

AN INTERTWINING OF HISTORY, EPISTEMOLOGY, MATHEMATICS AND DIDACTICS

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ABSTRACT.

This paper explores the intricate intertwining of history, epistemology, mathematics, and didactics through an examination of overlooked connections between the work of the Chinese philosopher Shao Yong and contemporary Western mathematical and scientific thought. Furthermore, it discusses the implications of Shao Yong's work for innovative a-didactic situations, as implemented with diverse learners in Chile

1 Introduction

We argue that History, Epistemology, Mathematics, and Didactics¹ can intertwine, metaphorically speaking, to form a four-stranded rope.

We intend to present and discuss a multifaceted example of this intertwining. This example involves activities carried out with a broad spectrum of learners at the University of Chile over the last ten years, ranging from undergraduate students to in-service primary school teachers.

To this end, we will first comment sequentially on some concrete instances of our four strands, while allowing for some revisiting. First, our historical strand begins with a string of random sevents starting with the Jesuit China Mission (1552-1715). Remarkable epistemological phenomena follow, involving sophisticated contemporary mathematical notions and the work of the neo-Confucianist philosopher Shao Yong (Birdwhistell, 1989; Kang, 2011; Ryan, 1966; Zhang, 2020; Zhang & Feng, 2008; Zhu, 2018). We highlight some relations and connections between the work of Shao Yong and later Western mathematics and science, relations which are often neglected in contemporary Western and Chinese scholarship (Li, 2019; Needham, 1956). We exemplify their implications for didactics, from an enactivist perspective, by discussing activities and *a-didactic situations* (Brousseau, 2002), suggested by

¹ The term "didactics" is used in the continental sense of "the art of teaching."

Shao Yong's work. We report on their implementation with a diverse group of learners. We conclude with a discussion, a caveat regarding the Whiggish approach to the history of mathematics, and a few questions.

2 *The historical strand*

This discussion commences with a significant historical and religious phenomenon: since its foundation, the Society of Jesus sought the global propagation of Christianity. China represented a pivotal mission field (Mariani, 2012). Their strategic approach involved first establishing linkages between Christian tenets, Western scientific advancements, and Chinese traditional culture, which they held in high regard. To this end, their missionaries undertook extensive study of the Chinese language and culture.

Among them was Joachim Bouvet (1656–1730), a prominent mathematician and astronomer, who arrived in China in 1685 and encountered the Yi Jing (Wilhelm, 1997), an intriguing ancient oracle based on combinatorial objects (the 64 hexagrams). He was particularly struck by the arrangement of these 64 hexagrams in an 8×8 tableau, attributed to the distinguished Chinese philosopher Shao Yong (1011–1077). Bouvet was a close friend to the German polymath Gottfried Wilhelm Leibniz (1646–1716), who, in 1689, had conceived the binary system, driven by his quest for a minimalist universal language capable of articulating all human culture. Leibniz conceptualized the fundamental elements of this system as the "Void" or "Nothingness" and the "Whole" or "One" (representing God), written as 0 and 1. Bouvet then immediately recognized the isomorphic correspondence between Shao Yong's 8×8 tableau of the hexagrams (when interpreted akin to a Western text) and the numbers 0 to 63 expressed in Leibniz's binary numeration.

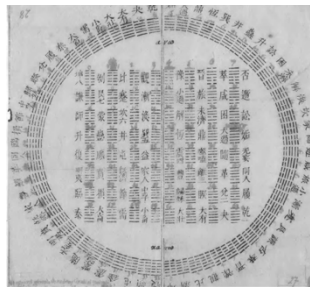


Figure 1. Shao Yong's original 8×8 tableau annotated by Leibniz.

Leibniz was enthused by Bouvet's discovery, as he had previously en-

countered difficulty in persuading his fellow mathematicians in Europe of the significance of his binary system. This finding allowed him to assert the profound importance of the binary system, evidenced by its independent development in China millennia earlier. Furthermore, this discovery aligned with his dream to bridge Chinese "natural theology" with Christianity. Bouvet was similarly captivated by the various diagrammatic renderings of the 64 hexagrams created by Shao Yong, whose intricate and metaphorical philosophical system integrated Confucian, Taoist, and Buddhist thought (Birdwhistell, 1989). Figure 2 illustrates two distinct synthetic representations of the 64 hexagrams: the rectangular and the circular Primeval Diagrams, *Xiantian Tu*² ((先天圖) (Marshall, 2015). The latter plays a pivotal role in Shao Yong's magnum opus, the *Huangji Jingshi* (皇極經世) or "Supreme Principles Governing the World" (Zhang & Feng, 2008). This complex and overarching work of cosmology and metaphysics endeavours to elucidate the entire universe and its unfolding through a comprehensive system of numerical cycles and correlations.

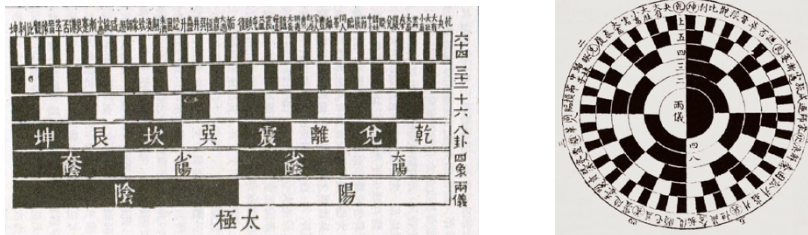


Figure 2. Rectangular and circular Xiantian Tu (Primeval Diagram)

Indulging in a degree of "whiggish overcoding" (Butterfield, 1931; Fried, 2011; Sinclair & de Freitas, 2019), a mathematician would readily identify a six-generation binary tree within the Xiantian Tu. Indeed, upon being presented with the rectangular *Xiantian* diagram, Bourbaki³ himself exclaimed: The

² *Xiantian* (先天) literally translates as "before sky" (rather than "before heaven"), a metaphor for "inborn", "primeval", the primordial state of wholeness and unity of the cosmos prior to the emergence of duality (the differentiation into Yin and Yang). *Tu* (圖) means diagram.

³ Or rather, one of his foremost avatars. See Corry (2004) for the history of Bourbaki.

binary tree! The historical-epistemological question arises then as to which are the different avatars of the binary tree throughout human history. Might the Xiantian Tu be the earliest one?

As previously noted in Soto-Andrade et al. (2022), another pertinent example emerges around 1600 in Dahomey, in the figure of Legba, the God of Chaos, also known as the God of the Crossroads. Legba's "forked path" is metaphorically represented by the iron fork in Figure 5 (Egash, 1999, p. 144). As explained by a Vodun divination priest (Ibid., p. 143), this fork embodies Legba because his responses to inquiries are binary—either "Yes" or "No"—with the specific path remaining unknown. Consequently, a series of such questions generates a sequence of bifurcations, forming a binary tree.

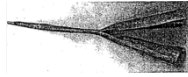


Figure 3. The forked path of Legba, the Vodun God of the Crossroads
Much later, in 2012, a four-step Xiantian diagram unexpectedly emerged in the context of “eternal inflation” within Western cosmology (Harlow et al., 2012). This diagram serves as a finite avatar of the "eternal symmetree," proposed by Stanford physicists as a *discrete combinatorial model* for a multiverse. See Figure 4.



Figure 4. (Left): A finite causal tree and (Right): the causal future of a in the infinite eternal “symmetree” with its boundary at infinity (Harlow et al. 2012).

3 The epistemological strand

From an epistemological perspective, it is indeed remarkable that both the ancient Chinese thinkers and Leibnitz shared a same insane and ambitious dream: to represent and manage the world's complexity through a language composed solely of binary elements (0s and 1s, or Yin and Yang, broken and unbroken lines). It is worth recalling that the Yi Jing oracle was supposed to give guidance and inspiration for any real-life problem using its hexagrams. This “insane dream” has, in fact, become a reality in contemporary times.

We also contend that a typical Western scientist, like Bouvet or Leibnitz,

does not feel the need for Xiantian like diagrams, once the 8x8 tableau is available. Nor do our students and in-service teachers, whom we have interviewed in this respect. Shao Yong, however, was profoundly attuned to change and *processes* within the Universe rather than static *structures*, an approach largely uncharacteristic of Western thought (with the possible exception of Heraclitus). As Birdwhistell (1989, p. 52) notes, he was “an alien, of another place and time”. Apparently, Shao Yong intended to present the 64 hexagrams as the outcome of a (binary) branching process, rather than merely a diverse collection of species of diagrams. Concurrently, he aimed to construct a synthetic (as opposed to analytic) view of the 64 hexagrams, one that could be comprehended at a glance. He thus emerges as a proto-scientific forerunner to Darwin, who conceptualized the diverse species of living beings on Earth as the outcome of a branching process, specifically, as the latest cohort of “the tree of life,” a metaphor whose profound inspirational power Darwin himself acknowledged (Gould, 2010). See Figure 5.

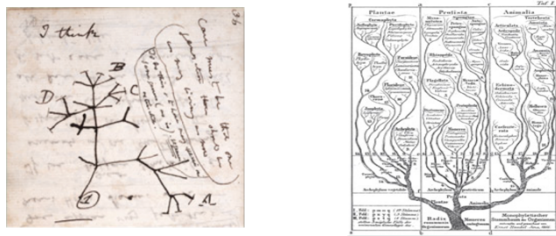


Figure 5. Sketch of a tree depicting phylogenetic relationships among species (Darwin, 1837) and a Tree of Life (Haeckel 1866).

The extent to which Shao Yong comprehended the binary system and the binary tree is a contentious issue in contemporary Chinese scholarship (Zhao, 2008). Some researchers posit that his construction of the 8x8 tableau relied on a recursive procedure rather than a direct binary expansion of numbers (Zhu, 2018). (Cf. Section 5.2).

4 The mathematical strand

Mathematically, the Xiantian Tu is a six-generation binary tree, hosting all whole numbers from 0 to 63. Since in mathematics “sky is the limit”, we would like to accommodate there *all* whole numbers. So, we just extend the six-generation tree into a binary tree with infinitely many generations, thereby yielding the “eternal symmetree” depicted in Figure 4(b). The rectangu-

lar *Xiantian Tu* further suggests a novel metaphor for integers: "Numbers are finite ascending paths within the infinite binary tree." Naturally, infinite ascending paths also exist, and any finite path can be canonically extended to an infinite one by consistently choosing the 'left' branch (representing *Yin*, or the *absence* of the corresponding power of 2). Notably, negative integers appear as infinite ascending paths that choose *right* from a certain node onwards. This suggests an amazing generalisation of the integers, as infinite ascending paths within the binary tree, or equivalently, as points on its *boundary at infinity* (Figure 4(b)). This offers a visual-geometric construction of Hensel's p -adic^[L]_{SEP} integers (Dickson, 1910), for $p = 2$, providing an intuitive alternative to his purely analytical algebraic approach, which cleverly manipulated infinite series of powers of 2 (Ibid.).

5 The didactical strand: Illustrative examples of related classroom activities

Our classroom activities, directly inspired by Shao Yong's work, are designed as *a-didactic situations*⁴ (Brousseau, 2020) and have been implemented since 2015 with the following cohorts of learners at the University of Chile:

- (a) First-year social sciences and humanities students enrolled in a one-semester mathematics course;
- (b) Prospective mathematicians, physicists, and mathematics-physics teachers undertaking a history and epistemology of mathematics course;
- (c) In-service primary school teachers participating in a professional development program designed to strengthening their mathematical formation.

All learners worked autonomously on pre-assigned or self-constructed problems within randomly formed small groups of three or four monitored by the instructor and teaching assistants. They uploaded a summary of their work, open questions and conjectures included, at the conclusion of each 90-minute session. It is important to observe that certain activities listed below preclude others, thereby naturally structuring a "didactical decision tree".

5.1 Discovery approach to the binary system and the hexagrams.

A class of students, without initially counting themselves, tried to figure out how to code their quantity, assuming they had no previous arithmetical knowledge. The idea emerged of *pairing*, where an "odd man out" may even-

⁴ In an *a-didactic situation* the teacher's didactic intent remains implicit: she steps aside to let the students explore and discover on their own.

tually remain. They continued the process by pairing the resulting pairs, potentially yielding a solitary pair, and so forth. They discussed then how to code the presence or absence of these residual individuals, pairs, quadruples, and so on. Through this exploration, they observed that a Chinese system instead of 1 and 0 would employ Yang and Yin for such encoding, thereby ultimately arriving at a representation of their quantity coded by hexagrams. We observed that students from group (b) did not have an embodied experience of the binary system, they had learned it in a purely formal way.

5.2 Archaeologic exploration of Yi Jing.

Learners received a copy of Shao Yong's 8×8 tableau, presented as an undated and unlocalised archeologic object, and tried to make sense of it. Several students from group (b) immediately saw it as an 8×8 matrix indexed by tri-grams! Others, from groups (a) and (c), slowly recognised the binary sequence of numbers by interpreting "unbroken" or "broken" in the n -th line as the *presence* or *absence* of 2^n , as Leibnitz did. Interestingly, we observed that some "somatically educated" learners (e.g., martial arts, tai chi or Feldenkrais practitioners, or dancers), despite being unfamiliar with the binary system, were able to intuitively grasp the binary sequence of the hexagrams. This understanding stemmed from feeling the "rhythm" established by the first four hexagrams: a pattern suggesting that after two forward steps, a return and "recapitulation" are necessary before a third step forward is taken. From this embodied insight, they could reconstruct the entire sequence or predict the subsequent hexagram when presented with any given one.

5.3 Synthetic view of the 8×8 square?

We asked the learners whether they were happy with the 8×8 tableau. Maybe they missed a more synthetic rendering, one that they could grasp in a glimpse? Learners from group (c), who are accustomed to working with concrete materials, spent approximately thirty minutes working in groups, cutting out paper printouts of the hexagrams and attempting to arrange them contiguously, side by side, in the sequence given by the 8×8 tableau (read as a Western book page). During this process, they observed that placing two broken lines side-by-side inadvertently created misleading unbroken lines. This observation led them to conceive the idea of coding broken lines as black

(Yin) and unbroken lines as white (Yang), within the hexagrams, thereby generating diagrams akin to the *Xiantian Tu*, and outwitting the other learners.

5.4 Grasping Xiantian in a glimpse

We asked our students to attempt to reconstruct the rectangular Xiantian diagram, after a brief viewing of approx. 200 milliseconds, and without any prior knowledge of its structure. Figure 6 shows the students' productions. The first diagram is due to a humanities student, who later went to Law school. The fourth one is due to a prospective math teacher. We did not observe a better performance among mathematically inclined students here. We have found that this reconstruction task is hard even for professional mathematicians.



Figure 6. Students' reconstruction of the rectangular *Xiantian* diagram

5.5 Numbers as ascending paths in a rooted binary tree

This was proposed to learners (b), who metaphorised numbers from 0 to 63 as ascending waks in the six-generation binary tree. They went further to the infinite binary tree (Needham, 1956, p. 276) and its *boundary*, subsequently re-discovering Hensel's notation of 2-adic numbers as series of powers of 2 (Dickson, 1910). Within this framework, they interpreted the far-right path (which consistently chooses the right branch) as representing the number -1.

5.6 A dice-based positional system (invented nowhere on earth)

Some islanders, somewhere on earth, invented dice, but did not develop the concept of 0. Remarkably, they manage quite well to efficiently record substantial quantities and perform arithmetic. Their primary school students, it seems, exhibit greater contentment than our own. The question then arises: how did they achieve this? This was proposed to learners (c), who, working in small random groups, were able to rediscover the islanders' dice-based positional system in less than an hour through hands-on manipulation of dice.

Figure 7 illustrates a sample of their work, showing the following “dice expansions” (beginning at the upper left):

$7=1\times6+1\times1$ =“one-one”, $30=4\times6+6\times$ =“four-six”, $42=6\times6+6\times1$ =“six-six”,
 $12=1\times6+6\times1$ =“one-six”, $36=5\times6+6\times1$ =“five-six” and
 $2024=1\times6^4+3\times6^3+2\times6^2+1\times6+2\times1$ = “one-three-two-one-two”.

Furthermore, the addition of 1 to 42 is demonstrated, by simply rearranging and flipping the dice to obtain “one-one-one”. The primary and friendly rule is that *a die showing a six can be transferred one column to the left by simultaneously flipping it to show a one*.



Figure 7. In-service teachers’ work on the dice-based positional system

A compelling epistemological question, often overlooked by cognitive archaeologists (Overmann, 2023), is why such a system apparently never emerged in any known terrestrial culture.

5 Discussion, caveats and open questions

We have merely scratched the surface of Shao Yong’s work and its mathematical, epistemological and didactical implications. Nevertheless, we have highlighted specific connections between his work and contemporary Western mathematics and science that seem to have been overlooked by contemporary scholarship. For instance, the *Xiantian Tu* can be interpreted as a diagram depicting the *Origin of the (64) Species* of hexagrams, conceived as the outcome of a *branching process*. Contemporary Chinese scholars, however, refer only in very general terms to the analogy between Shao Yong’s and Darwin’s evolutionary thinking. Also, Needham (1956, p. 276) specifically notes that “the [binary branching] process continues until the sixty-four hexagrams are formed, and could naturally go on *ad infinitum*” but does not connect this with Hensel’s analytic-algebraic construction of *p*-adic numbers. Moreover, the “symmetree” appeared in cosmology (one of Shao Yong’s main concerns) almost a century later. Students from group (b), interestingly, saw the transition from one *Xiantian* diagram to another as a change of coordinate system (e.g. rectangular to polar). Significantly, Shao Yong’s work emphasises syn-

thetic over analytical, as well as evolutionary over taxonomic, perspectives. Shao Yong's *Xiantian* diagrams inspired the design of various divergent a-didactic situations for our students, especially for those from group (b) following a distinct historical track (Fried, 2001). Their engagement in these history-track activities enhanced their mathematical understanding of subjects typically taught in an abstract and disconnected manner. Specifically, they constructed a novel metaphor for integers, as finite ascending paths in the infinite binary tree. Learners from the less mathematically inclined groups (a) and (c) reported having, for the first time, experientially understood the binary system, of which they previously held only vague recollections.

One limitation of our historical-pedagogical approach is that it requires ample time for students to explore and undertake the 'twist of mind' inherent in our divergent a-didactic activities and situations. One activity can take at least 60 minutes within one of our classes, which last 90 minutes. Especially first-year university students often come from a secondary school mathematics education that trained them primarily to merely execute prescribed tasks. Consequently, they often struggle to adapt to open-ended, problematic a-didactic situations where they are expected to explore, formulate questions, and configure their own problems.

Furthermore, our study needs an epistemic caveat. We wonder whether *Xiantian Tu* represents the first emergence of the binary tree in human history. To us, however, the binary tree is not a pre-existing mathematical object "standing out there", with various avatars throughout human history. Rather, we view it as a typical Western mathematical construction (hierarchical, per Deleuze & Guattari, 1980), which we automatically *project*, from a Whiggish perspective (Butterfield, 1931; Fried, 2001), onto the (rather *process* oriented) *Xiantian Tu*. Epistemological disorientation (Clark et al., 2018) might work as an antidote to this stance, which may be aptly metaphorised by Friedrich's (1817) 'Wanderer above the Sea of Fog' painting. In this picture the gentleman could represent Bourbaki⁵ retrospectively looking down on the history of mathematics from the vantage point of current structuralist mathematics. In

⁵ See Corry (2004) for the history of the mythical structuralist mathematician Bourbaki.

Soto-Andrade et al. (2022) we evoked the “Brownian mirage” of perceiving direction in our path backwards, which from the perspective of the past is indeed the path of a random walker (Fried, 2001).

Some open questions:

- Can concepts or methods that never emerged in human history nevertheless emerge in an enactivist classroom under minimal prompting? A cognitive archaeology fiction of sorts?
- In what degree is the history of mathematics a random walk?
- Is *epoché* (“bracketing”) genuinely possible in a contemporary classroom?
- To what extent can the enactivist historical-pedagogical approach exemplified in this paper be integrated into secondary or tertiary mathematics education as an antidote to the prevailing cognitive abuse (Watson, 2000) often observed in mathematics teaching for non-mathematically inclined learners?

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