THE PROOFS OF EUCLID ON GEOGEBRA

A step-by-step visualization

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ABSTRACT

This article presents an example of the use of technology, specifically *GeoGebra*®, to introduce Book I of Euclid's *Elements* (which ends with the Pythagorean Theorem and its converse) into the teaching of mathematics. This dynamic geometry software allows new approaches to Euclid's proofs, enabling students to have access to step-by-step constructed images, rather than just a static final image as seen in a book or a *PDF*. This facilitates a better understanding of the mathematical contents, as well as the argumentations used in each step of the proofs. This is an ongoing work; the ultimate goal of which is to have all the proofs from Euclid's *Elements* Book I accessible on *GeoGebra*® for free use.

1 Introduction

The History of Mathematics is an instrument which can have positive effects on mathematics teaching, as the vast literature produced by the HPM group has shown (Clark et al., 2018 and Clark, 2019). The aim of this work is to use Euclid's Elements as an instrument to illustrate the strength and rigour required to achieve mathematical results. Of course, it is also intended that this approach will teach students about this important work in the history of mathematics (i.e., also as a goal), and even *GeoGebra*® itself can be seen as a goal, since the use of dynamic geometry software is compulsory in many countries.

The technology is another instrument that has been discussed over the years, especially the role it should play in the context of mathematics education. In fact, this discussion is now again on the agenda with the release, for example, of *ChatGPT* in November 2022.

> Math education has been challenged by calculators, by *Google*, by homework and notes sharing platforms, by *WolframAlpha*, and more. This is another moment for us to think about how we want to adapt our instruction in light of an ever-changing landscape. (Eaton, 2023)

As for the simultaneous use of these two fields (history and technology) in education, studies are very scarce. For example, the article by Isoda and Aoyama (2000) discusses examples of the use of LEGO and dynamic geometry software to approach a mechanical/geometric problem by L'Hospital (1696) in a school context. In 2004, Isoda also demonstrated the potential of using historical tools and computer software by presenting an experiment carried out in secondary education:

> The historical tool and software, enabled students to remind developer's reasoning. Because the lesson started from a historical picture of ellipse compass and gave it meaning within a historical tool for reminding developer's perspective, students could understand mathematics as a Human Endeavor. (Isoda, 2004, p. 236)

2 The Elements of Euclid

The example presented in this paper has potential in both areas: the technology allows a better understanding of the content of Euclid's *Elements* and, on the other hand, it allows this work to be brought more easily into the school context, exemplifying and helping students to understand the rigour of a mathematical proof and the nature of the discipline itself. It should be noted that many of the propositions in this book are still taught today in elementary geometry, such as: how to construct an equilateral triangle (Elements I,1), SAS Criteria for equality of triangles (I,4), the equality of vertical angles (I,15), the triangular inequality (I,20); the transitivity property of parallel lines (I,30), the sum of the three interior angles of any triangle is equal to two right angles (I,32) and the Pythagorean theorem (I,47). All these results are proved based on some entities that are initially presented and asked to be considered as true: 23 Definitions, 5 Postulates and 5 Common Notions (as presented in Heath's version). For instance, the first common notion is "things which equal the same thing also equal one another", which is easily accepted by students. On the other hand, we must also be careful because some of Euclid's considerations are problematic in terms of today's mathematics, such as the common notion 5 which states that "the whole is greater than the part" (for example, the set of natural numbers has the same cardinality as the set of integers…; what does it mean in mathematics to be bigger?).

The structure of *Elements* shows how the 'mathematical building' is constructed and the origin of the strength of mathematical results, where, starting from a few premises that are assumed to be true, complex results are proved step by step, such as the Pythagorean Theorem and its converse (the last two propositions of Book I of *The Elements*). On the other hand, once a result has been proven, we can use this knowledge whenever we need it. And, regarding this, the first book is very interesting because it is very sequential in its proofs, since in the proof of a certain proposition, in general, the immediately preceding result is used. For instance, the proposition 1 is used in order to prove proposition 2; proposition 2 is used to show proposition 3; and so on. This structure makes it easy to visualise and comprehend the constructive and recursive nature of mathematical knowledge.

3 The Elements of Euclid: historical approaches beyond the traditional book

Euclid's *Elements* was the most widely circulated mathematical book for centuries, even before the introduction of the printing press. When this technology was created by Gutenberg (1430), naturally this work was one of the first math texts to be chosen for publishing. The first edition was published in Venice, in 1482, by Erhardus Ratdolt, but many others have emerged over time (in different places, at different times and in different languages).

One of the most interesting approaches to this work was done in Portugal in the 17th century, at the Jesuit College in Coimbra, which had several tiles in its classrooms with images of Euclid's proofs, reproducing figures from André Tacquet's version (Duarte, 2007). Figure 1 shows an example of one of these tiles, as well as the original image from which it was reproduced. These tiles were larger than usual and were intended to serve as a teaching pedagogical aid for maths lessons, in an example of innovation using Euclid's *Elements*. It should be noted that there were also tiles from other subjects such as astronomy and geography for the same purpose. Today, most of these tiles can still be visited at the Machado de Castro Museum in Coimbra (Portugal).

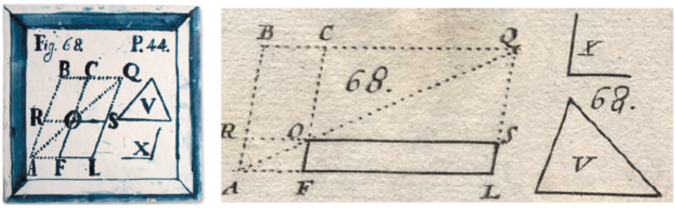

Figure 1. Tile with an image from Euclid's *Elements* edited by Andre Tacquet (I,44: To a given straight line in a given rectilinear angle, to apply a parallelogram equal to a given triangle) (Source: Duarte, 2007, pp. 40-41).

Another interesting approach to the work of Euclid was the book published by Oliver Byrne in 1847, where a coloured printout of the images was used to make them easier to understand (the colours black, blue, red and yellow were used). The objects are not defined by letters as usual, but by their colours; with this option, the proof is easier to follow and the whole structure is more visually appealing. Notice also that Byrne's version is slightly different from Heath's version indicated before; for instance, it considers 35 Definitions, 3 Postulates, 12 Axioms (instead of the term Common notions). At the website (Rougeaux, n.d.) it is possible to find a modernisation of Byrne's work. This website is not just a *facsimile* of the original, but an adaptation of this work into several languages (old English, modern English, Greek and Spanish), while maintaining the interesting layout of the original. This is a good example of how technology can help to spread original sources to a wider public, without losing the pedagogical aspects of the original. See figure 2 for an example of a Euclid proof, in Spanish.

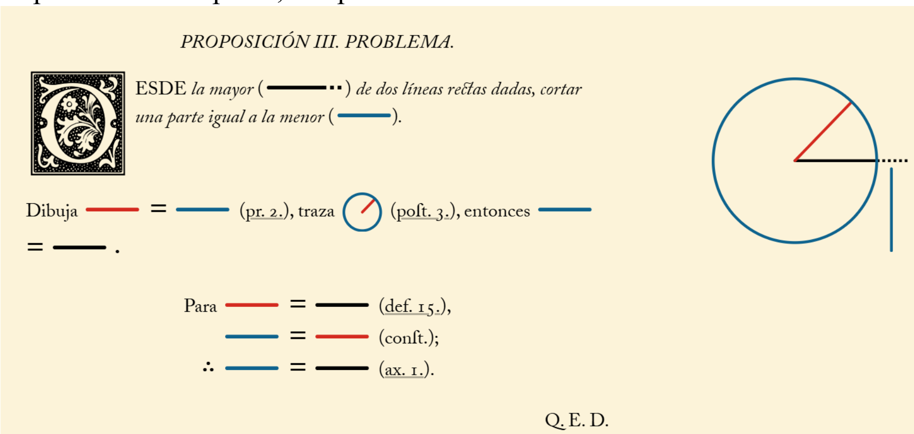

Figure 2. Online adaptation, in Spanish, of Proposition 3 of Byrne's book. (Source: https://www.c82.net/euclid/es/book1/#prop3).

The possibility of creating dynamic geometrical content has grown as computer possibilities have evolved. A well-known website was created by David Joyce (1996-97), where interactive applets of the proofs of Euclid could be found. The idea of this website is that users can move the starting points so that the images in the proofs can be adapted to the chosen data. In fact, the image that appears in a mathematical proof in a book is just a representation of what is being considered in generic terms. That is, for example, the image of the construction of the equilateral triangle on a segment is done for one specific segment (Elements I,1) but the considerations are valid for any segment that chosen by the user. This website, for each proposition, presents the final image that would appear in a Euclid book, which sometimes contains too many items and can hinder a full understanding of the reasoning used in each step of the proof (see, for instance, figure 3).

Unfortunately, the interactive aspect Joyce's website is not currently functional so that at the moment, the images appear statically, and it is not possible to move the indicated points. This is one of the reasons that the technology needs to be updated to guarantee interactivity in the approach to Euclid's proofs. On the other hand, the intention now is to create step-by-step images that would be built up alongside the used arguments. So, the process is gradual and, at first, only the initial data appears on the image, with everything else only appearing on the image when necessary for the proof.

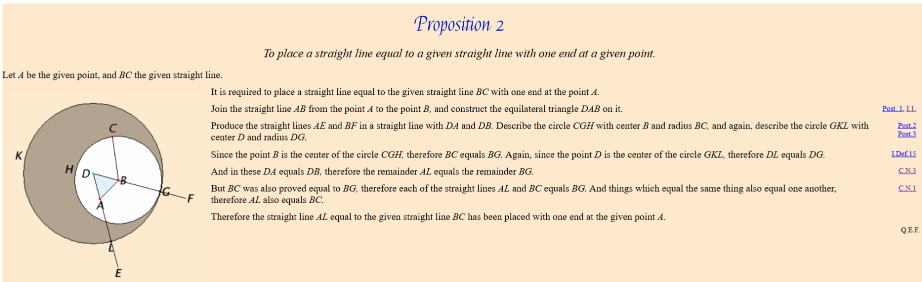

Figure 3. Applet with the proof of Elements I,2 by D. Joyce.
(Source: http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI2.html).

4 *Using Geogebra® to reach The Elements of Euclid: a proposal*

The main idea behind this project is to use Geogebra® software to construct dynamical proofs of the Book I of Euclid's *Elements*. It was decided to follow the set-up of David Joyce (1996-97), with the same exactly math notations. The advantage of this option is being freely accessible online and is based on the well-established version of the *Elements* by Thomas Heath (1956). Notice that this Book I deals with triangles, parallels, and areas, basic contents in the geometry for the first years of school and, as stated earlier, this book ends in Pythagorean Theorem. In fact, plane geometry is the content of the first six books of *Euclid's Elements* (for instance, Book III studies the circles and Book VI presents results about similar figures and proportions).

The selected software was chosen because it is free and because it is widely known among pre-university students and teachers, allowing them to adapt these contents to their needs. These GeoGebra® applets use a colour scheme to facilitate the understanding of what is being used at each moment of the proof: in black are the initial data, in red is what is being proven at that moment, in green is what is being used to deduce what is in red, and in blue are previous steps of the proof.

As an illustrative example of this work, the majority of the ‘steps’ of the proof of Elements I,1 are reproduced next (Figs. 4-8).

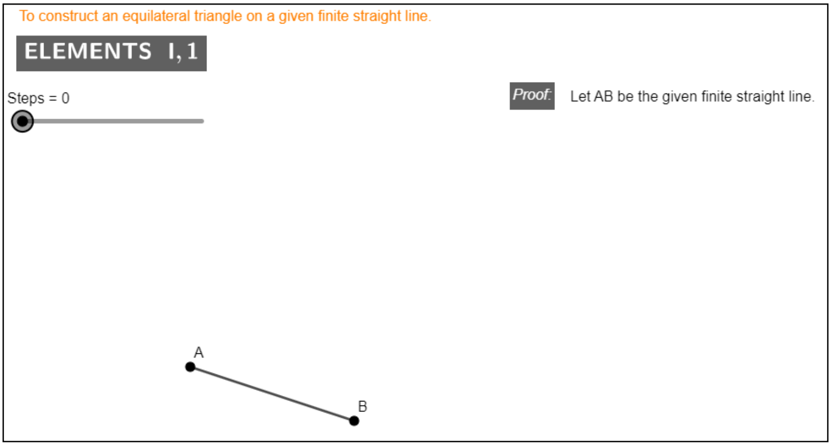

Figure 4. Initial data from Elements I,1. Points A and B can be moved to change the segment. The point ‘Steps’ is movable to proceed with the proof.

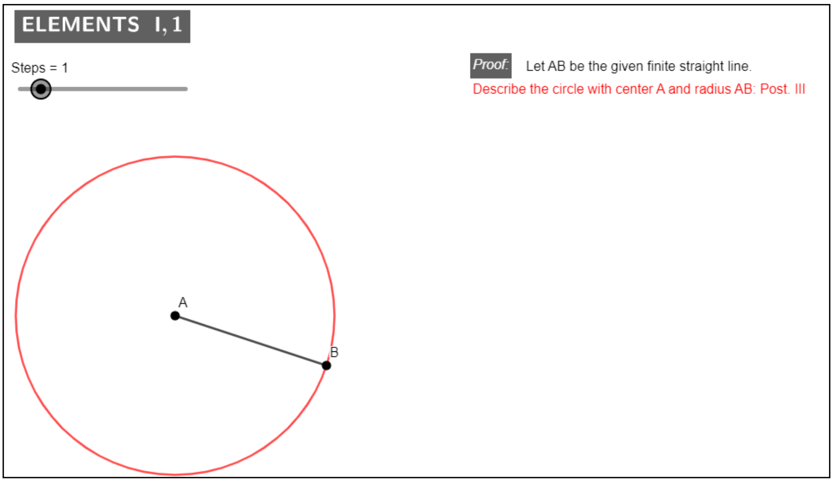

Figure 5. Step 1 of the proof of Elements I,1 and its justification on the right. Notice that is only possible to draw such circumference, given a center and a radius because Postulate III was stated at the beginning of *The Elements*.

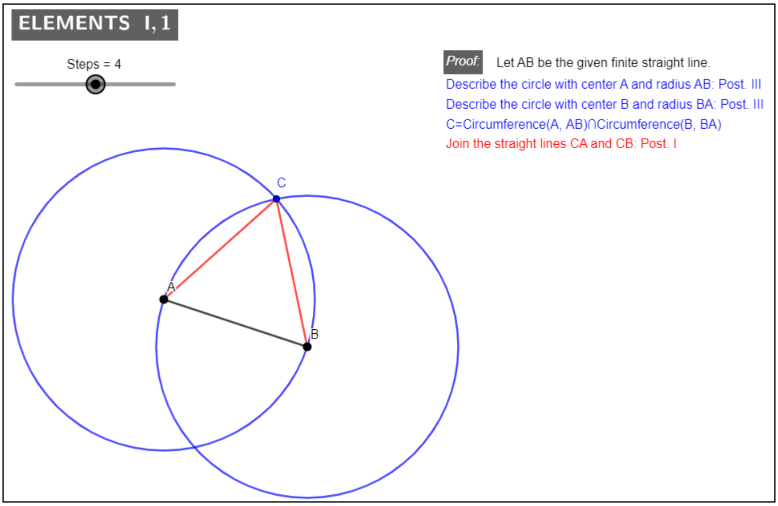

Figure 6. Step 4 of the proof of Elements I,1 and its justification on the right.

Postulate I asserts that, given any two points in the plane, it is always possible to draw a rectilinear segment joining them. In Joyce (1996-97), it is possible to find three historical criticisms of this proof, in particular, one about the question "Why does the point C exist?". This existence is not guaranteed by Euclid's axioms. In the XIX century, Pasch's axiom is stated to address this question.

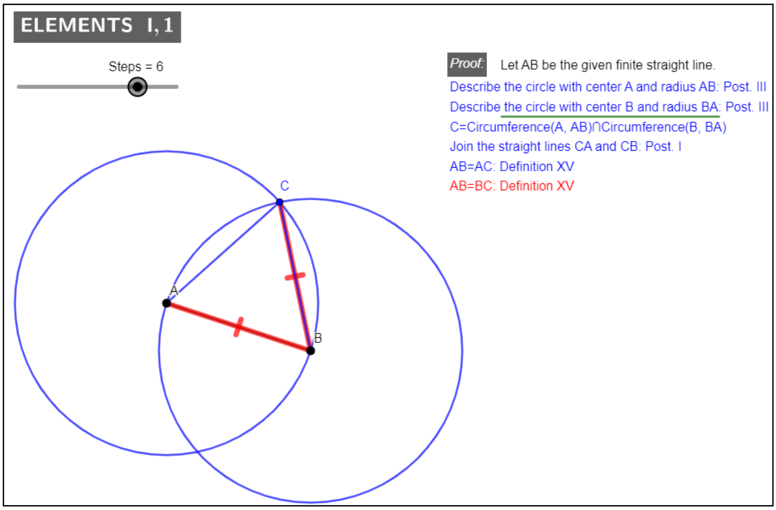

Figure 7. Step 6 of the proof of Elements I,1 and its justification on the right. By the definition of a circumference (XV), all its radii have the same length.

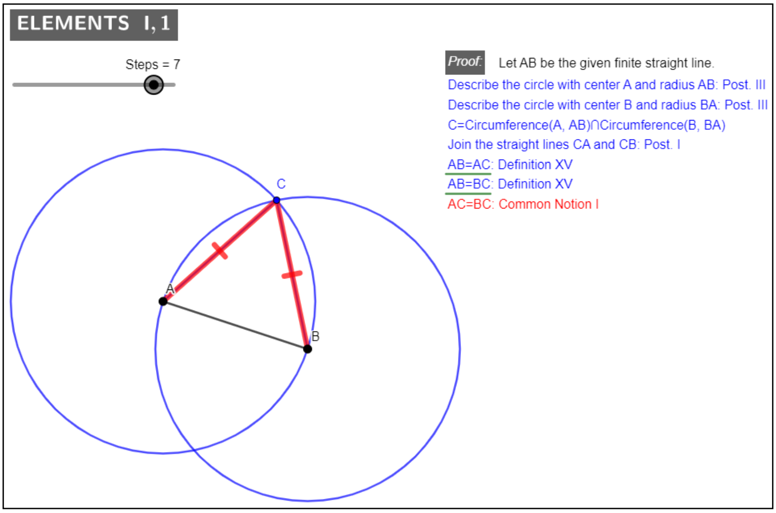

Figure 8. Step 7 of the proof of Elements I,1 and its justification on the right.

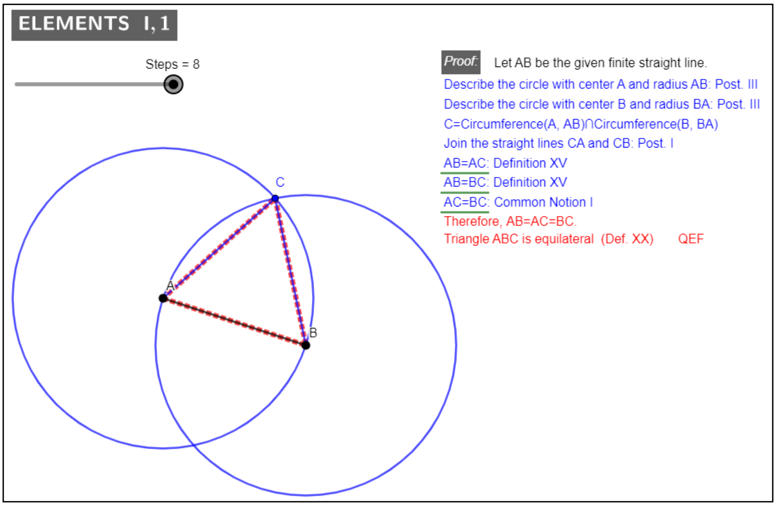

Figure 9. Final step of the proof of Elements I,1. Since the three segments are all equal, the constructed triangle is equilateral (definition XX).

The great advantage of these GeoGebra® applets is that the proof is being built step by step, in front of the students' eyes, and at each moment they can visualise the previous data that are being used. It should also be noted that in many of Euclid's proofs, the final static image shows various points and other auxiliary constructions that complicate the full understanding of the proof (see, for example, figures 10 and 11 to other propositions where the proofs use various auxiliary constructions to proof the desired result).

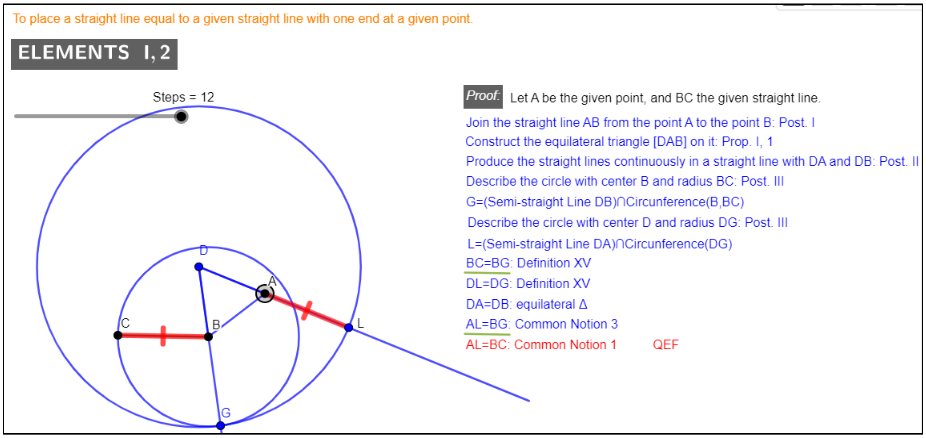

Figure 10. The proof of Elements I,2.

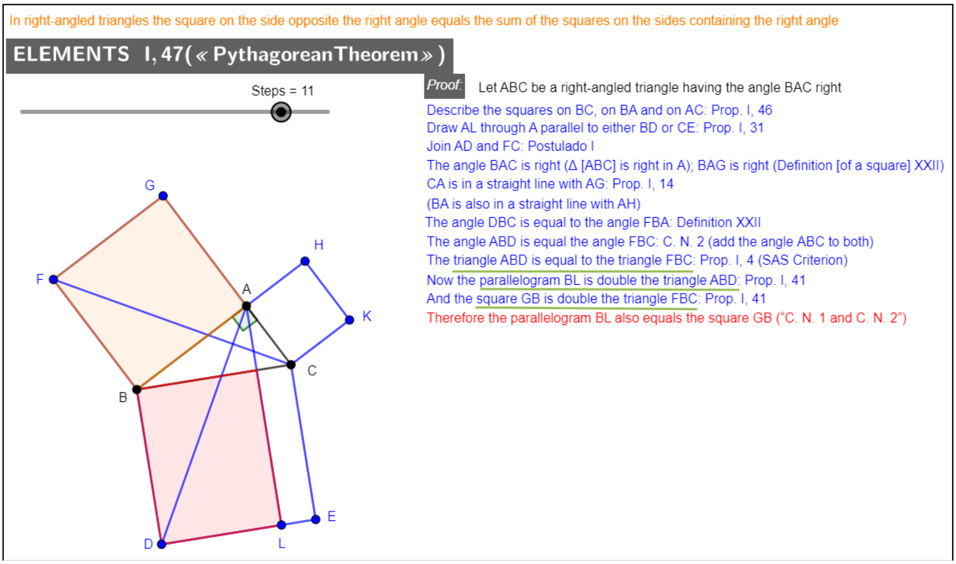

Figure 11. The 'step 11' of the proof of Elements I,47.

5 Final Remarks

Euclid's Elements was a success in part because it presents content that is still essential in elementary/school geometry in the early years. It's still a good example of how the construction of the 'mathematical building' works (what a mathematical proof is, what the rigour of a proof should be, where each step must be duly justified by previously proved results). In mathematics is still valid the metaphor of dwarfs on the shoulders of giants, with new truths/results based on the previously discovered truths. With today's technology, interactivity and dynamic geometry, it is possible to give a new look and new approaches to dispel some of the difficulty in reading an original text from the past and, hopefully, get a new audience for the History of Mathematics.

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