2.7).

In the case of geometry, moreover, we wonder if an improving of teaching can result from taking ownership of a resource designed by researchers in collaboration with teachers and teacher advisors. We used TDS as a tool to design and analyze the implementation in classes of mathematics-teaching units, aiming at a generic and epistemic student's learning of some particular mathematical knowledge. The Double Approach (Robert and Rogalski, 2005) – rooted in Activity Theory (AT) – with its concept of proximities (cf. articles 2 and 3, in this volume) could be used to analyze the distance between what students do and know and the teacher's goals for the students, and how students' responses influence the actions and mediations of the teacher in trying to reduce this distance. However, there is an important difference in the nature of the didactical devices: whereas TDS aims at *adidactical* functioning of the knowledge, and its evolution, by designing and managing an appropriate milieu, the theory of proximities aims at *didactical* actions that the teacher can use to bridge the gap between students' existing knowledge and the new knowledge aimed at.

In comparison, TDS is a tool for the teacher and the researcher to determine *conditions* necessary for a situation to make a generic and epistemic student need the knowledge aimed at – here, the focus is on purpose and utility of the knowledge; the framework of proximities is a tool for the teacher to determine *actions in the course of teaching* or to prepare for this action, and for the researcher to analyze the teacher's actions, where the actual students' answers and questions have an impact on the teacher's decisions – here, the focus is on purpose and utility of the teacher's actions. We find the two theories complementary and potentially useful in combination to study mathematics teaching situations.

In the two research cases presented in the paper, we had questions about the knowledge itself, the means to make it accessible to students and the needs of a generic teacher. Of this reason we could not limit ourselves to the analysis of the teachers observed, and that is why we resorted to TDS.

Conclusion

We presented the use of TDS in a collaboration between researchers and teachers in two contexts in which research questions concern teacher education. We saw that TDS was helpful for researchers and teacher educators not only to design situations to learn some precise piece of knowledge but also to analyze what happens in class during the progress of the actual implementation of the situation and to identify questions useful to develop teachers' practices. In the two contexts, the analyses in terms of TDS were carried out by the researchers but, through some examples, we saw that they fit some professional questions from the teachers. These questions concern mainly their teaching goal, the way to organise some task for the students (related to the knowledge at stake) in such a way the students can know by themselves something about the pertinence of their answers, and the way to manage students' work. These questions correspond partly with the researcher's ones, but are more practical: The teacher must translate the concepts of TDS in terms of what s/he usually does to prepare or analyze her/his class.

The comparison of the two contexts raise a relevant question for the research: to what extent does the teacher need to know the concepts of TDS in a theoretical way (as concepts of a theory) to be able to use them in practice? Direct collaboration may help teachers develop their practices. However, it is neither realistic nor desirable to expect that all teachers can collaborate directly with researchers.

In the case of multiplication, TDS helps identifying questions concerning the milieu of the proposed situations and their adidactical potential, appropriate for the knowledge at stake. This in turn, makes it necessary to discuss the properties of the target knowledge. In the analyzed episode, a conflict occurred between a property of the target knowledge (the commutative property of multiplication) and one of the proposed situations aiming at multiplication as a model. Comparison of the *a priori* and *a posteriori* analyses of the sequence (which is an important part of TDS methodology) reveals shortcomings in the identification of the target knowledge (done in collaboration between the researchers and the teacher): the didactical intention (as expressed during planning) was related to the non-commutative situation (Task 1); the commutative situation (Task 2) was not part of the didactical intention.

In the case of the production of resources, TDS helps researchers and teacher educators to identify (through collaboration) questions concerning the way teachers interpret the design of situations using TDS, and how they enrich teaching from the implementation of such situations—particularly the choice of didactical variables. These new questions emerging from class observations lead to modification and enrichment of situations in the sense of TDS in such a way that regular teachers may more easily use them. Indeed, an important perspective is the question of the use of such a resource by teachers with no contact at all with research. For that, during the experimentation of the resource, it is necessary to understand the origin of the changes made to the proposed situation, and how the teachers take into account, throughout the implementation, the link between the situation and the target knowledge, how they react to what is happening in class to achieve the mathematical goals, and to the way knowledge can progress in class. To analyze teachers' point of view, from their professional practice, the Double Approach derived from Activity Theory is complementary to TDS, as commented above.

Even if design takes into account regular practices, important questions about the use of the concepts of TDS remain for researchers and teacher educators. First: how may this use be explained to other teachers using the resources, teachers who are not familiar with TDS concepts? Second: what teacher education should accompany such resources? The teachers need mathematical and didactical knowledge but, above all, they need to be able to put them into operation. That is why we, as researchers, consider that the concepts of TDS may remain implicit for the teacher, and focus our attention on how they operate (or not) in the teachers' practices. Nevertheless, we hypothesize that making them explicit is valuable for teacher educators accompanying the implementation in class of situations designed using TDS.

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FRENCH AND ENGLISH THEORETICAL PERSPECTIVES IN MATHEMATICS EDUCATION RESEARCH: AN OVERVIEW AND DISCUSSION OF KEY ISSUES

Abstract. In this article we focus on issues related to theories in mathematics education as used in both French and English settings. As the final article in this special issue, we review the earlier articles and focus on the key ideas and issues which stand out for us. As with the other articles, we seek to address both common and contrasting perspectives, drawing on the examples which illustrate uses of theory. We end by pointing to issues of validation, scale and policy which challenge both groups and look towards facing such challenges jointly.

Keywords. Comparison of theories, mathematics education research, uses of theories, challenges for the research

Résumé. Des recherches en didactique des mathématiques anglaises et françaises : bilan et mise en discussion des perspectives théoriques et des principales questions abordées. Dans ce numéro spécial nous nous sommes centrés sur différentes théories utilisées dans des recherches anglaises et françaises sur l'enseignement et l'apprentissage des mathématiques ainsi que sur les formations des enseignants. Ce dernier texte revient sur les articles précédents, en mettant en perspective les théories et les principales idées et questionnements développés dans les différents exemples abordés. Nous nous attachons à dégager ce qui est commun et ce qui diffère. Nous terminons en revenant sur les problèmes de validations, d'échelles des recherches et de politique, qui constituent des défis partagés par les chercheurs des deux pays, en réfléchissant à des moyens communs d'y faire face.

Mots-clés. Comparaison de théories, recherche en didactique des mathématiques, usages de théories, défis pour la recherche

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Introduction

In concluding this Special Issue, focusing on French and English theoretical perspectives in research in Mathematics Education, our aims are twofold:

- To pick up threads from Article 2, in which we presented key aspects of the two perspectives, and to synthesise similarities, complementarities and differences;
- To reflect on the collection of Articles in the special issue and the richness of theoretical ideas that they bring to the overall picture.

In our discussion we weave together the substance and issues in these two aims, using Articles 3-6 to provide the rich examples to discuss issues in theory. The discussion is presented partially as a 'conversation' between the two authors, indicated explicitly by a name at the beginning of a paragraph or section. Otherwise the text is joint. The individual paragraphs/sections express a divergence in perspectives whereas, in the joint paragraphs/sections, we are largely in agreement.

1. Dialogue on our joint enterprise in this special issue

Aline: As we saw in Article 2, the development of the two scientific domains, the French and the English (in mathematics education and didactics) was quite different – to summarise their theoretical development with a Bernstein formulation, we can evoke a 'horizontal' development for the English part and a 'vertical' one for the French part. That is to say, for instance, that there were multiplicities of theories adopted in the research in the English case, in common with the other Education Sciences, and only few main ones in the French case, with a more indirect relation with the Education Sciences. In other words, the English development was built *inside* existing theories in Education Sciences, while the French one was built against (in contrast with?) the existing theories in the Education Sciences, and reinforcing the differences. Actually, the *a priori* stage of mathematical analysis (before research into didactics and pedagogy) contributes to the French singular approach and it still characterises almost all of the French research. Such mathematical analyses are less present in beginning stages of the English research, and may even be implicit in it.

Barbara: The idea of the horizontal and vertical makes sense. What seems an important difference is that French researchers throughout France are using the same theories, albeit in response to their own research questions and directions of study. In the English context, different researchers use different theoretical perspectives in relation to their research questions and directions of study and may not even agree on the use of certain theories in specific contexts. This makes for a complex theoretical debate within the English research community. With regard to the French focus on mathematics, conducting a mathematical analysis before embarking on other aspects in a study, I recognise a) that we generally do not do this, but (b) this does not mean that the nature of the mathematics is unimportant. As you have suggested, it is largely left implicit.

Aline: Here we could add something about the researchers in both cases: another difference may arise from the fact that most teacher educators in the UK have been teachers themselves. This is not the case in France, where, at the beginning of the development of didactics, the researchers were mostly university teachers teaching mathematics to undergraduate students with some of them teaching mathematics to in-service teachers. Perhaps this points to differences in the institutional positions and expected work, and maybe the French researchers had more opportunity to build theories, instead of using existing ones, not directly applicable to teacher training for instance. They have in mind the elaboration of a (missing) general theory for mathematics learning; they did not face directly the needs of applications to schools or for teachers.

Barbara: While it is true in the UK that most teacher educators have been (school-) teachers, there are UK mathematics educators who have not been. This raises the question of who are the people doing research in mathematics education in the UK. The answer is that, many are teacher educators who teach prospective mathematics teachers in the university, but not all. Teacher education is mostly a one year course leading to a PGCE – Post Graduate Certificate of Education. This does not include subject studies. So teacher educators do not usually teach mathematics, per se. Where research into mathematics learning and teaching in higher education is concerned, most researchers have not been teachers in schools, but they are likely to have become university teachers, teaching mathematics to undergraduate students. I certainly think that mathematics educators in the UK are more concerned with analysing teaching and learning, using theories which seem to make sense for their analysis, rather than working explicitly on the critique, development and unification of new theories.

Aline: To go back to the differences, we see that the presentation of the results in Section 3 of Article 2 is different: the English section is organised around five themes relating to what has been learned through research whereas the French section is organised according to what may be enlightened by each theory. But it is interesting to notice here that the English themes are also addressed in France. The main common one would be *meanings in mathematics*, as almost all the French research is concerned with many aspects of mathematical meaning. Equity studies in mathematics exist in France, but they are not as important in France as in the UK. *University studies* were begun from early years (1981) but have not represented an isolated theme until recently, with the development of transition themes, and particularly transition from high school to university. And, in France, there is not really a *policy* theme, apart from the recent studies about the international evaluations. So, behind a real diversity, there is much work on the same issues in the two countries.

Although our theoretical perspectives may seem quite different, we are concerned to study and know more about the same concerns and issues. We address now some particular examples of this.

2. Differing conceptions and practices using the same theoretical perspective – Activity Theory

2.1 Different conceptions of Activity Theory

Article 3 points out differences between researchers in both countries who use Activity Theory (AT): in brief, there are differences both in interpreting and using AT.

Barbara: The English interpretation is broader than the French one, involving a range of general categories in analysis of activity: for example, Engeström's categories of 'division of labour', 'rules' and 'communities', or Roth and Radford's 'contradictions', or Leont'ev's 'motives', 'actions' and 'goals'. These concepts and constructs have been in the public domain for many years, relating back to the origins of Activity Theory; they are well known and discussed. It has made sense to apply them, sometimes with adaptation, to issues in mathematics education, particularly in analyses of classroom interactions and the activity of teachers and students working with mathematics. Teachers and students can be seen as operating within quite different activity systems. Concepts of mediation, goal-oriented action and use of tools and signs allow analysis of complex educational settings, the tensions and contradictions which arise in practice, and the wider contexts and cultures that influence classroom activity.

Aline : In contrast, the French researchers are much more focused on the activity constituted by teachers and students as they work together on mathematics. They have introduced the 'double, ergonomic and didactic approach' to analyse the complexity of activity in classroom interactions. Taking into account this complexity leads them to broaden their analysis. They use cognitive and mediative components to describe the teacher's choices about content and classroom' implementation (where the activities are more visible). But they complete these descriptions including informations on the personal component, to take account of the teacher experience and knowledge, and on the institutional and social components about the corresponding adaptations of the teacher. As part of this same approach, the French researchers have operationalised Vygotsky's model of Zone of Proximal Development (ZPD) for mathematics, looking to bridge the gap between the teacher's *a priori* mathematical expectations and students' mathematical achievements expressed through the concept of 'proximities' in a mathematics lesson.

2.2 An example of different perspectives taken to study what occurs in a classroom

In Article 3, the common issue is a micro-level analysis of a real implementation of teaching, with studies of data from recordings, video (French) or audio (UK) made in the classes. The research aim is to understand what seems to occur when students hear and interpret the teacher's words and what may motivate the improvisations and comments of the teacher. In what ways do the students benefit, or not, from the discourse? Does their learning improve? How are the teachers' responses moderated towards her/his perceptions of student understanding of the mathematical concepts discussed? The two studies enlighten complementary aspects of these issues: from the English perspective, analysis of the contradictions that are inherent in the episode help to reveal and address issues in communication and their impact on classroom discourse. From the French perspective, analysis reveals the comments some teachers develop to draw students nearer to the teacher's mathematical goals. In the French perspective, the cognitive aspects are mostly taken into account, whereas in the English perspective, more global aspects of the situation become part of the analysis.

2.3 An example showing a possible use of AT as a lens to study a situation of classroom practice with technology

Article 4 discusses the use of technology in the classroom and theory related to this use. Although the theories to be applied in the two cases are different, the teaching-learning outcomes have many factors of similarity. Indeed the authors write:

> 'In some sense, our methods look at two sides of the same coin, teachers' classroom practices with digital technology, from our two different cultural perspectives'

In Article 4, the French case talks about opening up mathematics to student exploration in which the teacher is often in improvisation mode. The teacher has prepared the task carefully with expectations of what students can achieve by using the software as he has set it up. This seems like the equivalent of an *a priori* analysis of the mathematics. However, the student cognitive activity cannot be completely predicted – the expectations are punctured with challenges (tensions/disturbances) arising from the use of software leading to tensions in the planned cognitive route (in that students' activity does not fit with teacher's desired outcomes).

From an English perspective, it seems possible to theorise this through the third generation activity theory triangle of Engeström in which we see tensions between the tools used (the DGE[1] and open task) and the rules and division of labour. The rules of *a priori* analysis leading to tight control of pupil cognitive outcomes are challenged by the open nature of the task and by the DGE imposing its own dynamic in the activity; the expected division of labour, with teacher activity and student activity separate and well defined, is challenged by the need for teacher and students together to evaluate the reasoning deriving from unexpected DGE outcomes.

In terms of Vygotsky's ZPD, as used in French theory, we see the DGE as an important mediator with both students and teacher gaining new insights through their (joint) addressing of the unexpected geometrical outcomes.

3. Problems arising when researchers have similar aims but different theories

In Article 5, the three authors talk about theory, by means of examples from their practice of working with teachers. As they explain, Coles works from an Enactivist perspective whereas the two French authors, Chesnais and Horoks, use the Double Approach (DA) and seem influenced by Theory of Didactical Situations (TDS). They wanted to choose a video of classroom learning and teaching which all three could use to demonstrate differences in their practices and theoretical perspectives. All three of them believed that choosing a video which all could use was an important task in making clear to each other the very different objectives for each use of the video chosen. However, it was very interesting that they could not find one video that would work for all three.

<u>Barbara</u>: In the proposed UK video, the task was too open for the French – it did not lend itself to clear *a priori* analysis in order to articulate precisely the mathematics that students were supposed to come to know. Whereas a narrowing of the task would be more appropriate for this. I conjecture that, for Coles perhaps, the proposed French task was too narrow for his purposes.

<u>Barbara</u>: From my UK, very practical perspective, in designing classroom activity, we want to present a base for mathematical inquiry in which students can be challenged to think themselves into the problem posed which can be rather broad in scope. The mathematics is thus not narrowly defined. This requires a lot of the teacher since she has to deal with many possible ways in which the students interpret the situation – she has to respond to these in ways helpful to the students (supportive and challenging in varying degrees).

<u>Aline</u>: The pre-analysis of the mathematical task may help the teacher act in the classroom and it allows the researcher and the educator to have clear expectations of the mathematics to be learnt by students. It supports the complexity of teaching decisions and allows the teacher to keep the mathematical discussion focused. In this French perspective, if the task is an introductory one, its *a priori* analysis facilitates the teacher's telling of the knowledge at stake. The teacher is expected then to generalize it apart from the students' use of a contextualized form of the required knowledge on the problem. If it is another task, its *a priori* analysis facilitates the teacher's understanding of the students' precise work with the

required knowledge. It lets her modify the task if it does not fit well enough with her expectations. If it does, it lets her choose her interventions during the students' work, thanks to a deep interpretation and some adapted improvisation, taking into account what occurs, maybe detecting implicit factors as described in Article 3. Actually the teachers are not expected to analyse each task in such a way but it is important to be able to share some of them on important tasks and to enrich their awareness of the particular students' work.

Aline: According to the mathematical content, the tasks choices and their implementation are basics (essential) to let students actually experience in a precise context some of the knowledge to be achieved. Far from a reduction of the students'activity, it may be seen as a whole development process but it requires a lot of the teacher since she has to pick up in the students'work what may worthwhile generalisations or applications.

3.1. Different theoretial perspectives and what they can reveal

The differences in choice and use of theories in Article 5 allow us to reflect further on theory and its use in classroom settings.

Theory of enactivism (Barbara)

Comparing the practice and theory of the three authors of article 5, I think that the three researchers are trying to achieve different outcomes. The English researcher is using an enactivist frame to draw teachers into being enactivist practitioners through his work with them on video. For me, the use of enactivism here can be seen as follows:

Enactivism is sometimes described as 'a path laid while walking'. Students are presented with a very open task. It challenges them to engage and explore possibilities. As they engage, they 'walk'. As they walk they think about the task and start to make some sense mathematically. There may be several different paths for different students. If students discuss and collaborate, these paths can merge or cross, so that the challenge gets modified and the path becomes shared to an extent. We can see the teacher's role as a listener and guide, asking suitable questions, prompting and probing to support and/or challenge students (cf Jaworski's Teaching Triad). Students have to get used to the fact that there is not just one way or indeed one right answer – this is part of enactivism: becoming aware that there are many paths and that it is their own actions that can help them to find a path in fruitful directions. The teacher supports this in different ways. It is very challenging being a teacher within this theoretical frame. Coles uses the enactivist frame to challenge his teachers. They have to see the video and avoid putting their own interpretation on what they see. They cannot 'see' into the minds of teacher or students in the video. They have to limit their responses to the video in terms of what they can see literally. This forces them to be more aware of the choices a teacher faces and from which she chooses her responses to the students. The teachers observing start to be aware of this multiplicity of choices and perhaps become more aware of the complexity for the teacher and the responses that could be made. Which of these choices best supports or challenges the students is then open for discussion in the group. Their reflections on the video enable them to address their own practice and the choices that they make themselves, enabling them to make more informed, not 'better' choices since it is hard to define what is better. This can be a focus of discussion through which they again develop awareness. These layers of developing awareness form the 'path laid while walking' for these teachers.

Theory of the Double Approach (Aline)

The French researchers in Article 5 are trying to pass on some of the *a priori* analysis tools, built by reasearchers in mathematics education, that seem relevant to reflect on a mathematics session, before it and after it.

The DA does not inform directly students 'activity'. It informs teachers' activity by the way of their relation with students' activity. So these analyses may help to understand the students' activity by a better understanding of the teacher's choices. The teacher training involves a specific approach, based on DA for what concerns practices and on TA and TDS for what concerns the learning, according to the students' grades. For the training, the common idea is to let teachers appropriate some of the tools used for didactical analysis, taking into account that their practices are complex and are not only guided by the students' learning. There are different means to get it but they may be not 'direct'.

For us, it is more important to differentiate between the teacher's point of view and the researcher's one regarding the importance of an *a priori* mathematical analysis inside a whole conceptualising process and as a reference to study videos. A pre-analysis of the mathematical task allows the researcher and helps the educator to have clear expectation of the place of the task in the whole process leading to the mathematics to be achieved. There may be differences between tasks: some tasks facilitate the students' expected work before the teacher telling, some tasks are useful to reinforce the general presented knowledge by exercices, some tasks contribute to have available knowledge, as detailed above.

Some researchers prepare lessons 'ensuring' the knowledge to be achieved, particularly the TDS's researchers for primary level. For instance, they elaborate introductory tasks with a high potential of students' learning, leading to institutionalising the knowledge, provided the teacher's implementation fits the expected goal during the whole process. Using these tasks presupposes the way the teacher is going to intervene: the deal is to let students work by themselves on the tasks and then to make a bridge (to establish proximities) between what the students know or have done and the general knowledge to be achieved (cf. Article 6 – discussed below). It may be by displaying links (relations) between the contextualized knowledge, as used by students in exercises, and the general knowledge to be achieved, as told by the teacher in the specific moments of teacher telling. It may be before these moments or after, depending on the contents.

But not many teachers use such tasks, not only because it is difficult to implement but also as there are not such studies for each content, particularly in the secondary level, and it is difficult for the indivdual teacher to prepare such a corresponding scenario.

It is then useful to understand what occurs in ordinary classes, using AT theory (as exposed in Article 2), as a reference for analysing students' learning completed by the DA (Article 2) as a reference for analysing teaching practices. In these classes, some researchers study for instance the opportunities to get the students nearer the knowledge to be achieved, whatever the used tasks, and detect the missed occasions, trying to find reasons for them. These reasons may be tied to mathematics, for instance to the choice of the tasks, and/or to their implementations, for instance a student's difficulty may be unrecognised. It may lead to try to develop a kind of teachers' vigilance (care?) on some precise and problematic points, involving the students learning, tied to the tasks, the lessons and what occurs during the class. But these reasons may be also tied to the complexity of what the teacher has to do - managing heterogeneous students, with not enough time, and submitted to various personal, social and institutional constraints. The DA informs the researcher on what has to be taken into account to understand teacher's activity including this complexity.

Finally, the question on teachers' training involves the complexity of practices and some results of research based on the DA. For instance the stability of teachers' implementations, teachers' practice and the importation of the ZPD model for the practices' devlopment, leads us to take into account the teachers' implementations and to lean on the previous teachers' expertise to enrich it.

4. The same theoretical perspective but different situations

In Article 6, we find two different perspectives of using TDS to analyse teaching settings. The first comes from Norway, working within the English domain; the second is from France. In the first case, we see a researcher studying teaching practices in a teacher education setting in which the student teachers are learning mathematics in activity prepared by their mathematics teacher. Here the focus is on the way TDS may be a tool for the researcher to understand teaching practices and to help teacher development. In the example given, although no explicit *a priori* analysis of the mathematical content has been made, there is an understanding of what this mathematics consists of and of what is expected from the adidactical setting in which activity is rooted. The second case presents a collaboration between teachers, teacher educators and researchers giving them a common aim in designing resources for the teaching of geometry. The analyses with TDS show the way TDS may help the collaboration between researchers-teacher educators and teachers, in research on teacher development.

Barbara: I see in the Article an elaboration of TDS, explaining different aspects of the theory. Key concepts of *milieu*, both didactical and adidactical, and stages of *devolution* and *institutionalisation* were introduced. Although the contexts of the two examples were very different, it is possible to see how this theoretical perspective served an analysis of each of these settings. In this respect, having the different settings and seeing the same concepts related to each of the settings helped to make clear the main elements of TDS. In some ways, I see a value in the key concepts mentioned above for any setting in which a teacher wishes her students to learn specific mathematical knowledge. A difficulty arises when the tasks (didactical or adidactical) are predesigned by researchers or teacher educators with the expectation that a teacher can fulfil the designed teaching approach without having been a part of the original design/planning. In the first example, we see that the teacher is part of the design process, and in the second example, there is collaboration between teachers, teacher educators and researchers in the design. Thus this difficulty is avoided.

Aline: The framework TDS is particularly concerned with the design of learning situations, and also to analyse what happens in class during the progress of the actual implementation of the situation, in reference to the design, and, more recently, to identify questions useful to develop teachers' practices. But the main aim remains to study the cognitive potential of a given situation, that is the study of what the students may learn according to the contents' choices, mainly the tasks and their implementation, often to introduce a new notion. In both cases of Article 6, as the authors say, 'the focus was on the design of the situation itself and its study'. There are differences, however, in the objectives and research questions in the two contexts, which are training contexts. In each case, the teachers have to learn to use the chosen tasks, adapting them to their students but trying not to lose their potential. The use of concepts of TDS is more explicit for the teacher in the case of multiplication (first author) than in the case of geometry (others authors). In the case of geometry, there is a big difference between the small group (with researchers, educators and teachers) and the large group (of teachers): in the small group, gradually, there is a certain familiarisation, at least a use 'in action' of the concepts of TDS, without expressing them, in the exchanges during the design of the situations and the analyses of class observations; in the large group the focus remains on decisions focused on practice.

5. The use and influences of research results in the two domains

Although the development of the French and English scientific domains have taken different forms (Article 2) with the use different perspectives (cf. Articles 3, 4 & 5), or with the same perspective (Article 6), researchers in both contexts nevertheless tackle some common issues, and the kinds of outcomes we get are not so very different on a gobal scale (the two faces of the same coin, at a more or less general level). Both sets of researchers are concerned to develop the learning and teaching of mathematics, both in theory and in practice. What can be learnt from the joint enterprise, in these Articles, enriches the overall perspectives and emphasises the joint enterprise. However, these findings are not taken into account by the 'decision makers', either in France, or in the UK.

What is interesting is that this occurs in both countries, independently of the development of the domains. It is not (only) the proliferation of the theories that may explain this unwillingness of the institution and policy-makers to seek the advice of researchers or to pay attention to research findings. It is an important result of this common work: there have to be new ways to have some influence; perhaps there need to be international common results to make perspectives more visible to leaders in educational policy. This might, for example, follow the European synthesis of didactical results with the 'solid findings' such as those published by the European Mathematical Society ("Solid Findings in Mathematics Education", EMS Newsletter, September 2011).

However, in the domains in this special issue, the production of evidence (validation?) is not simple since most of the studies are qualitative ones and small scale. More generally, it is clear that there are no obvious means for 'assessing' such studies with quantitative evidence – as is confirmed in the Articles 3-6. Even though some international assessments inform on the state of students' knowledge, it is not directly possible for these to be turned into teaching changes. The relations between quantitative assessments and individual practices are not simple, there is often a lack of adjustment of the exercises to the corresponding teaching, and learning is a long process not reducible to a state that can be measured with a snapshot. It is well-known and concerns almost all Human and Social Sciences but it plays a role in the institutional reluctance. In these sciences, 'robustness' does not come from assessments.

However, in the case of teacher education, where the teacher educators are also researchers in mathematics education, there is growth of awareness of the outcomes and issues from research as researchers communicate both within and across national boundaries. The communication that takes place at national and international research conferences feeds into the professional knowledge base from where it is distilled by teacher educators in preparation for their work with teachers. It is possible to see this research knowledge permeating thinking and practice through teacher education opportunities. Teaching, as it can be seen in schools today, is influenced not only by policy decisions but also by the teaching of teacher educators, informed by their research knowledge. The 'solid findings' mentioned above can be an important contributor to this knowledge and we need to build this into our research and professional practice.

6. Relation of results of research to the contexts and focuses of the particular studies

We see another factor which weakens research results. The fact is that our results (outcomes) depend mostly on the contexts of the studies and their possible uses depend on situated learning, in a country or between countries. It is very clear in Article 6, where the adopted theory is exactly the same (TDS) but the institutional contexts and focuses of research are different – in one case the researcher studies real pre-service teachers training, and in the other case the researcher studies resources for in-service teachers' training. In the first case analysis reveals differences in the conceptualisation of tasks by the teacher and the mathematical activity of the students in working on these tasks. These have implications for the design of tasks more generally and for the work of the local practitioners more particularly. In contrast in the second case the issue is to find a resource available for many teachers. It leads to a first common analysis of the mathematics involved but then to a different analysis of the discussion on the variables and the way of presenting the research. This great dependence on the contexts may explain some lack of our influence, tied to the complexity of the way of adapting results to many factors. Programmes differ from one country to another, cultural habits too and even inside a country, teachers may develop some different ways of teaching to be comfortable in their craft; students are very different according to their family for instance, but not only. So one result has to be presented in the context it was obtained, with its limits and without obvious more general impact.

Another reason for a collective lack of influence is that local, qualitative analyses are more frequent than global ones. The shift from local studies to their global interpretation or use is difficult, precisely because of differences in context to the time they take and to the large amount of data to be gathered. Then it is hard to take into account all the variable parameters involved.

It is then difficult to infer global results from our local analysis for the students' learning. Researchers have only hypotheses on the quality of the scenarios. They suggest that the recurrence of teaching ways is an important factor for students' learning. An example can be seen in Article 3: in the French classroom episode, an *a priori* analysis of the mathematics in focus is done before activity takes place in the classroom, and informs the analysis of this activity. In the English episode, while mathematics learning and teaching is central to analyses, the mathematics itself is largely implicit in the analytical treatment of the episode.

A final reason for the lack of use of the research cited in this special issue, tied to the previous developments, may come from the fact that direct 'ways of doing' are not the aim but, rather, the aim is towards tools to understand what occurs and to elaborate and adapt the teaching as the lesson progresses, according to what occurs in the class with the students (cf. Article 5). It has to be somehow different from content to content, from one day to another, from one class to another and from one teacher to another. The results are not spectacular, there are no simple statements to pass on, they involve complexity which is not easy to communicate. While it is very important to understand what may be common in our works, not only for researchers to understand each other, but also for our readers, and specially the non-specialist ones. We see that Articles 3 to 6 allow us claim that there are many common issues addressed by the research, and that, in spite of differences in goals, long-term intentions, theories, methodologies, unit of analysis, data and contexts of studies to tackle these issues, the results may be considered as two aspects of the same reality. To say it in other words, it is possible to include these results in a one 'bigger' result. This may perhaps contribute to a better visibility outside the field of mathematics education.

However, one thing to observe is that we see cross-national studies in the EU which seem able to deal with a range of contexts, cultures and data collection, often with shared data and perspectives for analysis. A difference with what we are discussing above, these studies are conceived in advance, the theoretical perspectives are stated and agreed up front, as are methodologies and shared practices. These pre-arranged commonalities enable cross-national comparisons and wider impacting outcomes. One possibility from the insights that the joint activity for this special issue has revealed is for further joint research, although sources of funding are hard to acquire.

Conclusion

To conclude this article and the whole Special Issue, we have to describe some 'benefits' of this common work and open some perspectives. It is clear that the deepened discussion between researchers of different countries contributes to a deeper understanding of each point of view: we not only learn about each other's perspectives but we get new insights into our own perspectives. On the one hand, the discussion on the same themes, with the precise work on examples, was really very productive to let us enter the others' overall approaches and motivations. The contrasting of our micro-level analyses has been relevant to make us think about the issues, the methodologies and the results. Indeed to make others understand our work more exactly contributes to making us explain more deeply some elements we may never have made explicit and even to detect implicit chacteristics in our approaches which benefit from being made explicit.

On the other hand, one perspective may be to present in a single (simplified) form our various results – as two faces of the same regularity. For instance for teachers training, the main result is perhaps the necessity that all the researchers claim, of making the teachers become conscious of the students' needs, of the necessity of listening to them, and of giving them effective tools for their learning. It is also becoming clear (as evidenced in both English and French cases) that the collective study of videos may contribute to our main goals – whatever may be the way to reach this consciousness. The contrasting of the methods and of the fine results is perhaps less interesting for the rest of the world.

This unified presentation of our results may be easier in such a common work, in a second phase after the first phase of eliciting the contrasted approaches, and it may contribute to our visibility.

References

All the references in this article are within the references of Article 2 and Article 6, in this volume; the authors chose not to add references is this synthetic article.

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DES RECHERCHES EN DIDACTIQUE DES MATHEMATIQUES ANGLAISES ET FRANÇAISES : BILAN ET MISE EN DISCUSSION DES PERSPECTIVES THEORIQUES ET DES PRINCIPALES QUESTIONS ABORDEES.

Résumé. Dans ce numéro spécial nous nous sommes centrés sur différentes théories utilisées dans des recherches anglaises et françaises sur l'enseignement et l'apprentissage des mathématiques ainsi que sur les formations des enseignants. Ce dernier texte revient sur les articles précédents, en mettant en perspective les théories et les principales idées et questionnements développés dans les différents exemples abordés. Nous nous attachons à dégager ce qui est commun et ce qui diffère. Nous terminons en revenant sur les problèmes de validations, d'échelles des recherches et de politique, qui constituent des défis partagés par les chercheurs des deux pays, en réfléchissant à des moyens communs d'y faire face.

Mots-clés. Comparaison de théories, recherche en didactique des mathématiques, usages de ces théories, défis pour la recherche

Abstract. French and english theoretical perspectives in mathematics education research: an overview and discussion of key issues. In this article we focus on issues related to theories in mathematics education as used in both French and English settings. As the final article in this special issue, we review the earlier articles and focus on the key ideas and issues which stand out for us. As with the other articles, we seek to address both common and contrasting perspectives, drawing on the examples which illustrate uses of theory. We end by pointing to issues of validation, scale and policy which challenge both groups and look towards facing such challenges jointly.

Keywords. Comparison of theories, mathematics education research, uses of theories, challenges for the research

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Introduction

Pour conclure ce numéro spécial sur les perspectives anglaises et françaises de recherche sur l'enseignement des mathématiques[1] nous allons suivre deux pistes :

[1] Il peut être trompeur de traduire mot à mot le vocabulaire utilisé par les chercheurs anglais, nous avons opté selon les cas pour l'utilisation de termes français « équivalents » dans leur sens ou pour la reprise du mot anglais non traduit.

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- Tirer les fils de l'article 2, dans lequel nous avons présenté les aspects majeurs des deux perspectives, en synthétisant ce qui est analogue, complémentaire ou différent ;
- Revenir sur l'ensemble des articles de ce numéro et sur la richesse des éléments théoriques qu'ils apportent à la description globale.

Notre discussion permet de tisser ensemble contenus et questions en jeu, en nous basant sur les articles 3 à 6 pour illustrer par des exemples riches les problèmes théoriques qui peuvent se poser.

Nous avons opté pour présenter par endroits ce travail de synthèse sous forme d'un dialogue entre nous, les deux auteures, les prises de parole étant rapportées nominativement à leur auteure. Cela permet d'exposer nos divergences. Ailleurs c'est un texte commun qui résume nos positions, lorsqu'elles sont relativement analogues.

1. Premier dialogue sur la perspective générale du numéro spécial

Aline : Comme le montre l'article 2, les développements des champs scientifiques que les chercheurs anglais appellent « mathematics education » et les chercheurs français « didactique des mathématiques » ont été très différents. Pour résumer, on peut reprendre la formule de Bernstein en évoquant un développement théorique horizontal pour les premiers et vertical pour les seconds. Autrement dit, par exemple, les recherches anglaises ont multiplié les emprunts théoriques, en reprenant ce qui était développé en Sciences de l'éducation, alors qu'en France, on peut restreindre l'inspiration théorique à trois grands courants majeurs, en relation beaucoup moins étroite avec les Sciences de l'éducation. On peut encore dire que le développement anglais de notre champ commun de recherche s'est fait à l'interne du champ des Sciences de l'éducation, alors qu'en France il s'est construit d'une certaine manière « contre » les théories correspondantes, en renforçant certaines différences. En réalité ce sont les analyses a priori des contenus mathématiques en jeu, comme préalables à quasiment toute analyse didactique, qui contribuent à la singularité revendiquée de l'approche française. De telles analyses sont moins présentes dans les premières étapes des travaux anglais, et elles peuvent même y être implicites.

Barbara : Oui, cette idée de développement horizontal/vertical fait sens pour moi. Une différence importante tient à ce que partout en France, les chercheurs s'appuient sur les mêmes théories, même si leurs questions ou leurs objectifs diffèrent. Dans le contexte anglais, les différents chercheurs se placent dans des perspectives théoriques différentes en relation avec leurs questionnements particuliers et peuvent s'accorder sur l'utilisation de certaines théories pour étudier certains contextes spécifiques. Cela donne lieu à un débat théorique complexe dans la communauté des chercheurs. Comparant à la centration française sur les analyses des mathématiques en jeu, comme préalables à l'étude des autres aspects, je reconnais que a) nous ne procédons pas ainsi b) ce qui ne signifie pas que la nature des mathématiques ne soit pas importante pour nous ! Comme tu l'as dit, c'est laissé en large partie implicite.

Aline : Nous pouvons ajouter une autre différence qui tient à l'origine des chercheurs dans chaque contexte : la plupart des formateurs anglais ont été professeurs eux-mêmes. Ce n'est pas le cas en France, où, au début du développement de la didactique, les chercheurs étaient surtout des universitaires, enseignant les mathématiques à des étudiants ou à des professeurs en formation continue. Peut-être les différences institutionnelles correspondantes, et notamment les attentes en termes de recherches, ont pu jouer dans le travail théorique spécifique des chercheurs français qui avaient plus d'opportunités pour construire des cadres théoriques au lieu d'en utiliser d'autres existants, qui ne sont pas directement applicables par exemple en formation des enseignants.

Barbara : C'est vrai qu'au Royaume-Uni la plupart des formateurs d'enseignants ont été eux-mêmes des professeurs en primaire ou secondaire. Cependant ce n'est pas toujours le cas pour les formateurs en mathématiques. Cela soulève la question de l'origine professionnelle des chercheurs anglais en « mathematics education ». De fait, beaucoup sont formateurs et enseignent les mathématiques à l'université mais pas tous. La formation à l'enseignement se fait essentiellement en un an, et conduit à un PGCE[2] (certificat supérieur d'éducation). Cela n'inclut pas l'étude des mathématiques. De ce fait les formateurs d'enseignants ne font pas de cours de mathématiques séparés. En ce qui concerne les recherches sur l'enseignement et l'apprentissage dans l'enseignement supérieur, elles sont le plus souvent faites par des chercheurs qui n'ont pas été enseignants dans le premier ou le second degré mais par des enseignants universitaires, qui enseignent les mathématiques aux étudiants. Je pense que les formateurs anglais sont davantage intéressés par l'utilisation de théories qui puissent les aider à analyser l'enseignement et l'apprentissage que par un travail explicite sur la critique, le développement et l'unification de théories nouvelles.

Aline : Pour revenir aux différences nous pouvons voir que la présentation des résultats dans l'article 2 (section 3) n'est pas la même ; la partie anglaise est rédigée autour de cinq thèmes sur ce qui a été acquis dans les recherches alors que la partie française résume ce que chaque théorie est en mesure d'apporter. Il est toutefois important de souligner que ces thèmes de travaux anglais sont aussi des objets de recherche française. Le thème le plus partagé est sans doute celui qui s'intitule « le sens des mathématiques », dans la mesure où, en France, presque

[2] Post Graduate Certificate of Education

toutes les recherches sont concernées par tel ou tel aspect de ce sens. Les travaux sur l'équité, un des thèmes anglais signalé, existent aussi en France (on parle plutôt de recherches sur les inégalités sociales) mais de manière moins importante. Contrairement à ce qui s'est passé au Royaume-Uni, les études sur le niveau post-bac (initiées dès les années 81 en France) n'y ont pas été structurées comme un thème isolé des autres jusqu'à récemment, grâce notamment aux travaux sur les transitions et particulièrement la transition lycée-université. Enfin, même si on ne peut pas parler en France d'un thème centré sur la « politique », alors que c'est le cas au Royaume-Uni, des études françaises très récentes à partir des évaluations internationales ont un peu modifié cette donne.

En fait, derrière une diversité réelle, il n'en demeure pas moins que beaucoup de travaux portent sur les mêmes problèmes dans les deux pays. Cela nous engage à des comparaisons légitimes bien que nos perspectives théoriques soient vraiment différentes. Nous allons en donner des exemples.

2. Des conceptions et des usages différents d'une même théorie : la Théorie de l'Activité.

2.1 Différentes conceptions de la Théorie de l'Activité

C'est dans l'article 3 que sont relevées les différences entre les chercheurs des deux pays se réclamant de la Théorie de l'Activité (AT) : différences d'interprétation et d'usage.

Barbara : L'interprétation anglaise de la théorie et les emprunts qui en sont faits sont dans une certaine mesure plus larges que du côté français, mettant en jeu des catégories générales d'analyse de l'activité. Par exemple la division du travail, les règles et les communautés, autant de catégories que développe Engeström, ou encore les contradictions de Roth et Radford, ou les motifs, actions et buts de Léontiev. Tous ces concepts, issus des débuts de la théorie, relèvent du domaine public depuis des années ; ils sont bien connus et ont été discutés. Cela a du sens de les utiliser, quitte à les adapter aux problèmes d'éducation mathématique, et particulièrement en ce qui concerne les analyses des interactions en classe et de l'activité des professeurs et des élèves faisant des mathématiques. On peut considérer que ces derniers sont des opérateurs de différents systèmes d'activité. Les concepts de médiation, d'action orientée par des buts et l'utilisation d'outils et de signes permettent l'analyse de systèmes complexes en présence, avec les tensions et contradictions qui apparaissent dans les pratiques et le poids des contextes et cultures qui influencent les activités en classe.

Aline : En revanche, les chercheurs français sont davantage centrés sur l'activité des enseignants et des élèves restreinte aux moments où ils travaillent en classe de mathématiques. Ils ont introduit la Double Approche ergonomique et didactique

(DA) pour en analyser la complexité. Cela les amène à élargir (autrement) leurs analyses. Ils font référence aux composantes cognitive et médiative pour décrire les choix des enseignants sur les contenus et les déroulements (où les activités sont plus visibles). Mais ils complètent ces descriptions en introduisant des éléments sur la composante personnelle des pratiques, qui permet de tenir compte de l'expérience et des connaissances des enseignants, et sur les composantes institutionnelle et sociale, qui interviennent dans les adaptations correspondantes des enseignants. Ces chercheurs ont opérationnalisé dans cette approche le modèle de la Zone Proximale de Développement (ZPD) pour la classe de mathématiques, en introduisant le concept de proximités pour qualifier la qualité des liens explicités entre les attentes a priori des enseignants et ce que font effectivement les étudiants.

2.2 Un exemple d'adoption de différentes perspectives pour étudier ce qui se passe en classe.

Dans l'article 3, le problème commun abordé par les chercheurs est celui de l'analyse très locale de déroulements de séances effectives, à partir de données recueillies sur des enregistrements faits en classe, vidéo (en France) ou audio (en Angleterre). Le but est de comprendre ce qui semble se passer quand les élèves (étudiants) écoutent et interprètent les mots du professeur et ce qui peut motiver les improvisations et autres commentaires de ce dernier. Comment les élèves bénéficient-ils ou non de ce que dit l'enseignant ? Font-ils des progrès ? Comment les réponses du professeur sont-elles influencées par sa perception de la compréhension des étudiants ? Les deux études illustrent de manière complémentaire la manière d'aborder ce problème : coté anglais, l'analyse des contradictions inhérentes à l'épisode choisi aide à révéler et à aborder les problèmes de communication et leur impact sur le discours tenu en classe. Côté français, l'analyse révèle que les commentaires de certains enseignants rapprochent (plus ou moins) les élèves de ce que l'enseignant veut qu'ils apprennent. Ce sont surtout les aspects cognitifs qui y sont étudiés, alors que du côté anglais des aspects plus globaux de la situation font partie de l'analyse.

2.3 Un exemple montrant un usage possible de la théorie de l'activité pour étudier une séance de classe intégrant les technologies

L'article 4 discute des usages des technologies en classe et des théories pour les aborder. Bien que ces dernières soient différentes, les résultats en termes d'enseignement et d'apprentissage sont assez proches. Laissons la parole aux auteurs :

> « En un sens, nos méthodes examinent les deux faces d'une même médaille : les pratiques des enseignants utilisant les technologies en classe, à partir de nos deux perspectives culturelles différentes. »

Dans cet article, les chercheurs français évoquent une ouverture à l'exploration mathématique des élèves, l'enseignant étant souvent en mode improvisation. Cet enseignant a soigneusement préparé la tâche, en ayant des attentes précises sur ce que les élèves peuvent réussir grâce à l'utilisation qu'il a prévue du logiciel de géométrie dynamique. Ce travail préalable ressemble à ce qui correspond en didactique des mathématiques à une analyse a priori de la tâche. Cependant l'activité des élèves ne peut pas être complètement prévue – les attentes sont ponctuées de tensions et de perturbations qui proviennent principalement de l'usage du logiciel et qui amènent des modifications dans l'itinéraire cognitif planifié (c'est-à-dire que l'activité des élèves n'est pas celle que l'enseignant aurait voulue).

Du point de vue anglais, cela semble théorisable au sein du triangle associé à la troisième génération de la TA, comme le propose Engeström, qui place les tensions entre les outils utilisés (ici le logiciel et une tâche ouverte) d'une part et la division du travail et les règles d'autre part. Les règles de l'analyse a priori conduisant à un certain contrôle des acquis cognitifs des élèves sont mises en question par le fait que la tâche est ouverte et par la propre dynamique du logiciel de géométrie. La division du travail attendue, entre l'activité de l'enseignant et celle de l'élève, bien délimitée chacune, est mise en question par le besoin de l'enseignant et des élèves d'apprécier ensemble les raisonnements venant de résultats inattendus liés à l'usage du logiciel.

En termes de ZPD (comme elle est utilisée par les chercheurs français), nous pourrions dire que les technologies constituent une sorte de médiation entre enseignant et élèves, les deux parties progressant grâce à leur questionnement commun des résultats géométriques inattendus.

3. Des problèmes qui surviennent quand les chercheurs ont des buts communs mais des théories différentes.

Dans l'article 5, les trois auteurs parlent de théorie, à partir d'exemples tirés de leur pratique de formation d'enseignants. Comme ils l'expliquent, Coles s'inspire d'une théorie de l'enaction[3] alors que les deux auteures françaises, Chesnais et Horoks, utilisent la TA, enrichie de certaines idées de la TSD (Théorie des Situations Didactiques). Initialement ils auraient voulu choisir une vidéo tournée en classe qu'ils auraient pu utiliser tous les trois pour exhiber les différences entre leurs pratiques et leurs perspectives théoriques respectives. Ils étaient persuadés que choisir une telle vidéo, que tous pourraient utiliser, constituait une tâche importante donnant accès à l'explicitation de leurs différents objectifs à travers ce choix.

[3] Nous utiliserons selon les cas théorie de l'enaction ou enactivisme pour qualifier la source théorique dont Coles s'inspire.

Cependant il est très intéressant de constater qu'ils n'ont pas pu trouver une vidéo leur convenant à tous.

Barbara : La tâche de la vidéo proposée par Coles était trop ouverte pour les collègues français – elle ne permettait pas une analyse a priori mettant en évidence précisément les mathématiques visées pour les élèves. Une tâche plus restreinte aurait été plus appropriée. Je fais l'hypothèse qu'au contraire, pour Coles, la tâche proposée par les chercheures françaises était trop restreinte ! Dans ma propre perspective, vraiment pratique, lorsque nous concevons une activité pour la classe, nous cherchons à mettre en place une démarche de recherche qui puisse pousser les élèves à investir par eux-mêmes le problème proposé, qui peut être assez large. Les mathématiques en jeu ne sont pas précisées. Cela demande beaucoup à l'enseignant qui doit faire face à toutes les approches des élèves, en aidant chacun dans l'interprétation qu'il a choisie. Cela met en jeu à des degrés divers des aides et des questionnements.

Aline : L'analyse a priori de la tâche peut aider le professeur pendant la séance et elle permet aussi au chercheur et au formateur d'avoir des attentes claires sur ce qui est visé en termes d'apprentissage pour les élèves. Elle accompagne la complexité des décisions du professeur tout en lui permettant de garder le cap mathématique. En particulier s'il s'agit d'une tâche d'introduction, cette analyse facilite l'exposition des connaissances en jeu. C'est le moment où l'enseignant doit généraliser à partir de l'utilisation contextualisée des connaissances qu'ont faite les élèves dans le problème proposé. Dans les autres cas, cette analyse a priori facilite la compréhension que peut avoir l'enseignant des mises en fonctionnement précises des connaissances mathématiques de ses élèves. Cela peut lui faire modifier la tâche si ce qui se passe n'est pas assez conforme aux attentes. Ou alors cela lui permet de cibler ses interventions pendant la séance, grâce à une interprétation approfondie du travail des élèves et à des improvisations adaptées qui s'appuient sur ce travail, par exemple suite à la détection d'implicites comme ceux décrits dans l'article 3. En réalité il ne s'agit pas d'analyser ainsi chaque tâche, mais il est important que les enseignants puissent le faire sur quelques tâches importantes, ce qui enrichit leur prise en compte du travail des élèves.

Aline : En ce qui concerne les contenus mathématiques, le choix des tâches et les déroulements correspondants sont essentiels pour que les élèves mettent en fonctionnement, en contexte, les connaissances à acquérir. Loin d'être une réduction de leur activité, c'est au contraire un élément du processus global à mettre en place, mais cela demande que l'enseignant soit en mesure de repérer dans le travail des élèves ce qui vaut la peine d'être généralisé ou signalé comme une application.

3.1 Différentes perspectives théoriques et ce qu'elles peuvent révéler.

Les différences entre les théories exposées dans l'article 5 nous permettent d'aller un peu plus loin dans leur description et leur usage pour la classe.

La théorie de l'enaction (Barbara)

Comparant les pratiques et déclarations théoriques des trois auteurs de l'article 5, je pense que ces trois chercheurs essayent de faire des choses différentes. Le chercheur anglais utilise un cadre issu de la théorie de l'énaction pour installer chez les enseignants des pratiques enactivistes, grâce au travail qu'il mène avec eux sur des vidéos. Voilà comment je conçois cet usage.

L'énactivisme est quelquefois décrit comme « un chemin tracé en marchant ». On propose aux élèves une tâche très ouverte. Cela les engage à s'investir et à explorer des possibles. En s'engageant, ils « marchent ». En marchant ils réfléchissent à la tâche et commencent à lui donner un sens mathématique. Il peut y avoir différents chemins pensés par différents élèves. S'ils discutent entre eux et collaborent sur le problème, ces chemins peuvent se fusionner ou se croiser. Ceci peut modifier l'enjeu que se donnent les élèves et un chemin initial partagé peut être étendu. Le rôle de l'enseignant est d'écouter et de guider, en posant les bonnes questions, encourageant et sondant les élèves pour les aider et/ou les provoquer (cf. la « triad » de Jaworski). Les élèves doivent s'habituer au fait qu'il n'y a pas qu'une bonne manière de résoudre, ni même une seule bonne réponse – c'est cela l'énactivisme : prendre conscience du fait qu'il y a beaucoup de chemins pour arriver au but, et que ce sont ses propres actions qui peuvent aider à trouver un chemin qui mène dans une bonne direction. L'enseignant y contribue de diverses manières. Cela pose un véritable défi d'être enseignant adoptant un tel cadre. C'est ce que Coles propose à ses enseignants. Ils doivent regarder la vidéo en évitant d'y mettre leur propre interprétation. Ils ne peuvent pas voir ce qui se passe à l'intérieur des têtes. Ils doivent se limiter à ce qui est visible, au sens propre. Cela les force à être plus attentifs aux choix qui se posent à l'enseignant et à ce qui motive ses réponses aux élèves. Ces enseignants observateurs prennent conscience de la multiplicité des choix et peut-être de la complexité que l'enseignant doit gérer et des réponses qui peuvent être faites. Quels choix peuvent aider ou stimuler au mieux les élèves devient une question ouverte à la discussion du groupe. Ces réflexions sur la vidéo rendent ces professeurs capables de questionner leurs propres pratiques et leurs propres choix, en s'informant davantage, plutôt que de penser aux meilleurs choix, dans la mesure où c'est difficile de dire ce qui est mieux. Cela peut devenir un thème de discussion qui permet aussi de développer la prise de conscience. Ces niveaux de prise de conscience constituent « le chemin tracé en marchant » pour ces enseignants.

La théorie de la Double Approche (Aline)

Les chercheures françaises auteures de l'article 5 essaient de faire partager les outils de l'analyse a priori élaborée en didactique des mathématiques qui semblent pertinents pour réfléchir à une séance de mathématiques.

La DA n'a pas pour objet principal l'activité des élèves. Cette théorie s'intéresse à l'activité des enseignants en relation avec celle des élèves. En fait ses analyses peuvent contribuer à mieux comprendre les activités des élèves grâce à une meilleure compréhension des choix de leurs enseignants. La formation des enseignants met en jeu une approche spécifique, fondée sur la DA en ce qui concerne les pratiques des enseignants et sur la TA et la TSD en ce qui concerne les apprentissages, selon le niveau scolaire. L'idée générale pour la formation est de permettre aux enseignants de s'approprier un certain nombre d'outils utilisés en didactique, en prenant en compte la complexité des pratiques, et en particulier le fait qu'elles ne sont pas seulement dictées par les apprentissages des élèves. Il y a plusieurs manières d'y arriver, pas nécessairement directes.

Il est nécessaire de différencier le point de vue de l'enseignant et celui du chercheur en ce qui concerne l'importance des analyses a priori mathématiques, à l'intérieur d'un processus global de conceptualisation pour le second ou comme référence pour étudier une vidéo pour le premier. Une telle analyse permet au chercheur d'avoir une idée claire de la place d'une tâche donnée dans le processus amenant à l'apprentissage visé mais n'aide souvent le formateur qu'à l'étude locale de la vidéo. Il y a des différences entre les tâches qu'il est important d'apprécier pour tous, certaines contribuent à installer un travail contextualisé préalable à l'exposition de connaissances générales, certaines sont utiles à faire appliquer des connaissances générales dans des exercices, certaines servent à rendre des connaissances disponibles, comme nous allons l'expliquer ci-dessous. Ainsi certains chercheurs conçoivent des séances destinées à assurer la construction par les élèves d'une connaissance visée, c'est le cas notamment en TSD pour le primaire. Par exemple ils élaborent une tâche (un problème) d'introduction amenant à un moment d'institutionnalisation de la connaissance visée, ayant un fort potentiel pour l'apprentissage des élèves, pourvu que le déroulement des séances assuré par l'enseignant respecte le but visé tout au long du processus. Utiliser de telles tâches suppose, pour l'enseignant, de suivre une certaine démarche : laisser les élèves travailler de manière autonome, et ensuite s'appuyer sur ce qu'ils ont fait et ce qu'ils savent pour présenter la connaissance générale visée. Cela peut amener à établir des proximités – comme le montre l'article 6, discuté plus loin. Ce peut être en exposant les liens entre les connaissances contextualisées utilisées par les élèves en exercices et les connaissances générales visées, telles qu'elles sont exposées au moment des cours. Ce peut être avant ces moments de cours ou après, selon les cas. Mais pas tous les enseignants n'utilisent de telles tâches, non seulement parce que c'est difficile de gérer les séances, mais aussi parce que tous les contenus à enseigner n'ont pas été étudiés de cette façon, notamment dans le secondaire. Il est difficile pour un professeur isolé d'élaborer un tel scénario.

Il est alors utile de comprendre, grâce à la TA, ce qui se passe dans des classes ordinaires : cette théorie sert de référence pour analyser les apprentissages, complétée par la DA pour l'étude des pratiques (cf. article 2). Dans ces classes certains chercheurs étudient par exemple les occasions dont profitent les enseignants pour rapprocher leurs élèves des connaissances visées, suite à un travail sur des tâches, quelles qu'elles soient, et repèrent des occasions manquées, en essayant d'en identifier les raisons. Ces dernières peuvent tenir aux mathématiques elles-mêmes, notamment du fait des tâches choisies, mais aussi aux déroulements, par exemple certaines difficultés des élèves peuvent rester ignorées. Cela peut amener à développer une sorte de vigilance des professeurs sur des points précis, problématiques, liés aux apprentissages, aux tâches proposées, aux cours et aux déroulements. Mais ces raisons peuvent aussi tenir à la complexité de ce que l'enseignant a à faire, gérer l'hétérogénéité de ses élèves, tout en manquant toujours de temps, et respecter les contraintes sociales, institutionnelles et personnelles. La DA sert au chercheur à ne pas oublier ce qui doit être pris en compte pour comprendre les activités de l'enseignant en tenant compte de cette complexité.

Finalement la réflexion sur la formation des enseignants met en jeu cette complexité des pratiques et peut s'appuyer sur un certain nombre de résultats issus de la DA. Par exemple la stabilité des déroulements organisés par les enseignants et plus généralement de leurs pratiques, ainsi que l'emprunt du modèle de la ZPD pour le développement de ces pratiques nous amènent à travailler en formation sur les déroulements en classe et à nous appuyer sur l'expérience pour l'enrichir collectivement.

4. Une même perspective théorique mais des situations différentes

Dans l'article 6, on trouve deux manières différentes d'utiliser la TSD pour analyser des contextes de formation d'enseignants. La première démarche, incluse dans les recherches anglaises, vient de Norvège, et la deuxième de France. Dans le premier contexte, la chercheure étudie des pratiques d'enseignants dans un contexte de formation, où les enseignants débutants apprennent des mathématiques dans une activité préparée par leur formateur de mathématiques. Dans ce cas le focus est sur la manière dont la TSD peut être un outil pour le chercheur pour comprendre les pratiques d'enseignement et aider à leur développement. Dans l'exemple qui est présenté, bien qu'il n'y ait pas d'analyse a priori explicite du contenu mathématique en jeu, il y a une compréhension certaine de ces mathématiques et des attentes sur la situation adidactique qui débute la situation. Le second cas présente une collaboration entre chercheurs, formateurs et enseignants portée par un objectif commun, à savoir produire des ressources pour enseigner la géométrie. Les analyses issues de la TSD illustrent la manière dont cette théorie peut aider la collaboration entre chercheurs et formateurs (ou enseignants), en termes de développement professionnel des enseignants.

Barbara : Je vois dans cet article une forme élaborée de la théorie, avec une explicitation de différents aspects. Le concept clef de milieu, à la fois didactique et adidactique, et les étapes de dévolution et d'institutionnalisation sont introduits. Bien que les contextes soient très différents, on peut voir comment la même perspective théorique sert les deux analyses. Dans cette mesure, voir les mêmes concepts appliqués à deux contextes différents aide à clarifier ces éléments clefs de la théorie. D'une certaine manière, je vois l'intérêt d'utiliser les concepts clefs mentionnés ci-dessus dans n'importe quel contexte où un enseignant veut faire apprendre à ses élèves une connaissance mathématique spécifiée. Les difficultés arrivent quand la conception des tâches (didactiques ou adidactiques) est faite par le chercheur ou le formateur, en supposant que l'enseignant pourra adopter la démarche attendue sans avoir été associé à cette conception. Dans le premier cas l'enseignant est partie prenante du processus d'élaboration, et dans le second cas il y a une collaboration entre les enseignants, les formateurs et les chercheurs pour cette conception. C'est ainsi que les difficultés sont évitées.

Aline : Le cadre de la TSD est particulièrement bien adapté à concevoir des situations d'apprentissage et permet aussi d'analyser ce qui se passe en classe pendant la mise en œuvre des situations, en référence au projet. Plus récemment cette théorie a servi à identifier des questions utiles au développement des pratiques professionnelles des enseignants. Mais un des principaux objectifs reste l'étude du potentiel cognitif des situations à proposer, c'est-à-dire l'étude de ce que peuvent apprendre les élèves en relation avec le scénario proposé, essentiellement les tâches et les déroulements associés prévus, souvent pour introduire une nouvelle notion. Dans les deux cas traités dans l'article 6, comme les auteures le disent, le focus est sur la conception de la situation et son étude. Il y a cependant des différences dans les objectifs et les questions de recherche, liés aux contextes, même s'il s'agit de formation dans les deux cas. A chaque fois les enseignants ont à apprendre à mettre en œuvre les tâches choisies, en les adaptant à leurs élèves tout en essayant de ne pas en perdre le potentiel d'apprentissage. L'utilisation des concepts de la TSD est cependant rendue plus explicite pour les enseignants dans le cas de la multiplication (premier exemple) que dans le cas de la géométrie (deuxième exemple). En fait dans ce dernier cas, il y a une grande différence entre « le petit groupe » (formé des chercheurs, formateurs et de quelques enseignants) et le grand groupe (d'enseignants) : dans le petit groupe on installe petit à petit une familiarisation ou au moins une utilisation en acte des concepts de la TSD, durant les échanges sur la conception des situations et les analyses des observations de classes, sans explicitation formelle toutefois ; dans le grand groupe le focus reste sur les décisions en termes de pratiques.

5. Utilisation et impact des résultats des recherches dans les deux contextes

Bien que les développements des domaines scientifiques concernés aient pris des voies différentes (article 2), que ce soit dans des perspectives différentes (articles 3,4,5) ou non (article 6), les chercheurs s'attaquent en fait aux mêmes problèmes et les résultats obtenus ne sont pas si différents, à une échelle globale en tout cas – les deux faces d'une même pièce, considérées à un niveau de lecture plus ou moins général. Les chercheurs des deux pays essaient de développer les apprentissages des mathématiques en lien avec leur enseignement, en envisageant aussi bien les aspects théoriques que pratiques. Ce qu'on peut retenir de notre travail de mises en regard enrichit ainsi les perspectives globales et fait valoir notre entreprise. Cependant que ce soit au Royaume-Uni ou en France, nos résultats ne sont pas ou peu pris en compte par les preneurs de décision (politiques). Fait remarquable et indépendant du développement, pourtant différent, des domaines dans les deux contextes. Ce n'est pas la multiplicité des théories qui peut expliquer la réticence de l'institution et des décideurs de suivre les résultats des recherches et leurs conséquences. C'est un résultat important de notre travail commun : on doit trouver de nouveaux moyens pour avoir une influence ; peut-être y a-t-il besoin de montrer des résultats communs obtenus dans beaucoup de pays pour rendre plus visibles aux yeux des décideurs politiques les perspectives qui en découlent. Cela pourrait s'inscrire à la suite de la publication des synthèses européennes de résultats didactiques robustes publiées par la société européenne de mathématique (« Solid findings in mathematics education », septembre 2011).

Cependant, on le voit bien avec ce numéro spécial, la production de preuves de cette robustesse est rendue difficile par le fait que la plupart des études sont qualitatives et menées à une petite échelle. Plus généralement, il est clair qu'il ne peut pas exister d'évaluations classiques, quantitatives, pour de telles recherches (articles 3-6). Même si les évaluations internationales standardisées informent sur les connaissances des élèves, cela ne peut pas induire directement des changements dans les enseignements. Les relations entre les évaluations quantitatives et les pratiques individuelles des enseignants ne sont pas simples. Il y a souvent un manque d'ajustement des exercices proposés à l'enseignement correspondant, d'autant plus que l'apprentissage est un processus long qui n'est pas réductible à ce qui peut être apprécié à un moment donné. C'est bien connu et cela concerne toutes les sciences humaines et sociales mais cela peut quand même jouer un rôle dans les réticences de l'institution. Dans toutes ces sciences, la robustesse d'un résultat ne vient pas des évaluations (ou pas seulement).

Cependant dans le cas de la formation des enseignants, comme les formateurs sont souvent aussi chercheurs en didactique, il peut y avoir une sensibilisation croissante aux résultats et problèmes issus des recherches, dans la mesure où les chercheurs communiquent entre eux, au sein de chaque pays et internationalement. Les échanges organisés dans les conférences nationales et internationales nourrissent les connaissances professionnelles et peuvent se répandre ensuite chez les enseignants grâce aux formateurs. On peut voir cette connaissance issue des recherches diffuser, tant théoriquement que pratiquement, grâce aux opportunités offertes par la formation.

Cela amène à nuancer le propos initial. L'enseignement, comme on peut le constater aujourd'hui, est certes influencé par les décisions politiques mais aussi par les formations des formateurs, ayant un double statut, en relation avec leur connaissance de chercheurs. Les savoirs robustes (solid findings) évoqués ci-dessus peuvent contribuer de manière importante à cette connaissance et c'est à quoi nous devons participer dans notre pratique de chercheur et de formateur.

6. Relation entre contextes et résultats de recherches – un zoom sur des travaux particuliers

Il y a un autre facteur qui affaiblit les résultats de nos recherches. C'est le fait qu'ils dépendent largement des contextes de l'étude et que leurs usages dépendent aussi des situations, que ce soit dans un même pays ou non. C'est très clair dans l'article 6, où c'est exactement la même théorie (TSD) qui est adoptée mais où les contextes institutionnels et les focus des recherches diffèrent. Dans un cas le chercheur étudie une formation effective de futurs enseignants, dans l'autre cas, il s'agit de travailler sur une ressource pour les enseignants en exercice. Dans le premier cas l'analyse permet de constater des différences entre la conceptualisation de l'enseignant associée à certaines tâches et l'activité des élèves sur ces mêmes tâches. Cela a des conséquences plus générales pour la conception des tâches et pour le travail des enseignants de terrain. Dans le second cas en revanche le problème est de concevoir une ressource valable pour de nombreux enseignants. La première analyse des mathématiques en jeu est ainsi la même dans les deux cas mais celle sur les discussions sur les variables en jeu et la manière de présenter les recherches diffère. Cette grande dépendance du contexte peut expliquer une partie de notre manque d'influence, en relation avec la difficulté d'adapter les résultats à tous les facteurs qui varient selon les contextes. Les programmes diffèrent d'un pays à l'autre (voire d'une région à l'autre), les habitudes culturelles et scolaires aussi, les enseignants doivent trouver leur propre manière d'enseigner pour être à l'aise, les élèves sont très différents, notamment d'une classe sociale à l'autre mais pas seulement. Il en résulte qu'un résultat de recherche doit être présenté non seulement avec le contexte dans lequel il a été obtenu mais aussi en précisant ses limites, et sans que l'impact général qu'il pourrait engendrer soit évident.

Une autre raison qui peut amener ce manque d'influence constaté partout est que les analyses qualitatives locales sont beaucoup plus fréquentes que les globales et/ou quantitatives. Le passage d'une étude locale à une interprétation globale n'est pas évident, en relation avec la prise en compte nécessaire des différences de contextes, qui amènent à recueillir et dépouiller de nombreuses données, avec le temps que ça prend. Il est ainsi difficile de prendre en compte tous les paramètres (variables) en jeu. C'est donc difficile d'inférer des résultats globaux de nos analyses locales sur les apprentissages des élèves. Les chercheurs ne peuvent avoir que des hypothèses sur la qualité des scénarios à mettre en place. Il est suggéré que la récurrence des modalités d'enseignement est un facteur important d'apprentissage. Cela dit, il y a des différences dans la manière de les prendre en compte par les chercheurs. Par exemple dans l'article 3, dans l'étude d'un épisode de classe en France, une analyse a priori des mathématiques en jeu est faite avant que l'activité des élèves soit analysée, et la renseigne. Alors que dans l'épisode anglais, même si les mathématiques sont centrales, elles restent implicites dans l'analyse qui est montrée.

Une dernière raison du manque de diffusion des recherches comme celles qui sont présentées dans ce numéro spécial, tient au fait que ce ne sont pas directement les manières de faire qui sont en jeu. L'accent est plutôt mis sur les outils utilisés pour comprendre ce qui se passe, et pour élaborer les adaptations qui sont apportées par l'enseignant au fur et à mesure de la séance, compte tenu des élèves (cf. articles 4 et 5). C'est différent d'un contenu à l'autre, d'un jour à l'autre, d'une classe à l'autre, d'un enseignant à l'autre. Les résultats ne sont pas spectaculaires, mais ce ne sont pas non plus de simples constats, ils mettent en jeu toute la complexité de la classe et ce n'est pas facile à décrire. Or c'est très important de comprendre ce qui peut être commun à nos travaux, pas seulement pour que les chercheurs puissent se comprendre entre eux, mais aussi pour nos lecteurs, et particulièrement les non-spécialistes. Les articles 3 à 6 nous permettent d'affirmer qu'il y a en fait beaucoup de questionnements communs abordés dans nos recherches, et que, malgré des différences de buts, d'intentions à long terme, de théories, de méthodologies, de grains d'analyse, de données et de contextes d'étude pour attaquer les problèmes, les résultats peuvent être lus comme deux aspects d'une même réalité. Autrement dit il est possible d'inclure ces résultats dans quelque chose de plus gros qui les unifie. Cela peut contribuer peut-être à une meilleure visibilité hors de notre champ.

D'ailleurs, nous pouvons aussi constater que des études transnationales européennes arrivent à mettre en place des travaux intégrant les différences de contextes, de cultures et de recueil de données, réussissant souvent à partager ces données et à élaborer des perspectives communes pour les analyses. La différence avec ce que nous avons présenté ici tient au fait que ces études sont préparées à l'avance, les perspectives théoriques sont explicitées et acceptées en amont tout comme les méthodologies et les pratiques à partager. C'est ce travail en amont qui permet à ces communautés de chercheurs d'établir des comparaisons et d'élargir l'impact des résultats. Mener ensemble une future recherche serait ainsi une possibilité à envisager pour faire fructifier les idées issues de nos réflexions en vue de ce numéro spécial, même si nos fondements respectifs sont difficiles à partager.

Conclusion

Pour conclure nous allons justement décrire quelques bénéfices que nous avons pu retirer de cette entreprise et tenter d'ouvrir quelques perspectives. Il est clair que les discussions approfondies entre chercheurs de différents pays contribuent à une meilleure compréhension réciproque : non seulement nous nous familiarisons avec les perspectives des autres chercheurs mais encore nous pouvons avoir de nouvelles idées sur nos propres perspectives. D'une part la discussion sur des thèmes communs, étayée par le travail précis sur des exemples, a été vraiment productive et nous a permis d'entrer dans les approches globales et les motivations des autres chercheurs. Les mises en regard des analyses très locales ont été propices à la réflexion sur les problèmes, les méthodologies et les résultats. En vérité faire comprendre aux autres « en vrai » nos travaux a conduit à expliciter certains aspects plus précisément que nous ne l'aurions jamais expliqué sinon et même à détecter des caractéristiques implicites qui gagnent à être explicitées sans l'avoir été jusqu'ici.

D'autre part une perspective pour la suite pourrait être de présenter dans une forme unifiée (voire simplifiée) des résultats initialement variés – comme les deux faces d'une même régularité. Par exemple pour les formations, le résultat le plus important est peut-être la nécessité, sur laquelle tous les chercheurs s'accordent, de rendre les enseignants conscients des besoins des élèves, de la nécessité de les écouter, et de leur donner des outils effectifs pour leur apprentissage. Il est aussi devenu très clair (et prouvé dans les deux cas) que l'étude collective de vidéos peut contribuer à nos objectifs principaux, quelle que soit la manière de déclencher les prises de conscience. Le contraste des méthodes et des résultats fins est peut-être moins intéressant pour le reste du monde.

Une telle présentation unifiée de nos résultats peut être plus facile à faire à l'occasion de travaux communs, dans une seconde phase qui suit la première phase où on met à jour les différences, et elle pourrait renforcer la visibilité de nos recherches.

Références

Toutes les références citées dans cet article figurent parmi les références des articles 2 et 6 de ce volume ; les auteures ont choisi de ne pas les reprendre dans cet article de synthèse.

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