THE CONCEPTS OF CURVES AND EQUATIONS IN EARLY AMERICAN & BRITISH TEXTBOOKS ON ANALYTIC GEOMETRY

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ABSTRACT

Curves and equations, also known as the Cartesian Connection, is an important and fundamental concept in plane analytic geometry, but it has been shown that students lack a rigorous understanding. Focusing on the concept of curves and equations, we investigated 84 American and British analytic geometry textbooks published from 1826 to 1963. The study found that three types of non-rigorous definitions were prevalent in early textbooks published before the 20th century. The first type merely recognized a connection between curves and equations but did not express the nature of the strict correspondence. The second type, although intentionally emphasizing that each point needs to be satisfied, examined only one aspect of the correspondence, neglecting the bidirectional nature inherent in the relationship between curves and equations. The third category of non-rigorous definitions confused the concepts of curves with functions. Definitions of curves and equations have become increasingly rigorous in post-20th century textbooks, and these rigorous definitions can be divided into descriptive definitions, definitions based on the concept of set, and definitions based on sufficient and necessary conditions. It is noteworthy that the second category of non-rigorous definitions was not eliminated until the middle of the 20th century. Epistemological barriers in the historical development of the curves and equations concepts can become cognitive barriers for students in the classroom as well. Instructional strategies can be developed to guide students in recognizing non-rigorous definitions, leading them through a historical reconstruction of how concepts related to curves and equations have evolved. This approach facilitates a natural progression from qualitative to rigorous understanding. In addition, early textbooks also provide a variety of methods for verifying the Cartesian Connection, which provide abundant materials for teaching.

1 Introduction

The development of analytic geometry was a gradual process. Before its formal establishment, Apollonius (c. 262 B.C.-190 B.C.) and N. Oresme (1323-1382) used coordinate axes to study curves, while F. Viète (1540-1603) applied algebraic methods to solve geometric problems. In the 17th century, R. Descartes (1596-1650) and P. de Fermat (1601-1665) combined these approaches, establishing the connection between curves and equations within a coordinate system. This lengthy evolution suggests that students may inevitably encounter difficulties when learning about curves and equations.

Curves and equations, also known as the Cartesian Connection, refers to the principle that “a point is on the graph of the line l if and only if its coordinates satisfy the equation of *l*.” (Moschkovich et al., 1993). Despite its significance, research on how to effectively teach the Cartesian Connection remains limited. For instance, a study of high school students in Shanghai found that while textbooks introduce curves and equations, teachers rarely require students to verify the “if and only if” conditions for deriving equations, leading to confusion (Ruan et al., 2012). Similar findings by Knuth et al. (2000) and Moon et al. (2013) confirm that curves and equations remain challenging for both students and teachers.

To explore this further, we analyzed 84 American and British analytic geometry textbooks published between 1826 and 1963. The following research questions guided our investigation:

How were curves and equations defined in early textbooks?

How did these definitions evolve over time?

2 Methods

This research is part of a program on studying early American and British textbooks (Figure 1), organized by the HPM Community in Shanghai, China. The program aims to provide resources for teaching from a historical perspective, promote a comprehensive understanding of mathematical concepts among pre-service and in-service teachers, and offer insights for the development of curriculum materials. The program primarily focuses on textbooks published between the 18th and 20th centuries, a period during which modern mathematical knowledge systems gradually evolved from exploratory constructions into standardized forms. Textbooks from this era not only preserve the historical traces of conceptual developments in fields such as algebra, geometry, trigonometry, and analytic geometry, but also reflect the early shaping of modern instructional systems.

For this research, we focused specifically on analytic geometry textbooks, which became established as a distinct genre relatively late in the evolution of mathematics education. Based on a search in the HathiTrust Digital Library using the keywords “Analytic Geometry” and “Coordinate Geometry,” the earliest suitable American or British textbook we identified was The Principles of Analytical Geometry, published in 1826.

A total of 84 analytic geometry textbooks published between 1826 and 1963 were selected. The textbooks were grouped into 20-year intervals, with their publication dates illustrated in Figure 2. In cases where a textbook was republished by the same author, the earliest version was selected unless substantial changes were made in later editions; in such cases, the revised edition was treated as a distinct textbook. Textbooks published after the mid-20th century were generally excluded, as analytic geometry by that time had become a mature and highly standardized subject, rendering later works less informative for the purposes of this historical and conceptual study.

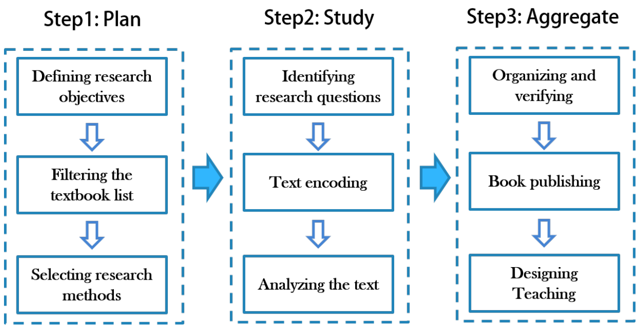

Figure 1. Flowchart of the program

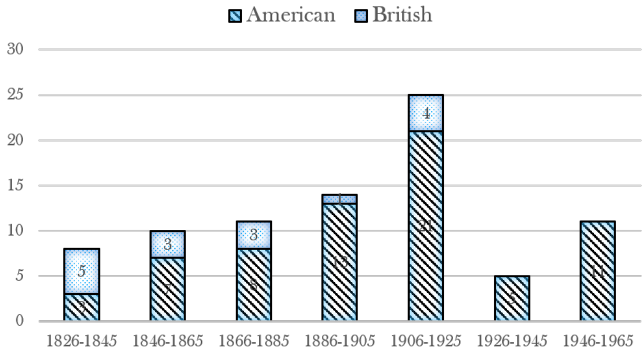

Figure 2. Temporal distribution of 84 early textbooks

3 Findings

Briggs (1881) emphasized the foundational role of this connection in his textbooks, noting: “This close relationship between curves and equations is the foundation of our discipline and warrants careful study. Once this concept is understood, the subject becomes natural and accessible; however, for anyone who neglects it or has an unclear grasp, analytic geometry will be incomprehensible.” (Briggs, 1881, p. 12).

Although the idea of connecting curves and equations emerged with the works of Descartes and Fermat, their use of coordinates was limited to positive values. It was not until mathematicians such as J. Wallis (1616-1703) began to consciously incorporate negative coordinates that a more complete one-to-one correspondence between plane curves and bivariate equations became possible, as noted by Kline (Kline, 1990, p. 322). The rigorous understanding of this connection, however, took much longer to develop, as evidenced by the definitions found in early textbooks, which transitioned from non-rigorous to rigorous over time.

3.1 Non-rigorous Definitions

Some early textbooks offered definitions of the connection between curves and equations that were not rigorous enough. These can be categorized into three types.

3.1.1 Type 1: Qualitative Definitions

The first category, qualitative definitions, recognized the connection between curves and equations without expressing the strict or formal nature of their correspondence. A total of 15 textbooks adopted this approach. For example, Biot (1840) described the mutual representation of curves and indeterminate equations as follows: “We may regard every line as susceptible of being represented by an equation between two indeterminate variables; and, reciprocally, every equation between two indeterminates may be interpreted geometrically and considered as representing a line, the different points of which it enables us to determine.” (Biot, 1840, p. 27). This explanation addressed the relationship in a general and rhetorical manner, rather than in a symbolic or algebraically rigorous way.

3.1.2 Type 2: Single-direction Definitions

The second category, found in 14 textbooks, fell into the trap of single-direction definitions. Although these definitions emphasized that each point must satisfy the equation, they focused on only one direction of the correspondence, neglecting mutuality. For instance, Young (1830) stated: "The line which any equation represents, or in which the variable point (*x,y*) is always found, is called the locus of that equation, or of the point (*x,y*)."(Young, 1830, p. 36). This definition focused on the equation determining the curve but overlooked the converse.

A similar asymmetry appears in Hardy's definition, which stated: "The equation of a locus is the equation which is satisfied by the coordinates of every point on the locus, and by no others" (Hardy, 1897, p. 14). While precise, this description emphasized sufficiency but did not explicitly require that all solutions of the equation must lie on the locus—thus potentially compromising completeness.

3.1.3 Type 3: Definitions Confused Curves with Functions

The third category confused the concepts of curves and functions. For example, Riggs (1911) conflated the terms "equation" and "function" by stating: "In each of the examples to be next studied, some simple locus of points will be considered, and the equation which expresses the dependence of the ordinate of any point of the locus upon the abscissa of the point will be derived. This equation will be known as the equation of the locus."(Riggs, 1911, p. 41). Similarly, Dowling (1914) argued: "The equation of the locus defines y as a function of x, and the locus itself is the graph of this function."(Dowling, 1914, p. 53).

The confusion between equations and functions likely arose from historical developments. The function concept, as introduced by J. Bernoulli (1667-1748) and L. Euler (1707-1783), bore a formal similarity to equation expressions in 19th-century algebra textbooks. Early algebra textbooks often defined functions through equations, and F. Klein (1849-1925) later emphasized unifying mathematical content under the concept of functions. This historical entanglement between "equation" and "function" has been well-documented (Liu et al., 2021).

3.2 Rigorous Definitions

Over time, textbooks increasingly provided rigorous definitions of the relationship between curves and equations. These definitions can be grouped into three types.

3.2.1 Type 1: Definitions by Describing

The first type of rigorous definition is derived through direct description. For example, Peck (1876) defined it as follows: "The equation of the locus of points satisfying a given condition is an equation in the variables x and y, representing the coordinates, such that the coordinates of every point on the locus satisfy the equation; conversely, every point whose coordinates satisfy the equation lies on the locus." (Peck, 1876, p. 42).

3.2.2 Type 2: Definitions Based on Set

A total of 6 textbooks utilized the theory of set to define the relationship between curves and equations. Hamilton (1826) was the first to propose: "Let $f(x,y)$ be an indeterminate equation between x and y; then, the set of points (x,y) will form a curve, called the locus of the equation $f(x,y) = 0$."(Hamilton, 1826, p. 52).

Although Hamilton (1826) offered a rigorous definition, his proof that the locus of a linear equation is a straight line fell into circular reasoning, possibly due to the immaturity of set theory at the time (Hamilton, 1826, pp. 55-58). With the later widespread acceptance of Cantor's (1845-1918) set theory, some textbooks began using set-theoretic notation. For example, Taylor (1962) represented the intersection and union of two curves using set operations (Taylor, 1962, p. 3). However, none of the surveyed textbooks used set notation to define the correspondence between curves and equations explicitly.

3.2.3 Type 3: Definitions Based on Sufficient and Necessary Conditions

Some textbooks provided concise definitions using logical terminology. For instance, after offering a descriptive definition, Taylor (1959) stated: "In other words, for a specific point (x,y), the ordered pair (x,y) satisfies the equation if and only if the point (x,y) lies on the curve." (Taylor, 1959, p. 20).

4 The Temporal Distribution of Chapters on Curves and Equations

The analysis above highlights the widespread misconceptions surrounding the concept of curves and equations in early textbooks, as reflected in the prevalence of non-rigorous definitions. In contrast, rigorous definitions demonstrate a variety of approaches converging on the same underlying principles. Figure 3 illustrates the evolution of definitions in early textbooks, grouped by 20-year intervals.

From the figure, it is evident that prior to the 20th century, more than half of the textbooks provided non-rigorous definitions, with qualitative definitions being the most common. This indicates that many authors of early textbooks approached the relationship between curves and equations primarily from a qualitative perspective. Entering the 20th century, the proportion of textbooks offering rigorous definitions gradually increased, with descriptive definitions remaining the predominant approach.

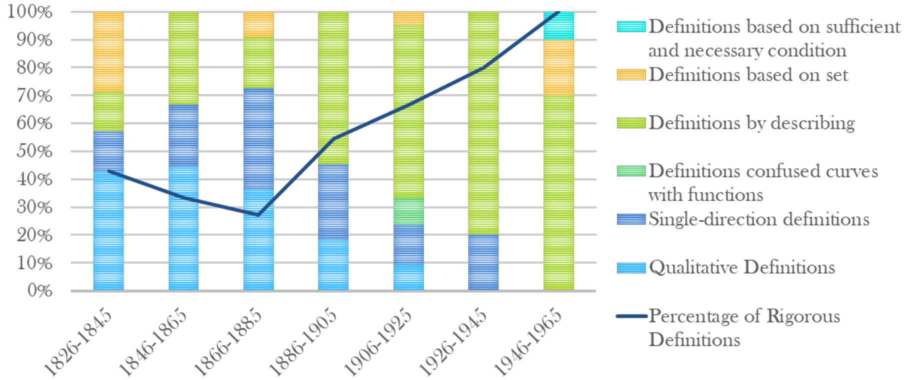

Figure 3. Evolution of the concepts of curves and equations

Notably, while more textbook authors attempted to rationalize the relationship between curves and equations quantitatively, many fell into the trap of focusing on only one direction of the correspondence. This misconception persisted well into the mid-20th century, underscoring the enduring challenges in achieving a comprehensive understanding of this foundational concept.

5 Verification

Peck (1876) emphasized that “The statement of the definition must be demonstrated, and the derived equation of the locus must be verified.” (Peck, 1876, p. 42). The verification methods in early textbooks can be broadly categorized into three types.

5.1 Type 1: Proof by Contradiction

Tanner (1898), after deriving the equation $3y-x-3=0$ for the line passing through the points $P_1(3,2)$ and $P_2(12,5)$, provided two types of proof by contradiction to demonstrate that "any point not on the line does not satisfy the equation."(Tanner, 1898, pp. 61-63).

Method 1: In the derivation process, he used the property of proportionality between corresponding sides of similar triangles. If a point is not on the line, it is impossible to form a similar triangle, and hence the proportionality $\frac{y-2}{5-2}=\frac{x-3}{12-3}$ does not hold.

Method 2: It was assumed that $P_3(x_3,y_3)$ was not on the line passing through P_1P_2. A perpendicular was drawn from P_3 to the x-axis, which intersected the line P_1P_2 at $P_4(x_4,y_4)$, where $x_3=x_4$ but $y_3\neq y_4$. Substituting into $3y_4-x_4-3=0$, it was found that $3y_3-x_3-3\neq 0$. Therefore, it was evident that no point off the line satisfied the equation $3y-x-3=0$.

5.2 Type 2: Backward Reasoning

Young (1936) used backward reasoning to verify the derivation of the standard equation of an ellipse. Starting from the equation of the locus,

$$PF_1+PF_2=2a \qquad (1)$$

he derived the standard equation of an ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (2)$$

He then verified that points satisfying equation (2) necessarily meet the condition in equation (1). By substituting values from (2) into the derivation steps, he showed:

$$\pm\sqrt{(x-c)^2+y^2}\pm\sqrt{(x+c)^2+y^2}=2a\ (a^2-c^2=b^2),$$

Four possible cases arose for the coordinates (*x,y*): Case (a): ＋＋; Case (b): －＋; Case (c): ＋－; Case (d): －－. Young (1936) then demonstrated that only case (a) satisfied the condition $PF_1+PF_2=2a$, as in the other cases, the triangle ∇PF_1F_2's two sides differed by more than the third side, 2c. This confirmed that the standard equation must meet the original condition. (Young, 1936, p. 36).

Cell (1951) further noted: "The verification in the second part can often be simplified by reversing the numerical steps, as this ensures equivalency in the derivation process without repeating each step in detail." (Cell, 1951, p. 33).

5.3 Type 3: Direct Substitution

Smith (1954) while deriving the equation $2x-y-1=0$ for the locus of points equidistant from $P_1(3, 0)$ and $P_2(-1, 2)$, set a point $P_0(x_0, 2x_0 - 1)$ satisfying the equation. He substituted P_0 into the distance formula to calculate the distances from P_0 to P_1 and P_2, verifying their equality and thus confirming the derived equation (Smith, 1954, pp. 20-22).

The above methods exhibit general applications and can be adapted based on specific contexts. Some 20th-century textbooks, such as those by Roberts (1918), acknowledged that exhaustive verification could sometimes be omitted but emphasized its fundamental importance: “This step is so similar in all examples that the student will not be required to give it, unless called for, but he should never lose sight of the fact that this is one of the essential conditions in the determination of the equation of a locus.”(Roberts, 1918, p. 48).

6 Discussion and Implications

As M. Kline (1990) observed, “The polished presentations in the courses fail to show the struggles of the creative process, the frustrations, and the long arduous road mathematicians must travel to attain a sizable structure.”(Kline, 1990, p. xi). In this regard, non-rigorous definitions, rather than being dismissed, can serve as valuable teaching resources to enhance students’ understanding of the connection between curves and equations.

First, qualitative definitions align well with students' initial conceptual understanding and can provide a foundation for deeper exploration. The second type of non-rigorous definition, which focuses on only one direction of the correspondence, highlights a common misconception. This can be utilized as an opportunity to guide students in identifying counterexamples and developing a more complete understanding of the bidirectional relationship. Finally, the third type, which conflates curves with functions, reflects a persistent confusion that many students also experience. This emphasizes the importance of distinguishing between the two concepts in instructional settings, fostering a clearer and more precise understanding.

As Schubring (2011) argues, using “historical errors” rather than exclusively celebrating historical successes can be more effective in helping teachers recognize the origins of common student difficulties. In this sense, the flawed or incomplete definitions in early textbooks are not merely historical artifacts, but potential pedagogical tools. By addressing these non-rigorous definitions systematically, educators can not only help students overcome common cognitive obstacles but also provide historical context that enriches the learning process. This approach underscores the value of integrating historical perspectives into mathematics instruction, enabling students to appreciate the gradual evolution of mathematical rigor while cultivating critical thinking and problem-solving skills.

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