From common thought to mathematics: the need for genetic theories

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Abstract

Part 1 of this paper shows that mathematical learning, rarely considered in its full extension from early childhood to adulthood, is in need of clear guidelines. Guidelines, the way we understand them here, are outlines of what might be called genetic theories. One shows what such genetic theories could look like. Part 2 develops a possible guideline, one whose title might be : from proportionality to linearity, or the evolution of the concept of ratio. In the conclusion, we revisit the notion of genetic theory to discuss further its nature and relevance.

In memory of H. Freudenthal,
to whom we owe so many lights
on mathematical learning.

PART 1

Guidelines: why and how?

1 Mathematics versus common experience

The road from common experience to mathematics becomes longer and longer with the passing of centuries[1]. First of all, whereas Babylonians and Egyptians dealt with rather isolated questions of arithmetic and geometry, the Greeks – particularly Euclid – elaborated wide range theories, with long chains of deductions. This kind of thought is far away from everyday life. All the more so that, in this setting, even obvious propositions had to

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[1] For a detailed development of this idea, see N. Rouche [2004].

be proved, because everything – with the exception of axioms – had to be proved.

The distance from common experience to mathematics became longer also by the introduction of letters in algebra, the representation of figures by equations in analytic geometry, the introduction of negative and complex numbers, the arithmetization of the continuum (the real numbers) and, in the course of XX^{th} Century, the constitution of mathematics as a single edifice, entirely deduced from the set theory axioms.

Another highly significant step was the creation of non-Euclidean geometries, leading to the coexistence of several contradictory theories, each one logically coherent. This major historical fact entailed a mutation of the nature of mathematical truth. Namely, for the Greeks, an axiom was true if obvious in some idealized (Platonic) world, whereas in contemporary mathematics, truth is identified with non-contradiction within some axiomatic setting.

Finally, there is the existence of structures such as groups, rings, fields, topological and vector spaces, categories, and their role in mathematical thought. These structures are abstract, they seem to say nothing about nothing, whereas they express the architecture and operation of a lot of particular situations, whether inside or outside mathematics. And that is why they have the double effect of illuminating these situations and of easing the transfer of intuitions between them.

These observations depict today's mathematics as they appear in the treatises, namely as a highly abstract monument, an artificial science. However, by contrast, mathematical practice and problem solving at any level are stimulating activities, not *a priori* deductive, relying on imagination, conjectures, search for examples and counterexamples, and of course deductions. They are a source of intimate satisfaction.

A consequence of the fact that mathematics are far aside from common thought is that most people have a wrong idea of that discipline. They see it as an immutable monument – they have no idea of its secular evolution –, they reduce mathematical activity to the application of rote computations, leading in every case to the unique solution through the unique good method. Recalling this is a trivial statement, but one that is unfortunately true. Such a wrong appreciation and ignorance of mathematics is by no means new, but it is growing worse with the passing of centuries and the evolution of mathematical knowledge.

An important proportion of teachers share this conception of mathematics, often inherited from their own teachers. Others, although aware of the genuine nature of mathematics and able themselves to do mathematics creatively, leave their students eventually convinced that this science is purely deductive.

Due to all these misunderstandings, it is difficult to bring forth a clear and efficient conception of mathematical teaching. Let us now outline the history of the reforms through which, in the course of the last fifty years, one has tried to improve this situation.

2 The coherence of the New Math

In the fifties and sixties of the XX$^{\text{th}}$ Century, the initiators of the New Math truly believed they had "the solution". And maybe they were right on some points, in spite of the difficulties they encountered. At any rate, their options were extreme and extremely clear and, for that reason, lend themselves to a clear evaluation. So let us recall the New Math movement and its foundations.

First, these initiators took a central fact into account, namely that mathematical science existed as a unified discipline, developed deductively from the axioms of set theory. Therefore, its presentation to students had to be axiomatic, deductive and focused on structures. The matters to be taught were – we mention the essentials only – sets, relations, and functions (in a naive way, for it would be impossible otherwise), then natural numbers, integers, rationals and reals, then vectors spaces and linear algebra, and then limits, continuity, derivatives and integrals. Figures were considered as misleading, the geometry of figures and solids almost disappeared, and with it a notable source of intuitions.

Teaching had to be rigorous. It was focused on structures like groups, rings, fields and vector spaces. To avoid confusion and a loss of time, concepts had to be introduced, as far as possible, in their definitive form, which automatically implied a high level of generality. Further, technical terms and symbols were abundant, more than in traditional teaching.

The New Math reform was initially conceived by well known mathematicians (Dieudonné, Choquet, Stone, Artin, ...) during two colloquiums, one in Royaumont in 1959 and the other in Dubrovnik in 1960 (see H.O. Fehr [1961] and O.E.C.E. [1961]). The Proceedings of Royaumont claimed that the whole of mathematical teaching had to be revised. But, paradoxically, Dubrovnik followed with a curriculum aimed at the end of high school only, and even in the scientific orientation. However, within the following few years, this program was extended, without substantial changes of principles, to the K-12 curriculum (see for instance G. Papy [1963]).

In this way, very general subject matters were proposed to very young students. But the more general a theory is, the wider its field of applications. Young people could not realize the huge variety of referents, within mathematics or not, of notions like sets, relations, groups, etc. Therefore, they had to content themselves with some examples, often artificial, deprived of interest at their level. Some concepts were almost inoperative in their field of experience. In such a context, stimulating student initiative in problem solving was rarely possible. By exception, a small number of them enjoyed this formal universe.

In spite of these shortcomings, the New Math reform followed a clear and well structured global plan. *Subject matters followed each other in an orderly, understandable way.* The teachers able to grasp the framework of the program could determine their position and the position of their students in the teaching project as a whole. Unfortunately, and by their very nature, the guidelines of this program were rarely grasped, especially by elementary school teachers.

Let us emphasize the following observation : the almost unique source of mathematics teaching during that period was contemporary mathematical science. And after all, starting from a science in view of teaching it seems quite natural. The rationale of mathematical teaching was that of mathematics themselves : axioms, rigorous proofs, definitive concepts, algebraic structures.

3 After the New Math, which coherence ?

What happened after the fading of the New Math? Some contexts of the reform survived to some extent, mainly geometric transformations and vectors. But there has been a revival of the traditional subdivisions of mathematics : basic arithmetic at the elementary level, algebra centered on equations and no longer on structures, geometry of figures and solids. New subject matters appeared, mainly the use of calculators, algorithms (sometimes), elementary statistics and data processing.

Another trend of the curriculums is the insistence on problem solving, a precious heritage of G. Polya. Moreover, there is the emphasis laid on the construction, or reconstruction, of knowledge, in all possible measure by the student himself.

The following observation is important : whereas – as mentioned above – the New Math followed a coherent global plan, the curriculums of today do not generally show coherence to the same degree.*It is not clear how students* *are gradually led to contemporary mathematics.* The New Math was an updating of the curriculum inspired by the progress of mathematical science as observed around the middle of XX$^{\text{th}}$ Century. The present curriculums are more like gatherings of old and new subject matters. *They are not inspired by a clearly identifiable scientific reflection.*

Nevertheless, the need for coherence is still there. It is witnessed by the very idea of the construction of knowledge, as mentioned above. But this idea refers more to the efforts of the individual student to understand and organize a piece of knowledge than to the elaboration of a recognized scientific corpus. The need for coherence is also witnessed, for example, by the NCTM's Standards, where three chapters are entitled *Mathematical Connections*. However, these chapters do not develop a concatenation of matters along the curriculum : they invite the students to multiply the various representations of concepts, to recognize connections between concepts and to see the relations with other disciplines. By contrast, the Proceedings of the Dubrovnik Colloquium (O.E.C.E. [1961]) proposed a firm sequence of interrelated mathematical topics, to be taught in the given order.

Summarizing, today's mathematical teaching seems to lack an explicit and firm reference, it knows no more what are its sources, its guidelines. This difficulty is understandable. Namely, if one recognizes that a globally deductive curriculum, one that is directly inspired by contemporary mathematical thought, does not suit a majority of students, then one has to look for another road, *starting from students thought and progressing stage after stage, through successive generalizations, towards today's mathematics*[2]. But such a progression is difficult to elaborate. Apparently, one has been seeking it gropingly for forty or fifty hears.

Hence the fundamental question : how to elaborate genetic theories[3] ? Let us now consider that.

4 Starting from common experience

A genetic theory ought to show sensible ways leading from common thought and language to a mathematical theory. Now, everyday experience implies no distinction between separate disciplines : mathematics, physics, chemistry, geography, etc. In this experience, there are objects with physical

[2]Still, one should distinguish which mathematics, to suit various categories of students.

[3]The idea of a genetic theory is strongly present, even if implicitly, in the work of Freudenthal. It does not coincide, but it relates to what O. Tœplitz[1963] called a *genetic approach*.

magnitudes (lengths, weights, volumes, ...), various motions, durations, sets of objects, patterns, rhythms, relations between things and people, etc. Some situations in the environment lead naturally to a mathematical explanation, others to a physical explanation, etc. And even mathematics and physics (and other couples of the same type) are not immediately disjoint. Mathematics has to emerge gradually, and for good reasons, as a distinct discipline.

This emergence does not follow an arbitrary order, even if some choices remain possible. Namely, some natural filiations of experiences and ideas impose themselves.

Let us consider, for example, geometrical thought near its birth. Most lines and planes in our environment are vertical or horizontal. All vertical lines are parallel, and so are all horizontal planes. Every vertical line is perpendicular to every horizontal plane and orthogonal to every horizontal line. It would be difficult to believe that these two physical directions (physical because they owe their existence to the field of gravity) do not play an important role in the birth of notions like straight line, plane, parallelism and orthogonality[4]. Of course, at some stage of geometry learning, every reference to physics will be abandoned. Geometry will then appear as an abstract theory, ready to be applied in various circumstances, including of course physical contexts.

Here is another example of a natural filiation. The practical origin of decimal numbers (numbers with a decimal point) is measuring lengths or other magnitudes within a decimal system of units. This is also their historical origin. Measuring magnitudes is a physical activity. Dropping the symbol for the unit of measure results in constituting these numbers as abstract entities, ready for a variety of applications. So the reasonable sequence is : first measures (i.e. concrete decimals, with an obvious use), then abstract decimals[5].

A key principle on the road from common experience to mathematics is that no concept should be introduced if it plays no role in some explanation. Mathematical concepts as they appear in axiomatic theories are endowed with technical characteristics, whose function is to allow the construction of complete and rigorous proofs, neglecting no particular case, avoiding logical

[4] As counter-examples, notice the negligible role played by the vertical and horizontal directions at the beginning of geometry in the N.C.T.M. Standards [1981] and in the French primary school program (see B.O. [2002].

[5] Again as counter-examples : measuring magnitudes is often presented not as the source of decimal numbers, but as one of their applications (see the two references in footnote 4).

pitfalls. In mathematical treatises, the role of definitions is less to say what things really are, than to serve as appropriate tools for proving theorems.

At the intermediate stages between common and mathematical thought, there are levels of rigor[6] and types of notions appropriate to the field of phenomena being studied. Such notions were called *mental objects* by Freudenthal[7]. If a concept appears prematurely in the construction of thought, i.e. at a stage where it seems like a complicated machine to execute simple actions, it lacks a raison d'être.

Under such conditions, what do we mean here by a *genetic theory* ? It is a rational construction progressing through stages from everyday contexts towards a mathematical theory, from simple questions and notions, possibly related to perceptions or manipulations, towards new more general, more abstract questions. Each stage of this construction brings answers to new questions and helps surmounting new obstacles. The mental objects and later the mathematical concepts brought into play at each stage are appropriate to the theoretical context that is reached. Examining a genetic theory from beginning to end, one should grasp the raison d'être of every new theoretical development.

These general considerations deserve to be illustrated by a substantial example. Let us now look at that.

PART 2

A tentative genetic theory: from proportionality to linearity

Let us now outline, as an example, a tentative genetic theory covering mathematical learning from K to 12. We choose to trace the notion of linearity, i.e. the successive generalizations of the concept of *ratio*, or the construction of what might be called the *linear structure*. This choice, which would deserve to be discussed at length, can be roughly justified by the pregnancy of this structure within mathematics in general. Further, it enhances the concept of function, also central in mathematics[8].

[6] Cf. H. Freudenthal [1973] : "There are levels of rigour, and for each subject matter there is a level of rigour adapted to it ; the learner should pass through the levels and acquire their rigour."

[7] See H. Freudenthal [1983], p. 31.

[8] On this respect, cf. F. Klein [1933] : "Wir, man nennt uns wohl die "Reformer", wollen in den Mittelpunkt des Unterrichts den Funktionsbegriff stellen, als denjenigen Begriff der Mathematik des letzten 200 Jahren, der überall, wo man mathematisches Denken braucht,

A warning is appropriate: in mathematics, a structure is something which exists and does not change. All history of mathematics resulted in eliminating from this discipline any temporal connotation. Structures and theorems are fixed. How then can we consider *an evolution* of the linear structure?

As a metaphor, consider a seed becoming a sprout, then a small tree and eventually a large tree. It produces buds, leaves, flowers and fruits. Season after season, it is never exactly the same, and even it changes considerably. Nevertheless, it remains the same living being, it retains its identity. It is a being in evolution. Here we would like to consider *linearity* as a being in evolution, born in the context of magnitudes (before any idea of measure) and becoming eventually a large tree in the context of vector spaces. An important difference however is that, by nature, this evolution consists of successive mutations, each mutation being a generalization. At each stage, some new – fixed, still – structure is generated. Such discontinuities are probably the main obstacles on student's way.

In the sequel, we avoided mathematical subtleties as often as possible. Mathematicians will have no trouble sorting out the occasional short cuts we have taken in this presentation.

5 Magnitudes and sets

Magnitudes. Lengths, areas, volumes, weights and durations are examples of magnitudes[9]. Magnitudes do exist before being measured. Some of their principal properties are:

> a) Given two magnitudes of the same type, either they are equal, or one is larger than the other : *magnitudes are ordered*.
>
> b) For each type of magnitude, *there is an addition.* For example, disposing two sticks end by end along a line realizes the sum of two lengths; putting together two heavy bodies realizes the sum

eine zentrale Rolle spielt." (We, who are often named the "reformers", want to place the concept of function at the center of the instruction, for it is this mathematical concept of the last 200 years which everywhere, when mathematical thought is needed, plays a central role.).

The didactical principle witnessed by this quotation was known in Germany, at the beginning of the XX^{th} Century, as *"das funktionales Denken" (the functional thought).* For more details, see K. Krüger [1999]. The notion of function would deserve consideration as a clue for the elaboration of guidelines.

[9] For a rigorous definition of magnitudes as equivalence classes, see N. Rouche [1992].

of two weights; pouring two quantities of water in a vessel realizes the sum of two volumes.

c) Adding 2 equal magnitudes together is the same as multiplying a magnitude by 2. The same holds for 3, 4, ... So *there exists a multiplication of magnitudes by natural numbers.*

d) Any magnitude can be divided into 2, or 3, ... equal parts. So, *there is a division of magnitudes by natural numbers.*

Sets. Let us now consider sets (specifically finite sets). For the sake of clarity, we retain only sets of identical objects. Sets share with magnitudes properties a), b) and c) above. More precisely,

a′) Two sets being given, they are equal[10] or one is larger than the other (*equality* is the possibility of one-to-one correspondence).

b′) Two sets can always be *added* (put together to make a single set).

c′) Every set can be multiplied by 2, or 3, or 4, etc.

On the other hand, sets do not share property d) with magnitudes. Indeed:

d′) A set can be divided into n equal parts only if n divides its number of elements.

Sets are *discrete* whereas magnitudes are *continuous*.

Physics or mathematics ? The practical operations on magnitudes and sets (of material objects) require physical manipulations. Further, these manipulations suffer serious limitations. First and foremost, comparisons of magnitudes are not entirely precise, mainly for lack of acuteness of our sense organs. But also, for example,

too large or too small objects, as well as sets containing too many objects, cannot be handled;

whereas plane surfaces made of cardboard can be superposed for area comparison, solid objects cannot, because they do not penetrate each other;

two durations cannot be compared directly if they do not begin or end at the same instant.

[10]Here, we use the term *equal* in its everyday meaning : expressed more precisely, it refers to the equality of the cardinals of the sets.

More examples could easily be produced. At the very outset, there is no distinction between physical and mathematical properties. But *as soon as one reasons about them*, one assumes – even if only implicitly – that they exist and are exact. They are idealized. Otherwise, many propositions would remain inconclusive.

6 Proportionality before measures

A first notion of ratio. In order to introduce some situations of proportionality, we need first to know what a ratio is. Let a and b be two magnitudes. It may happen – although this is a rare event – that there exists a natural number n such that $b = n \times a$. If this is the case, we will call *n the ratio of b to a.* The ratio expresses how much b is larger than a.

Let m and n be two natural numbers. If there is a natural number p such that $m = p \times n$, then we will call *p the ratio of m to n.* Such a ratio exists only if m is a multiple of n (one should remember that here and until further notice, a ratio is, by definition, an integer.

Three examples of proportionality. Here are now three examples of *proportionality* (proportionality is linearity at its birth).

I. First example : when water is poured into a cylindrical vessel, there is a one-to-one correspondence between heights and volumes of water (see figure 1, which shows drawings instead of real vessels). Observe that heights and volumes are magnitudes of different types. By this correspondence,

> the sum of two heights corresponds to the sum of the corresponding volumes;
>
> when there is a ratio between two heights, there is the same ratio between the corresponding volumes.

We summarize this by saying that heights and volumes are *proportional.* Figure 1 is a *table of proportionality.* These properties, stated above in a scientific language, are understood and expressed in daily life in more familiar ways. For instance, people say "two times the height, two times the volume", and this refers, but only implicitly, to the correspondence of the ratios.

The arrows on figure 1 illustrate the correspondence of the sums and the correspondence of the ratios. We will call the latter *internal ratios*,

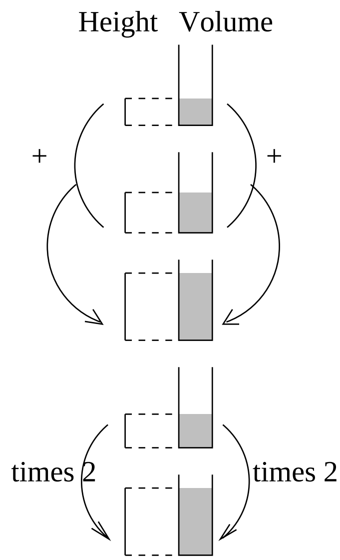

Figure 1:

because they are ratios between magnitudes *inside* a given column of the table (internal ratios will be opposed later to external ratios).

II. Another example, dealing this time with sets, is given by a situation of barter. I am willing to exchange 2 blue marbles against 3 red ones. So there is a one-to-one correspondence between the even sets of blue marbles and the sets of red marbles which are multiples of 3. Now, under this correspondence,

> the union of two sets of the first kind corresponds to the union of the corresponding sets of the second ;
>
> when there is a ratio between two disjoint sets of the first kind, there is the same ratio between the corresponding sets of the second.

III. As a third example, consider a balance with unequal arms : for instance one arm two times as long as the other, as shown in figure 2.
Let us establish a one-to-one correspondence between those weights which equilibrate the balance, one on the right pan and the other on the left one (both magnitudes of the same type). Again, by this correspondence,

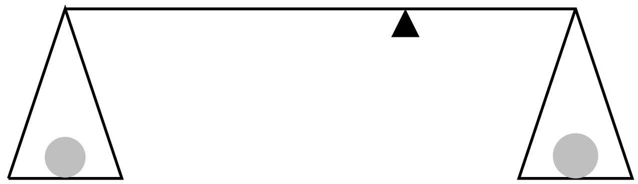

Figure 2:

> the sum of two weights of the first kind (those coming from the right pan) corresponds to the sum of the corresponding weights of the second kind (those coming from the left pan);
>
> when there is a ratio between two weights of the first kind, there is the same ratio between the corresponding weights of the second.

Constitutive properties of proportionality. The above examples illustrate the notion of *proportionality before measures,* of which the constituent elements may be stated as follows:

> there are two sets of magnitudes (or sets of sets);
>
> there is a one-to-one correspondence between these sets;
>
> every sum of two elements in one set (either one) corresponds by this correspondence to the sum of the corresponding elements in the other;
>
> when there is a ratio between two elements of one set (either one), there is the same ratio between the corresponding elements of the other (these ratios are called *internal ratios*).

The correspondence of the sums entails that of the ratios[11], via the definition of the product by a natural number as a repeated addition.

Comments. 1) Magnitudes and sets behave in the same way in this context.

2) They are first physical entities, but are idealized as they become thought objects.

[11] Remember that, up to here, the ratios are integers.

3) They are not measured. Measures will appear later in our construction of linearity.

4) Addition is probably the first, the simplest and the most fundamental binary operation encountered in the course of cognitive development: what in fact could be simpler than *putting things together*? One-to-one correspondences are probably the simplest one can imagine, for multiple correspondences imply undefined choices (several images corresponding to a single original). The conservation of sums and ratios via a one-to-one correspondence is something orderly and reassuring. Hence, perhaps, the natural character of proportionality.

7 Measure as proportionality

In this section, we consider magnitudes of any given type.

Measures in natural numbers. We know that there does not always exist a ratio (in the sense proposed at section 6 : ratios up to here are integers !) between two magnitudes. In particular, the ratio of b to a fails to exist whenever $b < a$.

In spite of this limitation, let us try to introduce some notion of measure. Let u be a magnitude, chosen as a *unit of measure.* Then, consider all magnitudes a of the form

$$a = n \times \mathrm{u},$$

where n is any natural number. In other words, we consider all magnitudes having a ratio to the unit u.

There is a one-to-one correspondence between these magnitudes and the natural numbers. We call the latter the *measures* of the former in the unit u. Under this correspondence,

> the sum of two magnitudes corresponds to the sum of their measures;
>
> if there is a ratio between two magnitudes, there is the same ratio between their measures.

Such a system of measures is interesting, because *magnitudes and their measures are proportional*: measures faithfully represent magnitudes. This means that the usual operations with magnitudes have a faithful translation into measures. Further, the addition of magnitudes and their multiplication by a natural number can now be performed on their measures. *Mental – or paper and pencil – operations replace physical ones.*

But this system of measures suffers two shortcomings :

only magnitudes of the form $a = n \times \mathrm{u}$ have a measure ;

there is not always a ratio between two magnitudes of this form.

In all cases where a is of the form $n \times \mathrm{u}$ for no n, one may look for an n such that

$$n \times \mathrm{u} < a < (n+1) \times \mathrm{u}.$$

Such an n always exists, but expresses a measure only approximately.

A partial answer to this difficulty consists in changing the unit u, choosing a new one, very small, in such a way that more magnitudes can be measured, and be measured with greater precision. But it always remains that by far not all magnitudes can be measured in whole numbers, which stays true however small u is chosen.

Of course, this statement is theoretical. In fact, with u very very small, the limitations of our sense organs are such that one does not perceive any difference between $n \times \mathrm{u}$ and $(n+1) \times \mathrm{u}$. Under such circumstances, one may always believe the measure to be a natural number.

Now the question is : how is it possible to improve this system of measures ?

Measures in (positive) fractions. *One tries to generalize the notion of ratio.* Let two magnitudes a and b be given. Instead of looking for a single natural number n such that

$$b = n \times a,$$

one looks for two numbers m and n such that

$$b = m \times (a : n),$$

where ":" means *divided by.* This is usually written in short as

$$b = (\frac{m}{n}) \times a,$$

and $\frac{m}{n}$ is called a *fraction*[12]. By way of generalization, when such an equality exists, we say that $\frac{m}{n}$ is the *ratio* of b to a.

Fractions are considered as a new type of number, generalizing natural numbers. They are endowed with an addition and a multiplication (we don't prove this here). The existence of these operations entails that the concept

[12] In the present context, *fraction* has to be interpreted as *positive fraction.*

of ratio can be extended to fractions. Namely, if three fractions are such that

$$(\frac{m}{n}) = (\frac{p}{q}) \times (\frac{m'}{n'}),$$

then $\frac{p}{q}$ is said to be *the ratio of* $\frac{m}{n}$ *to* $\frac{m'}{n'}$.

Observe that the notion of ratio went through a mutation. In fact, as long as we used natural numbers only, we said that the ratio of a magnitude (or a number) to another one expressed how much *larger* the former was than the latter. Now a ratio between two magnitudes (or fractions) expresses how much one is *larger* or *smaller* than the other : by this, we mean that, henceforth, a ratio can be smaller or larger than one.

Two observations are appropriate here :

> first, two magnitudes a and b being given, *sometimes there is no fraction* $\frac{m}{n}$ *such that* $b = (\frac{m}{n}) \times a$ *or, in other words, no ratio of* b *to* a. A popular example is when b is the length of the diagonal of a square and a the length of its side;
>
> second, *between two fractions, there is always a ratio.*

We can now extend our system of measures. Let u be chosen as a unit of measure and consider all magnitudes of the form

$$a = (\frac{m}{n}) \times \mathrm{u},$$

where $\frac{m}{n}$ is any fraction or, in other words, all magnitudes having a ratio to u. There is a one-to-one correspondence between these magnitudes and the fractions. We call the latter the *measures* of the former in the unit u.

Under this correspondence,

> the sum of two magnitudes corresponds to the sum of their measures;
>
> the ratio of two magnitudes equals the ratio of their measures.

In other words, in this new system of measures, *magnitudes and their measures are still proportional.* This is no surprise, for addition and multiplication of fractions are defined in the only way ensuring this proportionality. The operations on fractions are the exact counterparts of the operations on magnitudes[13].

[13] To the extent that, in elementary teaching, the operations on fractions are defined via the corresponding manipulations of magnitudes, which is quite sensible.

Measuring in fractions is much more effective than measuring in natural numbers. In fact, a unit u being chosen, there are many more magnitudes of the form $(\frac{m}{n}) \times \mathrm{u}$ than of the form $n \times \mathrm{u}$.

However, this system of measures suffers a shortcoming:

> namely, *only magnitudes of the form* $a = (\frac{m}{n}) \times u$ *have a measure.*

When a magnitude a cannot be measured as a fraction, then one can look for a fraction $\frac{m}{n}$ such that

$$(\frac{m}{n}) \times \mathrm{u} < a < (\frac{m+1}{n}) \times \mathrm{u}.$$

Such a fraction always exists, and – even better –, n can be chosen arbitrarily large, such that, u being chosen, the measure of a can be estimated with an arbitrary precision.

Now, in spite of the improvements pointed out, the question remains: what can be done to improve, to complete, this system of measures ?

Measures in (positive) reals. *One again tries to improve the notion of ratio.* We will not here enter into details. A new type of numbers, called the *reals*, is created, of which naturals and fractions are particular cases[14]. They are such that, *a and b being two given magnitudes, there always exists a real α such that*

$$b = \alpha \times a.$$

By way of generalization, α is called the *ratio* of b to a.

The real numbers themselves are endowed with an addition and a multiplication. The concept of ratio can then be extended to real numbers themselves: given two reals α and β, there always exists a third one γ such that

$$\beta = \gamma \times \alpha.$$

We call γ *the ratio of β to α.* Let us now extend again our system of measures. Any magnitude u being chosen as a unit of measure, *every magnitude is of the form $a = \alpha \times u$ for some real α.* There is a one-to-one correspondence between magnitudes and reals. We call the latter the *measures* of the former in the unit u. Under this correspondence,

> the sum of two magnitudes corresponds to the sum of their measures;
>
> the ratio of two magnitudes equals the ratio of their measures.

[14]Here *real* has to be interpreted as *positive real*

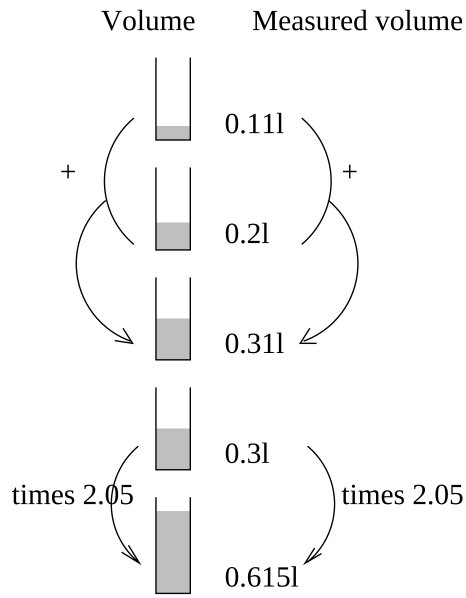

Figure 3:

Magnitudes and their measures are still proportional.

Figure 3 illustrates the proportionality of volumes and their measures in liters. The correspondence of the sums and that of the internal ratios are again illustrated by appropriate arrows.

Unlike the preceding ones, this system of measures suffers no more shortcomings. In fact, as already observed,

> all magnitudes are measured in reals;
>
> there is a ratio between any two magnitudes, as there is a ratio between two reals.

So this is, at least from a theoretical point of view, a perfect system of measures. Why from a theoretical point of view? First because practical measures fail as soon as too high a precision is required, and then only approximations are possible. Second because, whereas natural numbers are rather easy to add, multiply and divide, these operations are more complicated for fractions, and much more complicated for reals. *The real numbers yield a numerical characterization of the continuum,* but at the expense of a rather cumbersome theory, in several respects unnatural.

8 Proportionality of measures

Magnitudes of the same type. As a first example, consider two cylindrical vessels with unequal bases, put side by side on a horizontal table. Let us pour water up to the same level in both. There is a one-to-one correspondence between such volumes, also when they are measured (using the same unit). This is a correspondence between magnitudes, but also between numbers, namely those measuring the volumes. This is illustrated by a table showing only numbers (see table 1). This is a table of proportionality, with the now well known properties : the correspondence of the sums and the correspondence of the internal ratios.

first vessel : volume in liters	second vessel : volume in liters
0.2	0.4
0.4	0.8
0.6	1.2
0.8	1.6
1.2	2.4

Table 1:

But there is more. We have volumes on each side, and thus there is a ratio between any two corresponding volumes, equal to the ratio of the corresponding measures. In our example, the base area of the second vessel is 2 times the base area of the first, which entails that any measure on the right equals 2 times the corresponding measure on the left. This ratio – the same for all couples of measures – is called the *external ratio*, *external* because it organizes the passage from one column to the other. Of course, the passage from the right column to the left one requires the inverse ratio, $\frac{1}{2}$ in our example. In table 1, we might have drawn arrows (on the model of figures 1 and 2) to show the external ratio, as well as the correspondence of the sums and that of the internal ratios.

Generalizing from this example of volumes, we may state the following :

> in a table of proportionality between magnitudes of the same type measured in the same unit, there is an external ratio, which is the ratio between any two corresponding measures.

This ratio is also called the *coefficient of proportionality.*

An important property is that the existence of the external ratio entails the correspondence of the internal ratios, and conversely. This we don't prove here.

A second example will be of the utmost importance in the sequel. Consider, as in figure 4, a horizontal line OP and an inclined one OQ.

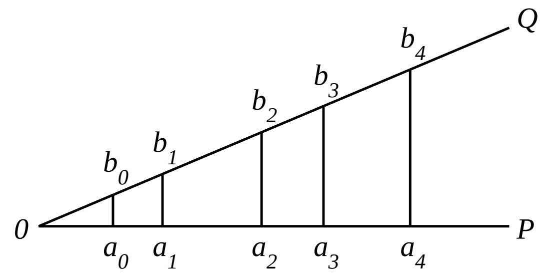

Figure 4:

The horizontal segments Oa_0, Oa_1, Oa_2, etc. are in one-to-one correspondence with the vertical ones a_0b_0, a_1b_1, a_2b_2, etc. The same is true of their measures using any unit, e.g. the centimeter. This correspondence is a proportionality. It is illustrated in table 2. Here, the external ratio is $\frac{1}{2}$. The

horizontal segments : length in cm	vertical segments : length in cm
1.5	0.6
2.5	1
4.5	1.8
5.75	2.3
7.5	3

Table 2:

following is a fundamental property of this proportionality:

> disregarding the line OQ, let us draw any number of *proportional* segments like Oa_0, a_0b_0, or Oa_1, a_1b_1, or Oa_2, a_2b_2, etc. Then it appears that all points b_0, b_1, b_2, etc. are aligned.

As a third – and also fundamental – example, let us mention the reproduction of objects at a given scale. If an object A' reproduces an object

A at scale 0.4, then the distance between any two points of A' equals the distance between the corresponding points of A multiplied by 0.4. Distances in A' are proportional to distances in A, and the external ratio is 0.4. In this context, *external ratio* and *scale* are synonyms.

Magnitudes of different types. Table 3 shows on the left a certain number of time intervals, and on the right the distances covered by a traveller during these intervals. This is a table of proportionality.

durations in hours	distances in kilometers
0.25	1
0.5	2
0.75	3
2	8
4	16

Table 3:

Let us now compare this table with tables 1 and 2. In table 1, we had volumes in both columns, and in table 2 we had lengths in both. There is no difficulty to define a ratio between two magnitudes of the same type. Therefore, we were able to define an external ratio for tables 1 and 2. In table 3, we have magnitudes of different types. There is no ratio between two such magnitudes. Nobody will ever discover a number which, multiplying an interval of time, would yield a distance. So far, so good, for pure magnitudes. But in table 3, the intervals of time and the distances are measured, and measures are numbers. And between two numbers (zero excluded), there is always a ratio. In table 3, the external ratio between measures is 4. But the numbers representing magnitudes depend on the units of measure and therefore the ratio changes if one changes the latter. One takes this dependence into account by saying that the ratio, also called the *velocity* of the traveller, is of 4 *kilometers per hour*. This is also written as $4\frac{\text{km}}{\text{h}}$.

One could also draw arrows on table 3, to show the external ratio, as well as the correspondence of the sums and that of the internal ratios.

Generalizing this observation, we may conclude :

> in a table of proportionality between two magnitudes of different types, both measured in a given unit, there is an external ratio, which varies with the units.

A consequence of this dependence on units is that, for magnitudes of different types, internal ratios are easier to understand and to manipulate than the external ratio[15]. Hence the success of *the rule of three* in the resolution of proportionality questions, this rule relying upon internal ratios only.

Straight line graphs. Figure 5 is a diagram of the traveller's motion above. It relies on the following proportionalities:

> first and foremost, distances and durations, both physical magnitudes, are proportional before any measures : see section 6 for this meaning of proportionality;
>
> durations are measured in hours, the measures being proportional to the durations;
>
> the durations in hours are transformed into lengths in cm via the proportional rule that 1.2 cm represents by 1 h; this transformation is also a proportionality;
>
> these computed cm are marked on the duration axis; this is the reciprocal action of a measure: passing from a measure to the corresponding physical magnitude, here a length along the axis;
>
> distances are measured in kilometers, the measures being proportional to the distances;
>
> the distances in km are transformed into lengths in cm, via the rule that 0.3 cm represents 1 km;
>
> these computed cm are marked on the distance axis.

There is proportionality at every stage. The result is a straight line diagram. Why ? We have seen above that an appropriate disposition of proportional lengths yields such a rectilinear diagram. In the example above, due to the whole chain of proportionalities, the lengths marked on the duration axis are proportional to the lengths marked on the distance axis. This explains the straight line.

This example of a motion is typical of a general practice: for the sake of representation, all kinds of magnitudes are transformed into lengths, via measurements and scales. Lengths are the most "readable" magnitudes. That is why most diagrams are based on lengths. The transformation of magnitudes into lengths is a central application of proportionality. No doubt

[15] This is common observation in schools and has been strikingly illustrated by I. Soto [1994], when observing illiterate people in Chile.

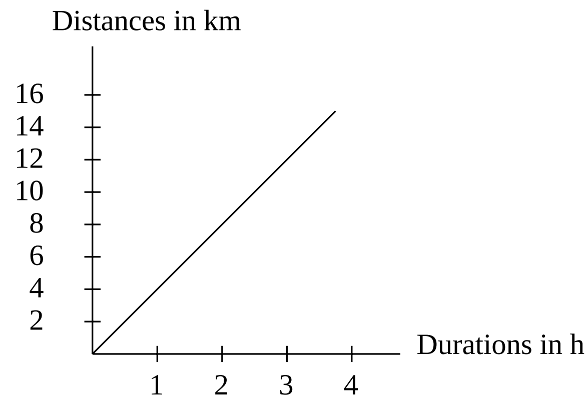

Figure 5:

that proportionality is a key to faithful representation of magnitudes and mappings.

9 Taking negative quantities into account

Figure 6 shows a road with distances in kilometers, measured from some point 0, positive towards East and negative towards West. We used positive and negative numbers to locate points on the line.

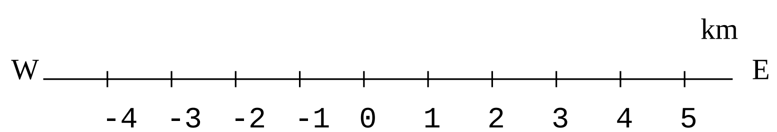

Figure 6:

This implies a new interpretation of the notion of order. Up to here, *larger* and *smaller* were interpreted in the common meaning. From now on, a number will be said to be *larger* than another one, if its corresponding point on the line lies at the right of this one. Whereas the abstract laws of order, e.g. transitivity, are still valid, the interpretation changes considerably. This is a drastic mutation. However, the previous interpretation of order is still valid for positive numbers.

Table 4 gives the position of a traveller on this road at various moments in time.

time in hours	position in kilometers
−0.75	+3
−0.50	+2
−0.25	+1
0	0
+0.25	−1
+0.50	−2
+0.75	−3
+1.00	−4
+1.25	−5

Table 4:

The situation is similar to that of the traveller of section 8 in this respect that in both cases, a person travels at $4\,\frac{\mathrm{km}}{\mathrm{h}}$. But there are important differences. For instance, concerning such a motion, one may ask the following questions:

> Knowing the position of the traveller at time (+0.50), where was he 2 hours earlier?
>
> Knowing the velocity of the traveller and his initial position, where is (or was) he at a given time, positive or negative?

To answer such questions conveniently, one introduces an addition and a multiplication on the new type of numbers. These numbers – the reals in their full extension – generalize the positive reals. This extension is a mutation, as is shown, amongst other things, by two spectacular changes:

> the sum of two reals is no longer necessarily larger than either of its terms;
>
> the product of two real numbers obeys the famous law of signs, irrelevant in the case of positive reals.

Further, *one generalizes once more the notion of ratio.* In spite of the major changes mentioned above, *its definition remains phrased identically* :

> if α, β and γ are reals, and if

$$\beta = \gamma \times \alpha,$$

then γ is called the ratio of β and α.

Further, and as before, any two reals (zero excepted) have a ratio.

Whereas the definition of ratio does not formally change, its meaning does. The ratio is now something more than an expression of how much larger or smaller a number is as compared to another. It also takes into account the fact that the numbers being compared can be on either side of the origin 0 on the number line.

Now, in spite of these drastically new interpretations of addition and ratio, a table of proportionality still exists and remains formally the same, or in other words, obeys the same definition. The permanence of the structure can be observed on table 4. Figure 7 shows that the graph of this proportionality is still a straight line.

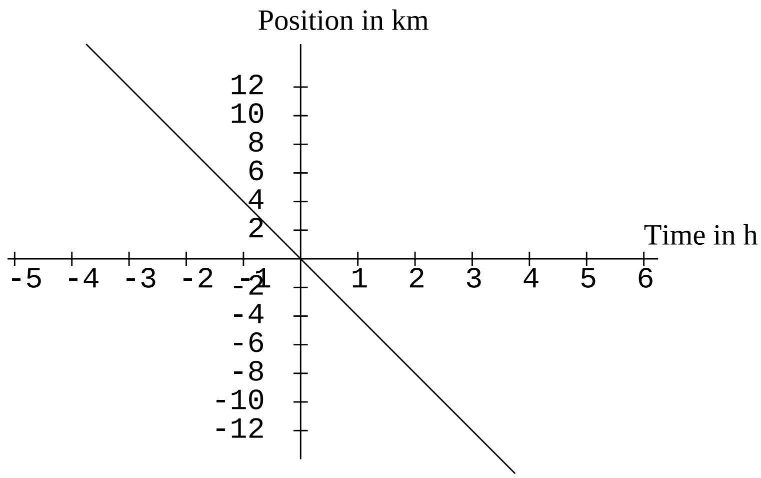

Figure 7:

Let us emphasize that, although the notion of proportionality remains structurally the same, its interpretation, its relation to real situations has changed considerably. However, whenever only positive numbers and ratios are concerned, nothing is changed.

10 Abstract proportionality, linearity

In our examples up to here, all numbers expressed measures of volumes, lengths, distances covered by a traveller, distances from the origin on an axis, durations, velocities, etc. Consequently, we took into account the units of measure, and our external ratios were sometimes composed magnitudes such as (measured) velocities. Let us now "cut the umbilical cord" of proportionality.

We drop any reference to measures and define a proportionality between pure numbers. Such a proportionality is a one-to-one correspondence between the system of real numbers and itself. It can be illustrated by a two column table, showing couples of real numbers (as many as one wishes) facing each other, with the following properties: the correspondence of the sums, the correspondence of the internal ratios, the existence of an external ratio, a straight line graph. This correspondence is also called a *real linear mapping.*

At first sight, such an abstract notion of proportionality says nothing about nothing, except itself. What is its use? It is a pure structure – something common in mathematics –, a kind of model ready to be applied either to concrete situations via appropriate interpretations, or to other mathematical contexts.

11 Oriented magnitudes, vectors, linear transformations

Some of the magnitudes studied up to here were measured by positive reals, and could be represented on a graduated half axis. Some others were measured by reals in general, they were directed magnitudes, representable on a graduated axis. Both types are one-dimensional. However, in everyday life as well as in geometry, physics, etc., there are magnitudes oriented in the plane or in space. For instance, the changes of position of a point, the translations of a solid object, the velocities of a point moving in a plane or in space, the forces, etc., all these magnitudes can no longer be measured by a real number, nor be represented by a point on a line. In addition to being large or small, they point to some direction. We call them *oriented magnitudes.* They can be represented by an arrow. Of course, the common notion of proportionality, the one studied up to here, cannot be applied as such to this new kind of magnitudes, it has to be adapted.

In principle, just as we did with ordinary magnitudes, we now ought to deal with *concrete* oriented magnitudes, i.e. we ought to explore the context in which this new kind of proportionality arises. Such a development, a must on the way towards vector spaces in every curriculum, would be too long here. So let us limit ourselves on the one hand to the displacements of a point, and on the other to the forces.

If a moving point passes from a fixed position A to another A', and then from A' to A'', its displacements can be represented, as on figure 8, by two arrows, namely $\overrightarrow{AA'}$ and $\overrightarrow{A'A''}$ (in straightforward notations). The resulting displacement is $\overrightarrow{AA''}$ (see figure 8). We call it the *sum* of the other two.

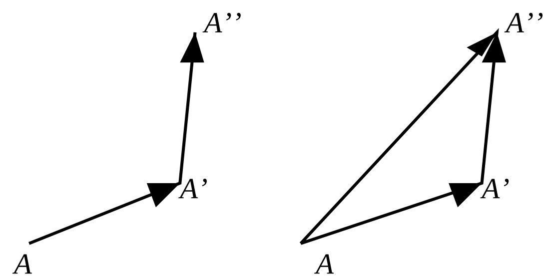

Figure 8:

Figure 9(a) represents two forces $\overrightarrow{f}$ and $\overrightarrow{g}$ applied to a given point P (e.g. a point possessing a mass). One may imagine that these forces are obtained by pulling at two strings attached to A. They have an effect on P (an acceleration). One obtains the same effect on P while replacing $\overrightarrow{f}$ and $\overrightarrow{g}$ by another force $\overrightarrow{h}$, the one appearing in figure 9(b). We call this third force the *sum* of the other two.

Let us now replace displacements and forces by "abstract" arrows: we call them *vectors*[16]. Two vectors can be added either like displacements or like forces. The result is the same. So we have an *addition of vectors*.

In the same way we defined the multiplication of an ordinary magnitude by a real number, we define the multiplication of a vector by a real number. This is known as the multiplication of a vector by a *scalar* (a new name for real number). We give no details here.

What now about ratios ? The ratio of a vector to another one "should be" a number which, multiplying the latter, yields the former. But such a

[16]We allow ourselves to confuse *arrow* (or *oriented segment*) and *vector*.

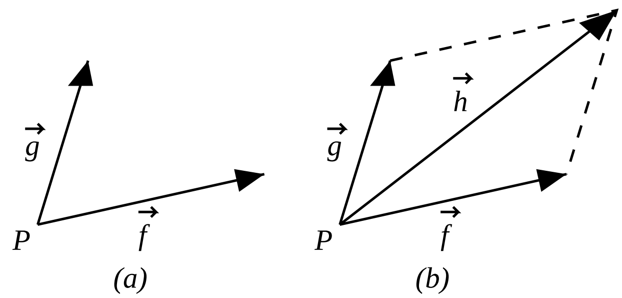

Figure 9:

number exists if and only if both vectors have the same direction. A deadlock ?

While, as we just saw, it is usually impossible (in a plane) to pass from a vector to another one using a single real number, it is usually possible to pass from *two* vectors to a third one using *two* real numbers. Let $\overrightarrow{c}$ be a vector, and $\overrightarrow{a}$ and $\overrightarrow{b}$ two non vanishing vectors with different directions. Then there exist two reals λ and μ such that $\overrightarrow{c} = \lambda\overrightarrow{a} + \mu\overrightarrow{b}$. Such an expression is called a *linear combination* of $\overrightarrow{a}$ and $\overrightarrow{b}$. So *the linear combination is, in some way, a generalized form of the ratio.*

Let us now see wether we can rely on the idea of linear combination in view of generalizing the idea of proportionality. As a preliminary, let us first look at the way a table of proportionality, in the usual sense, can be generated. One writes a real nonvanishing number a in the first column. Then one writes, in the same column, as many real numbers as one wishes. Each such number can be written in the form $b = \lambda a$, where λ is a real number. This number is the (internal) ratio of b to a. Then, in the second column and facing a, one writes a number a' chosen at will. At last, using the correspondence of the internal ratios, on writes the numbers $b' = \lambda a'$ in the second columln, facing the numbers $b = \lambda a$ in the first. The external ratio of the table is $\frac{a'}{a}$.

Let us now imitate this procedure in the context of vectors. In the first column, we write two vectors $\overrightarrow{a_1}$ et $\overrightarrow{a_2}$, nonvanishing and having different directions. In the same column, we write as many vectors as we want, of the form

$$\overrightarrow{x} = x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2}, \qquad (*)$$

where x_1 and x_2 are real numbers. Next, we write in the second column, facing respectively $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$, two vectors $\overrightarrow{a_1'}$ et $\overrightarrow{a_2'}$ chosen at will. At last, facing each vector of the form (*), we write, in the second column, the vector

$$\overrightarrow{x'} = x_1\overrightarrow{a'_1} + x_2\overrightarrow{a'_2}.$$

This table is such that any linear combination of two terms chosen in the first column corresponds to the linear combination of the corresponding terms of the second column. Let us put this in another, more suggestive way. Whenever two vectors $\overrightarrow{x'}$ and $\overrightarrow{y'}$ face $\overrightarrow{x}$ and $\overrightarrow{y}$, then $\overrightarrow{x'} + \overrightarrow{y'}$ face $\overrightarrow{x} + \overrightarrow{y}$. There is correspondence of the sums. Further, whenever $\overrightarrow{x'}$ faces $\overrightarrow{x}$, then $\lambda\overrightarrow{x'}$ faces $\lambda\overrightarrow{x}$, for any real λ. There is correspondence of the internal ratios, in this restricted sense (vectors of same direction).

We just defined a *linear mapping* for vectors in two dimensions. This notion generalizes the proportionality, which may thus be named *linear mapping* in one dimension. On can define in the same way linear mappings in three dimensions. In this passage from one to two and then three dimensions, there appears a large variety of new phenomena. Let us only mention that all classical plane and space geometries are developed in the context of linear mappings. We now see how proportionality, at the origin a modest seed (fed by many contexts), became a large (abstract) tree full of blossoms.

This however is not the end of the story. For linearity develops next from plane to space, then from 3-space to n-dimensional spaces, where algebra expresses geometrical facts by necessity, imagination falters, and spatial intuitions although still work. Not to mention the infinite dimensional spaces of functional analysis, with the dramatic discovery that not all linear mappings are continuous.

At the end of this long journey, let us remark that, when studying linear phenomena, one should not forget to contrast them with nonlinear ones. Unfortunately, we don't have enough space here to develop this pedagogically fundamental viewpoint.

CONCLUSIONS

In this paper, we outlined a genetic theory, the one of the linear structure. Relying on this example, let us try to understand more precisely the nature of a *genetic theory* and its relevance. As we have seen, a genetic theory consists in a sequence of notions, theories, structures, starting from common experience and thought and aiming at constituted mathematics. Essentially, these notions are of an increasing generality, each one being relevant in a context broader than the preceding one. Passing from one to the next is strongly motivated by questions, observed shortcomings, obstacles or the need of a new understanding. Each new theoretical stage appears as an adapted, efficient answer to the encountered difficulties, to the new contexts taken into account. It is rooted in everything that precedes, its main source of meaning and intuitions.

The genetic theory leads to formal theories (rationals, reals, vector spaces, ...). Underway, these theories cut their ties to common experience, to concrete contexts. But they remains capable to rediscover them at any moment, because they started therefrom. They are not like theories which, conceived at a formal level, seek their applications afterwards.

Therefore, a genetic theory does not proceed from axioms and definitions to lemmas, theorems and corollaries. It is not *globally* deductive. But then, one may rightly wonder where are, in such a scheme, the proofs, the deductions. The fact is that, as stated before, each new notion or theory results from a mutation (remember the observed successive meanings of the term *ratio*). And as there is at each stage change of a definition or a property, one has to reorganize things each time. And reorganizing means bringing some new deductive order. A genetic theory requires proofs at every stage.

Now, after all, what is the use of a genetic theory? Here are some tentative answers.

A genetic theory throws light on the relations between everyday things and phenomena on the one hand and their mathematical expressions on the other. It shows the strong, although sometimes hidden, reasons underlying the construction of mathematics. It helps to identify what might be called a mathematical culture, closely connected to its sources in common thought. If it were not a little ambitious, one might say that it is the result of an effort towards a *rational epistemology*[17].

But what can be the use of a genetic theory in teaching?

To avoid damageable confusions, let us stress the fact that *a genetic theory cannot directly inspire a curriculum.* Namely, things happen, in the family or school progression of a child, in an order devoid of such a degree of rationality. For example,

> children do not discover the properties of plain (not yet measured) magnitudes before getting acquainted with some mea-

[17] *Rational epistemology* as opposed to *historical epistemology*, the one studying the secular evolution of concepts, and to *genetic epistemology* (in Piaget's sense), studying the emergence of concepts amongst children. Rational epistemology would rather result from a search for the relations between concepts, as they appear to an adult using his reason. It would be a product of the *lumière naturelle* (the natural light) in the sense of Descartes or Pascal.

sures;

they do not learn all the essential properties at the basis of the natural numbers before meeting some simple fractions;

they observe negative numbers on the thermometer before learning how to use them fully in algebra;

they experience velocities and forces and certain of their properties before representing them by vectors.

After this warning, let us now examine the reasons justifying the elaboration of genetic theories.

Hopefully, if the authors of curriculums or textbooks clearly understand the construction of the subject matter, they will more efficiently introduce new concepts, and only in case of necessity or of reasonable usefulness.

Hopefully, a teacher having assimilated a well constructed guideline, on the one hand will have at her disposal some keys construct her course and to interpret the difficulties encountered by the students, and on the other will fully realize which stage her class has reached, what logically (not chronologically) comes before that stage and what is being prepared.

Hopefully, a teacher having thoroughly studied some genetic theory would realize that mathematics have some roots in reality, that their form and articulations are intelligible, and would have an increased interest in this science and would fear it less.

Let us conclude by a single suggestion: *that in each country, a permanent group of teachers of all levels and of mathematicians, studies the curriculum as a whole, from early childhood to adulthood, from common knowledge to mathematics, avoiding any premature concept.* We insist on *a permanent group*, for casual discussions are likely to be inefficient on such a subject, where mutual understanding has proved to be so difficult. Isn't it essential to have at work on the same project those people who know the children intimately and those who know mathematics intimately?

This paper comes as a continuation of a collective study of the CREM on the same subject (see N. Rouche ed. [2002]). I wish to express my gratitude to its authors: M. Ballieu, M.-F. Guissard, P. Laurent, C. Lemaître, L. Lismont, P. Tilleuil, T. Sander, E. Vanderaveroet, F. Van Dieren, M.-F. Van Troeye, J. Van Santvoort and P. Wantiez.

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Contents

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