EXPECTATIONS REGARDING FRENCH PROSPECTIVE TEACHERS' KNOWLEDGE IN GROUP THEORY

A (tentative) historical survey

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ABSTRACT

History of mathematics and history of mathematics education generate insights for didactical reflection (Artigue, 1991; Chorlay & De Hosson, 2016; Chorlay et al., 2022). This paper aims to present elements of history of mathematics education that will contribute to a PhD dissertation (A. Rolland) on the links between mathematical knowledge acquired in tertiary education and its use by secondary school teachers (a topic known as "Klein's second discontinuity"). The specific topic chosen is that of knowledge about group theory, and the aim of the PhD is to produce a capstone course, that is a programme for pre-service teachers training backed up by general methodological and theoretical reflections on Klein's second discontinuity. One of the challenges is to understand (and to help prospective teachers understand) the future role of abstract theories whose link with the content to be taught is not transparent. Rather than seeking to determine a priori (through mathematical and/or didactic reflection) what this role is in the case of groups, we are seeking, through a historical study of one of the main French competitive examinations for teacher recruitment (*agrégation*) from 1950 to 1990, to identify the expectations and motivations of the institution recruiting teachers with regard to future teachers. The choice of study period allows us to investigate variations in these expectations, showing that they may have been significant even before and after the New Maths period (Gispert, 2023).

1 Introduction

In France, prospective secondary school mathematics teachers typically pursue a degree in mathematics, which almost always includes an introductory course on group theory. This content is often perceived by students as disconnected from what they will be teaching (Zazkis & Leikin, 2010). This results in a (second) "discontinuity", as defined by Klein (1908/2016), with regard to this theory. He noted that upon completion of their university studies, pre-service teachers seemed unable to "discern any connection between [the task of teaching] and [their] university mathematics", and thus "[their] university

studies remain only a more or less pleasant memory which has no influence upon [their] teaching." (Klein, 1908/2016, p. 1)

This phenomenon, first identified in the late 19th century, has since been addressed through the design of adapted teaching resources (Sultan & Artzt, 2018; Usiskin et al., 2003; N. Wasserman, 2018) and by research in mathematics education (Planchon, 2022; Winsløw & Grønbæk, 2014). These studies investigate a range of approaches, including the explicit delineation of the connections between secondary and tertiary mathematics, as well as the identification of teaching practices transferable to secondary education settings. Nevertheless, this remains a topical issue in pre-service teacher education today.

This communication draws upon the ongoing PhD of Agathe Rolland, supervised by Nicolas Grenier-Boley (Mathematics Education) and Frédéric Brechenmacher (History of Mathematics). The aim of the PhD is to is to understand (and help prospective teachers understand) the future role of abstract theories whose link with the content to be taught is not transparent.

The question the authors seek to address in this paper is as follows: how can history of mathematics education inform reflection on the role of group theory in preservice teacher education?

Firstly, we will describe the type of course design that has been selected in order to address Klein's second discontinuity regarding group theory. Secondly, we will examine historical sources to clarify the expectations of the institution recruiting teachers, in relation to group theory.

2 Addressing the second discontinuity: capstone courses

One way to address the challenges of Klein's second discontinuity is by the incorporation of capstone courses for prospective teachers. Winsløw and Grønbæk explain the use of such courses as follows:

> (…) the term 'capstone course' is used to indicate a study unit which is located towards the end of an academic study program, with the aim of concluding or 'crowning' the experience, and to link academic competence and training with the needs of a professional occupation (Winsløw & Grønbæk, 2014, p. 4).

Wasserman et al. (2023) emphasise that the link between tertiary mathematics and the needs of secondary school teachers can be of varying natures, either focused on mathematical content, disciplinary practices, or on didactical practices. Murray and Star (2013) identify two types of capstone courses focused

on mathematical content. The first type, labelled "secondary mathematics from an advanced standpoint", aims to revisit secondary mathematics content with the depth provided by tertiary mathematics. The second type, labelled "tertiary mathematics with connections", runs through the curriculum of tertiary mathematics, focusing on the importance of some notions for secondary mathematics.

The main goal of the dissertation is to identify design principles for a capstone course of the first type described above on groups, then design – and possibly implement – such a course. This entails attempting to answer two questions: *what* should pre-service teachers know about groups? And *why* is it useful, necessary or advisable that they do so, for their professional practice as future mathematics teachers?

The PhD will not address two additional questions: the impact of such a capstone course on actual teaching practices and the impact of a "group-informed" or "group-rich" secondary school teaching on student learning. They will be tackled after the PhD.

3 A heuristic historical detour

3.1 The French context

In order to take a step aside from our own preconceptions and biases regarding groups, the PhD aims at uncovering the answers given to the "what" and "why" questions by the French Ministry of Education in the 1950-1990 period. The material presented here is still a work in progress, and makes use of historical sources in a heuristic manner.

The specific French context should make it easy to spot explicit answers to the two questions: firstly, the curriculum for secondary education is nationally defined. Secondly, the majority of secondary teachers are public servants. They are hired via a national competitive examination - the *agrégation*[1] - which is designed like a university examination (with a clearly defined official curriculum, published written and oral examination questions and annual reports of the selection board), but with a limited number of places, and which serves as a selective recruitment procedure.

[1] There is another national examination which serves the same purpose, but with a lighter curriculum, the *CAPES*. We will not address it, given that the sources are too incomplete.

A key factor to take into account is whether groups were explicitly taught in secondary school. During the 1950-1990 period, groups were the subject of study in secondary education in an extended "New Maths period". Regardless of whether or not groups were explicit objects of study in secondary education, some knowledge of groups was expected from future teachers, at least since the 1960s, that is even before the New Maths reform (Gispert, 2023).

3.2 A heuristic use of historical files

Among the sources we studied, two stood out as particularly relevant. In 1956-1957, the APMEP (Association of Secondary School Mathematics Teachers) organized a series of lectures for its members, entitled "Algebraic structures and topological structures[2]". The contributors were professional mathematicians advocating a more "modern" teaching of mathematics: H. Cartan, J. Dixmier, R. Godement, C. Pisot, L. Schwartz and J.-P. Serre were members of Bourbaki, and G. Choquet, A. Lichnerowicz and A. Revuz played an active role in the New Maths reform in France. P. Dubreil, P. Lelong and L. Lesieur also contributed to the lecture series.

In the introduction to the lectures series, G. Choquet and G. Walusinski argue (Cartan et al., 1957, p. 3-4):

> The rapid expansion of research and the ever-more rapid progress of discovery mean that even those teachers who teach at elementary levels have to renew their theoretical knowledge. (...) Their very teaching must take advantage of new scientific discoveries: today one must teach the mathematics of today. This will better prepare students for the studies and research that await them[3].

Thus, it seems that one goal of introducing structures to secondary and elementary levels is to pave the way for the pupils' future academic careers. The professional mathematicians mostly care about the education of the more mathematically inclined and advanced secondary students, with a view to smoothing out the first Klein discontinuity, i.e. between secondary and ter-

[2] *Structures algébriques et structures topologiques.*

[3] "L'extension rapidement croissante de la recherche, le progrès de plus en plus rapide de la découverte imposent, à ceux mêmes des professeurs qui enseignent à des niveaux élémentaires, de renouveler leurs connaissances théoriques. (…) Leur enseignement même doit profiter des nouvelles acquisitions de la science : il faut enseigner aujourd'hui les Mathématiques d'aujourd'hui. Ainsi mieux préparer les élèves aux études ou aux recherches qui les attendent." Free trans. Deepl and R. Chorlay.

tiary education. However, it is unclear why pupils who will not pursue further studies in mathematics should also be exposed to these concepts.

In his introductory lecture on "algebraic structures", H. Cartan formulates cautious but rather undecipherable suggestions. He claims that "This use [of algebra] is by no means new; what is especially new is our awareness of it." This means that modern mathematics is more algebraic than ever, not so much because of the extension of a specific domain called "algebra", but because modern mathematicians are now more "conscious" of the pervasiveness of algebra. His opinion is as follows (Cartan et al., 1957, p. 5):

> The teaching of mathematics in secondary classes should reflect this development, at least in the final classes; not so much in terms of a change in the syllabus as in the way classical theories are presented, taking into account this new perspective[4].

Thus, this lecture series sheds light on the fact that, for these professional mathematicians, it was necessary to bring a modern viewpoint on well-known objects to school children. Yet, it is not exactly clear why.

The other important sources we studied are the annual reports of the selection board. These reports help us better understand the scope of the mathematical culture expected from the applicants. We have identified several relevant excerpts, dating from the New Maths period:

> The study of geometry seems to be too neglected. However, general theories (which are sometimes derived from it!) have a wide range of practical applications. Any study of geometry is inseparable from the notion of a group operating on a set[5]. [Report, 1975, p. 91]

Here, it is made clear that groups should not be considered solely for their most general aspect, but also for their connection with geometry. Thus, multiple points of view on groups coexist.

> Group theory. The candidate should not develop a general theory if it cannot be supported by non-trivial examples[6]. [Report, 1978, p. 75]

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[4] "L'enseignement des Mathématiques dans les classes secondaires doit se ressentir d'une telle évolution, tout au moins dans les classes terminales ; moins sans doute dans le sens d'une modification des programmes que dans la façon de présenter des théories classiques, en tenant compte d'une nouvelle optique." Free trans. R. Chorlay.

[5] "L'étude de la géométrie semble trop délaissée. Pourtant les théories générales (qui en sont parfois issues !) y trouvent un large champ d'applications concrètes. Toute étude géométrique est inséparable de la notion de groupe opérant sur un ensemble." Free trans. R. Chorlay.

[6] "Théorie des groupes. Le candidat ne doit pas développer de théorie générale s'il ne peut pas l'étayer par des exemples non triviaux" Free trans. R. Chorlay.

The selection board emphasizes the importance of examples, which are often left out by applicants who focus exclusively on general group theory.

> In the "paths" of mathematical thought, definitions are very often crossroads; unfortunately, many candidates seem to place them in the middle of the desert because their introduction is so lacking in relevance and their use so wanting in scope. How else can you deal with topics such as (2)[7], (11), (28), if not by showing how the notion, however elementary, of quotient, dimension or ideal provides the answer to varied and difficult questions, and then - this is the problem of structure - why it makes it possible to solve them, why it makes them their common substance[8]? [Report, 1979, p. 59]

The selection board wanted the applicants to be aware of the fundamental role of structures: to provide answers to a wide range of problems.

> This compartmentalisation of knowledge appears to be a general fact; even the best candidates hesitate to bring together concepts which are clearly related but which they have studied at different times, thus depriving their plan of its most fruitful examples and reducing it (…) to an enumeration which reveals little about the structures and lacks the spirit of synthesis[9]. [Report, 1979, p. 59]

The applicants were expected to draw connections between different domains of mathematics.

> Before going into detail on the various aspects, let us first stress the two imperatives that must be met in every lesson: to illustrate and to unify. (...) To refuse, for example, on the pretext that we are presenting a lesson on groups, any recourse to linear algebra, analysis, geometry or topology, is a damaging attitude in every respect, a scientific nonsense as well as a pedagogical blunder[10]. [Report, 1980, p. 66]

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[7] (2) refers to "Examples of quotient algebraic structures." ("Exemples de structures algébriques quotients.")

[8] "Dans les « chemins » de la pensée mathématique, les définitions sont bien souvent des carrefours ; nombre de candidats semblent malheureusement les placer en plein désert tant leur introduction est dénue d'à propos et leur usage de portée. Comment traiter des sujets tels que (2), (11), (28), sinon en y faisant ressortir comment la notion, pourtant élémentaire, de quotient, de dimension ou d'idéal fournit la réponse à des questions variées et difficiles, puis - c'est le problème de la structure - pourquoi elle permet de les résoudre, pourquoi elle en fait leur commune substance ?" Free trans. R. Chorlay.

[9] "Ce cloisonnement du savoir apparaît d'ailleurs comme un fait général ; les meilleurs candidats eux-mêmes hésitent à rapprocher des notions dont la parenté est pourtant claire mais qu'ils ont étudiées à des moments différents, privent ainsi leur plan de ses exemples les plus fructueux et le réduisant (…) à une énumération peu révélatrice des structures et dont l'esprit de synthèse est absent." Free trans. R. Chorlay.

[10] "Insistons d'abord, avant d'en détailler divers aspects, sur deux impératifs qui s'imposent dans chaque leçon : illustrer et unifier. (…) Se refuser, par exemple, sous prétexte que l'on présente une leçon sur les groupes, tout recours à l'algèbre linéaire, l'analyse, la géométrie ou la topologie, est une attitude dommageable à tous égards, un non-sens scientifique en même temps qu'une maladresse pédagogique."

During the New Maths period, it seems that with regard to groups, the selection board expected applicants to be able to highlight the generalizing and unifying power of the structural approach, but more besides. They had to be able to flesh out the concept, with applications to domains of mathematics outside of algebra, such as geometry, but also with concrete examples.

The documents we studied show that the "what" question is always only *partially* answered in official documents; the *actual scope* of what is expected from applicants varies in time and needs to be studied as a social construct involving several stakeholders (as we know is the case for school structure, curricula etc.); this scope can be documented by studying the official reports of the selection board, whose purpose is to make its expectations more explicit for future applicants. Moreover, the answer(s) to the "why" question is almost nowhere to be found when it comes to the specific needs of future teachers. While some evidence justify the presence of group theory in the curricula – citing goals such as unifying mathematical knowledge or bringing school mathematics closer to contemporary research - nothing justifies why this content seems relevant specifically for future teachers.

4 Conclusion

The historical study showed that the expectations regarding the content prospective teachers need to know about groups have evolved between 1950 and 1990. Notions about groups were not explicitly taught in secondary school outside of the "New Maths" period. However, some knowledge about groups has been expected from the *agrégation* applicants since 1958. In particular, the role played by groups in geometry seems to have changed over the period: first introduced to give a modern viewpoint on a "classical theory", the transformations of the Euclidean plane, groups became "inseparable" from the study of geometrical objects during the "New Maths" period. Moreover, the selection board stressed the importance of drawing connections between domains of mathematics. This aspect seems relevant for a capstone course. However, why specific knowledge about groups is expected from secondary school teachers remains unclear. The current situation (2024) presents a typical contrast. Indeed, there is no mention of groups in the secondary syllabus[11],

[11] One mention, actually: "Sans introduire explicitement les structures algébriques, cet enseignement introduit et étudie certains exemples fondamentaux : corps des nombres

yet some knowledge about groups is required for the written part of the competitive examination. However, in the actual written exams, there were no questions on groups in 2021, 2022, 2023 or 2024. Nothing in official documents (curricula or reports from the selection board) explains this state of affairs.

In order to inform the design of a capstone course in group theory, it thus seems necessary to supplement this heuristic study with other sources. Interviews with mathematicians who are either experts in group theory or who wrote books about group theory aimed at prospective teachers preparing for the *agrégation* should help shed some light on the "what" and "why" questions.

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Secondary sources

complexes, groupes des nombres complexes de module 1 et des racines n-ièmes de l'unité, anneau des entiers relatifs, d'une manière suffisamment approfondie pour préparer à des généralisations. De même, on aborde la notion générale d'équation algébrique, mais pas celle de polynôme formel. Le professeur peut mettre en évidence l'apparition dans divers contextes de notions communes : élément neutre, opposé ou inverse." (Grade 12, "maths experts").

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