CALCULUS WITH HISTORY-BASED MANIPULATIVES: PROBLEMATIZING THE TRACTRIX

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ABSTRACT

We share our attempts in reviving historical aspects of calculus through the exploration of a geometric-mechanical artifact. This allows for the experience of significant steps through hands-on activities, from Leibniz's early conception of calculus (tractional motion, 17th century) to the Enlightenment (demonstration machines, 18th century) and the realization of practical mathematics instruments ("integraphs," 19th-20th century). Although the adopted history-based artifact has already been introduced in the literature, its effective use in education remains unclear. To explore this, we will focus on the first construction of a curve by the inverse-tangent problem, the tractrix. Despite its significant historical value, this task left participants quite unsatisfied during some previous workshops. Therefore, the need to define useful and engaging problems related to this construction has emerged. We will reflect on how we adopted such an approach in teacher training.

1 *Introduction and background*

In this work, we provide an example of how to integrate history and historical instruments into a workshop on infinitesimal calculus, coherently with Italian research on workshop activities using mathematical mechanical tools, as described in the literature as *mathematical machines* (Maschietto & Bartolini Bussi, 2011). In such a context, manipulatives are named machines and not instruments to suggest something that takes an input, performs a process, and produces an output. We adopted a new device that is deeply rooted in historical ideas related to the mechanical implementation of the solution to inverse tangent problems, which analytically corresponds to the resolution of ODEs.

Unlike contemporary teaching approaches, our device aims to facilitate the introduction of various calculus concepts through geometric constructions without limits, reminiscent of Leibniz's initial approach to calculus. Although geometry is considered today mainly a visualization tool, from ancient Greece to the 17th century, geometric constructions were essential for justifying the existence of mathematical objects. From this perspective, it was necessary to define the primitive class of acceptable tools in geometric constructions: this is the problem of "exactness." In this context, machines (also beyond the ruler and compass) offer an approach in which the concept of a curve is treated in a way that is new to modern students: curves are seen not as a set of points, but as a continuous geometric trace. Such a notion of exactness was crucial in Descartes to justify the introduction of algebra in geometry (cf. Bos, 2001). By extending Descartes' definition of acceptable curves, Leibniz validated the inclusion of transcendental curves through machines, and a key content was the possibility of constructing curves given the property of their tangent, i.e., as the solution of inverse tangent problems (see Blåsjö, 2017).

Inverse tangent problems are compelling for several reasons. Beyond their historical significance in the development of calculus (also considering problems in physics), these problems maintain a strong connection with material implementations, such as the creation of scientific instruments for demonstration, education, and practical application. A rich overview concerning both theoretical content and material devices is visible in Tournès (2009).

2 A history-based device

Our device collects the legacy of instruments for the inverse tangent problem, summarizing them with a simple design. Although the main ideas behind the device have been evident since its first version (cf. Maschietto & Milici, 2024), improvements have been proposed based on user feedback (e.g., see Milici et al., in press). The aim is to make users focus on the mathematical role of the components, minimizing technical complications. The prototype of such an instrument has been built using typical FabLab tools, including laser cutting and 3D printing, along with some additional hardware components, such as screws, ball bearings, and rubber O-rings. Let us name the components of the device according to the numbering in Figure 1. With these components, the device can be assembled in different configurations (Section 3).

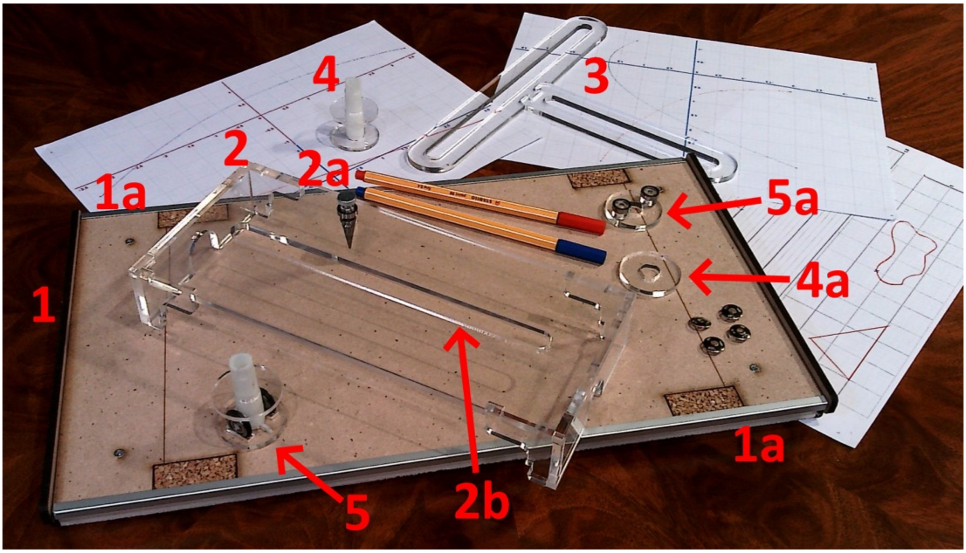

Figure 1. The pieces of the device.

(1) The “base.” It is a wooden base with four cork rectangles to fix the corners of a paper sheet with thumbtacks. It has two “rails” (1a) along its long borders.

(2) The “plate.” It is a rectangular plate of transparent plexiglass that can slide along the rails (1a). The “peg” (2a) is fixed to the plate. On its top, there is a ball bearing to pass through one of the rods (3), and on the bottom, there is a spike. The “slot” (2b) is carved in the middle of the plate.

(3) “Rods.” They are linear guides to be put on the plate. The rods can be joined to form a T in which a rod is the perpendicular bisector of the other.

(4) The “simple pointer.” It is a piece that can slide inside the slot (2b) and a rod (3). On its top, one can put the “head” (4a), which helps the user move the piece. It features a hole that can be used as a viewfinder to move the pointer along a curve and as a marker to leave a trace.

(5) The “wheeled pointer.” It is a simple pointer (4) with two parallel wheels at its bottom that can rotate at different speeds. It has a “head” (5a): unlike (4a), it has two ball bearings to constrain the direction of the rod (3) through which it passes, ensuring it is parallel or perpendicular to the direction of the wheels (the “head” can be right-angle rotated).

3 *Possible activities*

Coherently with our previous research, we adopt the theoretical framework of the Theory of Semiotic Potential (Bartolini Bussi & Mariotti, 2008) to set hands-on activities on the history and epistemology of inverse tangent problems. The fundamental idea of the Theory of Semiotic Mediation is that an instructor uses a specific artifact as a tool of semiotic mediation for constructing mathematical meanings. Activities are organized within didactical cycles, including group activities with the artifact, individual activities, and collective mathematical discussions. In this framework, a crucial notion is the semiotic potential of an artifact; its analysis is fundamental for conceiving tasks and is at the basis of interaction with participants. The semiotic potential of our device got deepened in Maschietto & Milici (2024, §6). To provide an idea, we introduce some possible activities.

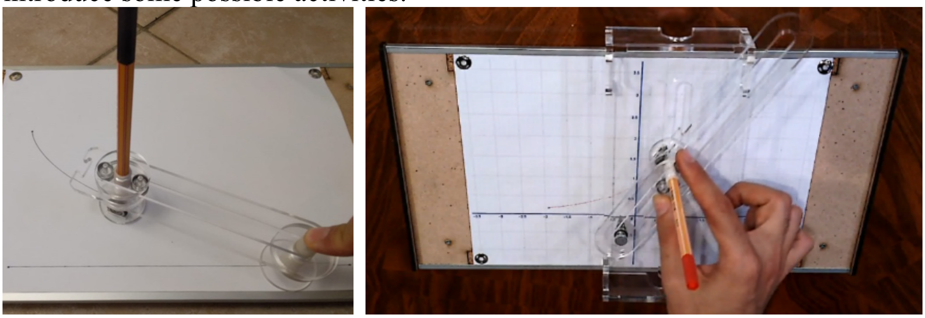

Figure 2. The device is assembled to trace Perrault's curve (left) and an exponential curve (right). Cf. https://www.youtube.com/watch?v=LMLt90R8zHA 0:14-0:50 and https://www.youtube.com/watch?v=kqtU9GpcN78 0:00-0:15.

We can introduce the problem of the existence of curves in the 17th century, specifically Perrault's construction of the tractrix (cf. Section 4.1; for more references about the various activities, see Crippa & Milici, 2023, §2, or Maschietto & Milici, 2024, §2). By introducing the device assembled as shown on the left side of Figure 2, we can propose tracing the curve. The aim of this activity is to link the mechanical properties of the device and the geometrical properties of the tractrix, thus reinterpreting the idea of tangent (in this case, the tangent exists before the curve).

Machines tracing an exponential curve by solving the inverse tangent problem implement the geometrical property of having a constant subtangent, a property that is generally neglected by modern students. The request for the second activity is to invent a machine for the exponential function (the reconstruction of an exponential machine starting from historical sources in a math education workshop is illustrated in Plantevin & Milici, 2022). The assembled machine is visible on the right of Figure 2. As a historical counterpart, we can recall the related historical machines of the 18th century (cf. Poleni, 1729). To foster this construction, we propose focusing on the geometrical properties of the tangent to the exponential using GeoGebra. Then, we suggest reflecting on the components of the device (e.g., which component can we use to guide the tangent?) to trace the sought curve.

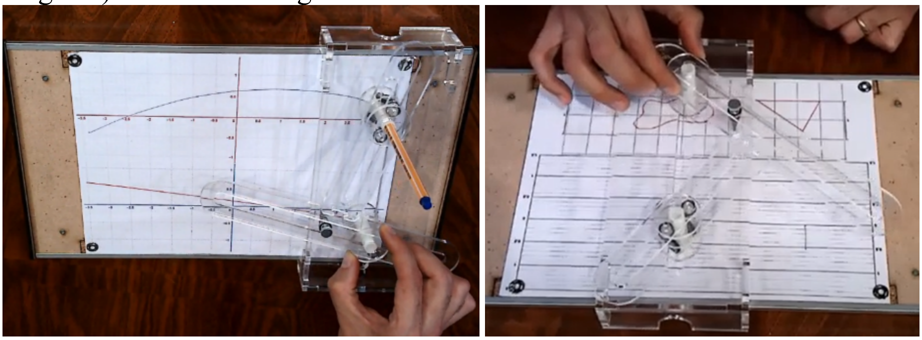

Figure 3. The device for derivatives and anti-derivatives, featuring two reference frames (left) and an integraph-mode sheet (right) – cf. https://www.youtube.com/watch?v=TyxCAR317HE, 0:00-1:05 and 1:07-3:45.

After introducing the role of the tangent and practicing a bit with the components of the device, we are ready to propose an activity related to the Fundamental Theorem of Calculus by introducing the device in the configuration as in the left of Figure 3 (with a background sheet with two Cartesian references). To recognize that the device is performing derivatives and antiderivatives (considering participants who already know Calculus), we propose focusing on the gestures necessary to move the wheeled pointer along straight and general curves (to move this pointer along a curve, the direction of the wheels has to be parallel to the tangent to the curve). Then, focus on simple pointer displacement when moving the wheeled pointer on straight segments. That should help in generalizing when moving along the graph of a function, thus realizing that the simple pointer describes the derivative. Similar reasoning, starting from moving the simple pointer on a horizontal segment and generalizing to the motion along the graph of a function, should lead to the fact that the wheeled pointer moves along one of the anti-derivatives. A crucial element is to notice that the device transmits the direction of the wheeled pointer (i.e., the direction of the tangent) to the ordinate of the simple pointer. Furthermore, by simply changing the paper sheet (Figure 3, right), if we move the simple pointer along the perimeter of a figure, we can calculate its area by measuring the displacement of the wheeled pointer, as in the historical instruments called “integraphs” (cf. Abdank-Abakanowicz, 1886).

Even though we marginally tested such a part, we can also use the device to introduce the concept of differential equations. We can note that, from a historical perspective, Leibniz’s geometrical insight into calculus originated from the inverse tangent problem, which analytically corresponds to solving differential equations. We propose to explore the device assembled as shown on the right side of Figure 2, but with the wheeled pointer rotated by a right angle (cf. https://www.youtube.com/watch?v=kqtU9GpcN78 0:17-0:50, but without the Cartesian system). That defines a parabola by implementing the property of having a constant subnormal (parabolas are orthogonal trajectories of exponential curves). To analytically explore the trace, we propose that users introduce a Cartesian system and convert the mechanical constraint into a simple differential equation.

4 Problematizing the tractrix

The tractrix holds a significant place in the history of mathematics as a paradigmatic example of a transcendental curve. Indeed, its construction comes from the first mechanical solution of an inverse tangent problem. First emerging in the 17th century, the tractrix became a focal point for early modern geometers, including Huygens and Leibniz. The tractrix was not only a theoretical curiosity but also a practical challenge: constructing it through continuous motion required innovative mechanical devices.

4.1 Historical sources

As a starting point for understanding the historical and conceptual development of mathematical instruments for transcendental curves, Giovanni Poleni’s 1728 *Letter to Hermann* (published in Poleni, 1729) offers a rich and revealing case. This letter is not merely a theoretical reflection but introduces tangible, functional mathematical instruments (not just idealized mechanisms or prototypes) for solving the inverse tangent problem, particularly in the construction of the tractrix and logarithmic curves. Poleni's machines, described in meticulous detail and physically built, represent a significant step in the material culture of mathematics. They were designed to trace curves through a single, continuous motion, in an efficient and reliable way.

Poleni's letter also serves as a historiographical pivot. According to him, the tractrix was effectively discovered on three separate occasions. The first discovery is attributed to Claude Perrault, who introduced the following problem to a select group of Parisian nobles and scholars in the 1670s, as shown on the left of Figure 5: consider a pocket watch whose chain extremity moves slowly (to avoid inertia) along a straight line called the "directrix." Which is the curve traced by the clock, and for what reasons?

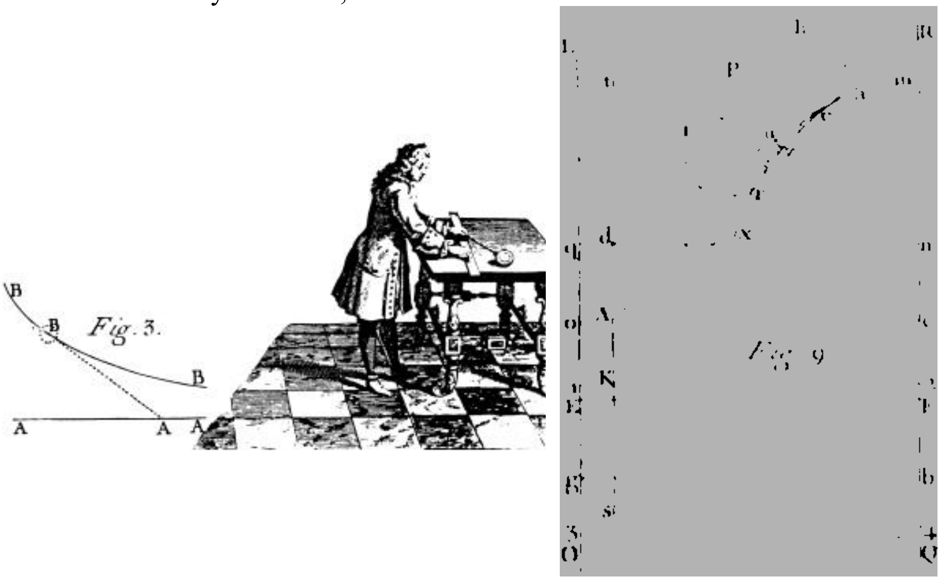

Figure 4. The tractrix in Poleni (1729). Left: representation of Perrault's construction [table BB]. Right: Construction of the tractrix by tangent properties [table DD].

The second was by Christiaan Huygens, who, in a publication from September 1693, described the geometric properties of the curve and, through the definition of a machine (he replaced the pocket watch and its chain with a weight attached to a rod), helped make the concept more widely accessible. He justified the motion by a "purely geometrical principle:" On a horizontal plane, consider a physical point attached to an inextensible string or an inflexible rod. While pulling the other extremity, if the point makes some resistance with the plane by its weight or other physical properties, this point moves along a trajectory in which the taut string or the rod is always tangent to the described curve.

Lastly, Gottfried Wilhelm Leibniz is considered the third discoverer. Although his publication came after the others, he was the first to mathematically formalize the curve's nature using the tools of modern analysis and differential calculus.

Poleni, however, emphasized that none of these three thinkers succeeded in building a fully functional and accurate machine to trace the tractrix. Each of their devices had limitations or flaws. This perceived shortcoming left an opening for a fourth contributor, Poleni himself, who believed he had finally achieved what the others had not: a precise and theoretically sound method for constructing the curve. Perrault's demonstration with a pocket watch (left of Figure 4), Huygens' rigid rod mechanism, and Leibniz's use of a taut string all attempted to embody the curve's defining property: a constant-length tangent to a fixed axis. However, Poleni critiques these earlier efforts as incomplete or imprecise, particularly in their reliance on physical forces like gravity or friction, which compromised the geometrical purity of the construction. His innovation, the use of a wheel to implement tangent direction, offered a more stable and reproducible solution. This idea, however, had a precedent in the work of John Perks, a British schoolteacher whose 1706 and 1715 papers in the *Philosophical Transactions* described similar mechanisms. Although Poleni never cited Perks, the conceptual overlap is striking and has been explored in detail in (Crippa & Milici, preprint).

Perhaps most significantly, Poleni's letter contains what is likely the first formal justification of why a wheel's direction can be used to guide the tangent to a curve. He draws on Leibniz's notion that a curve can be seen as composed of infinitesimal straight segments, each tangent to the curve at a point. Poleni explains that a wheel rolling along a curve naturally aligns its direction with these tangents. As shown in his diagram (right of Figure 4), the wheel's contact point traces the curve such that the direction of motion at each instant is tangent to the path. (Notice that the wheel is represented in various positions and, due to the 2-dimensionality of the diagram, is drawn on the plane although it is perpendicular to the plane.) This insight not only grounded the mechanical construction in sound mathematical reasoning but also marked a conceptual leap in the design of mathematical instruments.

4.2 Teachers' training proposals

Perrault's construction of the tractrix holds historical importance precisely because it does not rely on advanced mathematical knowledge (Perrault himself was not a mathematician). Yet, his mechanical approach offered a compelling solution to a new mathematical problem: defining a curve through the inverse tangent problem. From an epistemological perspective, this marks a pivotal shift. Traditionally, geometry dealt with direct tangent problems, where the curve is given and the tangent is derived. In contrast, Perrault's construction reverses this logic: the curve is generated based on the behavior of its tangents. This shift required a conceptual rethinking of what a tangent is and how it can define a curve. Such conceptual changes often seem obvious in hindsight, but they are difficult to achieve without a fundamental reorientation: in our previous workshop activities participants struggled to grasp the tangent-based reasoning without guided intervention.

Building on previous feedback, we integrated the device, particularly in its tractrix configuration (left side of Figure 2), into the course "Critique of the Principles" for pre-service mathematics teachers (Palermo, Fall 2024). Together with other mathematical machines, our device was adopted to introduce approaches to mathematics that differ from today's mainstream view. One of the aims was to remind that, for a long period, geometric constructions were at the basis of the foundation of mathematics. After introducing the "issue of exactness in geometric constructions" in antiquity and for Descartes, we proceeded to Perrault's construction without providing the mathematical reflections on the role of the tangent. Participants firstly used the device to draw Perrault's curve and were then given a sheet with another pre-drawn Perrault's curve traced using an unknown arm length (the constant distance between the pointer on the directrix and the one tracing the curve). The drawing included the directrix (base segment) but not the cusp point. We asked:

- How long is the arm of the drawn curve? How did you find it and why?
- In general, what mechanical constraints of our instrument and what geometric conditions/invariants define this curve*?*

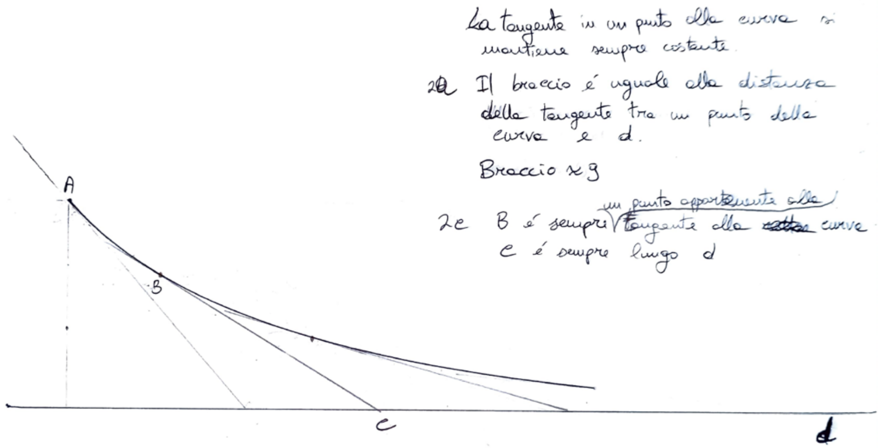

Figure 5. Problem: Given a part of the tractrix (without the cusp) and the directrix, find the length of the “arm.”

Participants worked in little groups of about four people, and were encouraged to freely use the tools provided, which, in addition to our device, included a ruler. After some reasoning, many naturally drew the estimated tangents and made measures with a ruler, thereby gaining an intuitive understanding of the tangent’s role in the construction. For example, in the answer of Figure 5, we can see some traced tangents (note that their first attempted solution consisted of drawing the perpendicular to the directrix *d* at endpoint *A*). To find the length of the arm, they clearly recall the role of the tangent: the solution is given by the distance of the tangent line from any point on the curve and the intersection with the directrix (points *B* and *C* in the figure). This activity enabled participants to justify the construction (that they materially performed before this problem) by linking mechanical components to geometric constraints with minimal instructor intervention.

It is important to emphasize the significance of the material components in this reasoning. Indeed, even though it is sufficient to trace lines with the ruler to find the length of the arm, the relation with the device tracing the tractrix is essential to provide concrete significance to the curve. For example, some participants walked along the printed curve using the tractrix device by roughly adjusting the arm length. (To materially implement a different arm, they used a finger to keep the distance between the pointers constant, even though one of the pointers was no longer at the extremity of the connecting rod.) Such a modification of the device was used to gain insight into a solution or to verify the exactness of their ideas. In such a setting, the tangent to the curve is not only a theoretical mathematical object but becomes the modelling of a material object (representing the direction of the wheeled pointer).

After this activity, it was possible to introduce the history of the tractional motion without making the passage from direct to the inverse tangent problem appear obvious. The discovery of a strong connection between mechanical components and geometric reasoning (e.g., the direction of the wheeled pointer) laid the groundwork for further activities, such as those described in Section 3 (exponential and derivatives/antiderivatives).

5 Conclusions

In this paper, we focus on the problematization of the tractrix in hands-on activities that adopt a historically based artifact. We aimed to minimize instructor intervention and allow participants to grasp the underlying geometric principles independently. The use of the physical device proved to be a natural and effective way to introduce historical concepts through tangible, hands-on activities. This approach not only facilitated engagement but also offered a different perspective on the foundations of calculus, bridging historical insight with conceptual understanding. People interested can reproduce the activity by building the adopted device using digital factory tools. Indeed, we freely share online the sources to reconstruct the device, including assembly instructions and printable sheets, at https://www.thingiverse.com/thing:5532958. Some videos are visible at https://www.machines4math.com/ .

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