EVOLUTIONARY ARGUMENTS FOR HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION: A CRITICAL AND CONTRUCTIVE DISCUSSION

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ABSTRACT

In research done in the field of history and pedagogic of mathematics, theories from other areas of didactics of mathematics are often an integral part. The vice versa is not the case, and when history of mathematics finds its way into general mathematics education research, it is often in forms related to so-called evolutionary arguments. In this paper, we discuss this by analyzing three influential theories in didactics of mathematics, which have been informed by history of mathematics: works by Anna Sfard (1991, 1995), Guy Brousseau (1997), and Guershon Harel and Larry Sowder (2007). We analyze their work with respect to how they use the history of mathematics and for what purposes in order to invite a discussion of the potential influence of HPM in more general mathematics education research.

1 Introduction

When we look at theories and theoretical frameworks and constructs within the HPM research area, which deal with the roles of history of mathematics in mathematics education and the significance of history for the teaching and learning of mathematics, a variety of theories from other areas of mathematics education form an integral part of much of the research. If we look at it from the other side, from mathematics education research at large, the history of mathematics does not play a significant role, and when it finds its way into the more general mathematics education research literature it is often in a form which is related to so-called "evolutionary arguments" (Jankvist, 2009). In this paper, we take a closer look at this by displaying some examples from mathematics education. We look at three firmly rooted and influential theories in didactics of mathematics, which have been informed by history of mathematics: Anna Sfard's model for learning of mathematical concepts, Guy Brousseau's work on epistemic obstacles, and Guershon Harel's and Larry Sowder's development of students' proof schemes. We analyze their work with respect to how they use the history of mathematics and for what purposes. Before presenting our analyses, we provide a brief introduction to evolutionary arguments resorting to the history of mathematics in mathematics education. Upon the analysis of the three cases, we invite a critical and constructive discussion of the role of history of mathematics in mathematics education research and point towards other ways in which history (may) inform theoretical developments in the field.

2 Evolutionary arguments in mathematics education research

The idea that "ontogenesis recapitulating phylogenesis" permeated educational thoughts, by transferring the idea from biology to psychology and cognitive development, and philosophies from the turn of the nineteenth century and well into the twentieth century—eventually known as the "genetic principle". In mathematics, the German mathematician Felix Klein[1] advocated for the genetic principle in teaching, that (quoted from Schubring, 2011, p. 82):

> … teaching should, by tieing to the natural disposition of the youth, lead them slowly to higher things and eventually even to abstract formulation, by following the same path on which the entire mankind struggled to climb from its naïve primitive state upwards to more developed insight.

In the early 1960s, leading mathematicians from North America published a memorandum "On the Mathematics Curriculum of the High School", formulating "fundamental principles and practical guidelines", such as their 5th principle, labeled the "Genetic method" (Ahlfors et al., 1962, p. 190-191):

> It is of great advantage to the student of any subject to read the original memoirs on that subject, for science is always most completely assimilated when it is in the nascent state" wrote James Clerk Maxwell. There were some inspired teachers, such as Ernst Mach, who in order to explain an idea referred to its genesis and retraced the historical formation of the idea. This may suggest a general principle: The best way to guide the mental development of the individual is to let him retrace the mental development of the race—retrace its great lines, of course, and not the thousand errors of detail. This genetic principle may safeguard us from a common confusion: If A is logically prior to B in a certain system, B may still justifiably precede A in teaching, especially if B has preceded A in history. On the whole, we may expect greater success by following suggestions from the genetic principle than from the purely formal approach to mathematics.

[1] For a discussion on Felix Klein and the genetic principle, see Jahnke et al. (2022).

This plea for the genetic method was commented on by the director of the School Mathematics Study Group (SMSG), Edward G. Begle from Stanford University, who made it clear that most of the guidelines formulated in the Memorandum agreed with the texts of the SMSG, except, probably

> … the "Genetic Principle" stated in the fifth guideline. This principle, as stated, would, for example, deny to our students the efficiency of using algebra in the first course in geometry and would require children to learn to compute with Egyptian, Babylonian, Greek and Roman numerals before being introduced to the historically later but far more efficient place-value decimal system. (Begle, 1962, p. 426).

Nonetheless, the genetic principle was put forward by at least the 65 mathematicians in the USA and Canada, who signed the memorandum that was published in *The American Mathematical Monthly* in 1962.

3 Three cases of theory building in mathematics education drawing on the history of mathematics

The genetic principle re-entered the discourse in mathematics education in the 1980s with Jean Piaget and his studies on the relation between psychogenesis and history of science (Schubring, 2011, p. 84). In the following, we will look at three cases from the past 30 years where evolution and genesis in the history of mathematics is used to advocate theoretical constructs.

3.1 A model for concept development informed by history

In a well-known series of papers, Anna Sfard (e.g. 1991, 1995) presented a theoretical model for concept development. The model is based on observations from the history of development of mathematical concepts and is applied to the individual learning of mathematical concepts. Sfard (1995, p. 15) wanted to find the "roots of the difficulties experienced by students" when they are confronted with abstract mathematical concepts.

She distinguishes between two different ways of conceiving mathematical objects: operational conception and structural conception. Her thesis is that there is an ontological gap between the two, which she found to be an explanation for the difficulties experienced by learners of mathematics in conceiving mathematical objects. By structural conception she meant being able to see and understand a mathematical entity as an (abstract) object, which means being able to refer to it as a "real" thing. In contrast, or rather dual, to this conception, by operational conception she referred to processes, algorithms and actions, meaning, she explained: "interpreting a notion as a process implies regarding it as a potential rather than an actual entity, which comes into existence upon request in a sequence of actions" (Sfard, 1991, p. 4). She argued that "*In the process of concept formation, operational conceptions would precede the structural*" and claimed that "this statement is basically true whether historical development or individual learning is concerned" (Sfard, 1991, p. 10, italics in the original).

Sfard took this as a basic conjecture from which she deduced her model for concept acquisition. It led to a model describing a cyclic process consisting of three phases: (1) The preconceptual stage, where mathematicians accustomed themselves to certain operations on already known (i.e. constructurally conceived) objects. (2) A period of predominantly operational approach. In this phase the coming, new object begins to emerge. (3) The structural phase, where what has been emerging in the previous phases, becomes recognized as a full-fledged new mathematical object (Sfard, 1991, p. 13).

She connected her model to students' individual learning, quoting Piaget: "the [mathematical] abstraction is drawn not from the object that is acted on, but from the action itself." (Sfard, 1991, p. 17). Comparably to the three phases above, she distinguished between three stages in individual learners' concept formation. (1) Interiorization: the stage where "a learner gets acquainted with the processes which will eventually give rise to a new concept [...] These processes are operations performed on lower-level mathematical objects." (2) Condensation: the stage where "a person becomes more and more capable of thinking about a given process as a whole, without feeling an urge to go into details. [...] The condensation phase lasts as long as a new entity remains tightly connected to a certain process." (3) Reification: happens "when a person becomes capable of conceiving the notion as a fully-fledged object. [...] Reification, therefore, is defined as an ontological shift." (Sfard, 1991, p. 18-19)

In her model, reification requires that one have tried to make operations with the notion as a whole. This is exactly, she argued, why reification is so difficult for mathematics learners, and this is her answer to her initial question of why students experience such difficulties when they are confronted with abstract mathematical concepts. Sfard called it a vicious circle: In order to reify a mathematical object, one must already have used it as a (reified) object in higher-level interiorization processes.

In 1995, she further elaborated on her claims of historical and psychological parallels (Sfard, 1995, p. 17). She collected instances from history of algebra and from classroom experiments to argue that it is inherently difficult to transition from an operational to a structural approach to mathematical thinking (Sfard, 1995, p. 22). Because of this, she concluded,

> for those who teach, therefore, familiarity with the history of mathematics is not just optional; rather, it seems indispensable to make them alert to the deeply hidden difficulties concerned with new concepts. (Sfard, 1995, p. 34).

3.2 Unavoidable epistemological obstacles

Our next case stems from Guy Brousseau's (1997) well-known Theory of Didactical Situations, where history of mathematics comes into play when talking about *epistemological obstacles*—a notion Brousseau has borrowed from the French philosopher, Gaston Bachelard. Obstacles are often identified through students' difficulties with or errors related to certain concepts. According to Brousseau, an obstacle is to be considered as a piece of knowledge, wrong as it may be, and not as a lack of knowledge.

Brousseau distinguishes between obstacles of ontogenetic, didactical, or epistemological origin. The first ones are those due to limitations (e.g. neurophysiological ones) of a student at a given time. The second ones are those which depend on choices made within an educational system. The third kind is the ones of most interest to us here. According to Brousseau, these obstacles play a formative role and should not be avoided. They may, he wrote, be identified in the history of the concept itself. Brousseau's general hypothesis was that "certain of the students' difficulties can be grouped around obstacles attested to by history" (Brousseau, 1997, p. 96):

> It is in the analysis of resistance and in the debate […] one must look for elements which will allow the identification of obstacles for the students. In any case, it will never be enough to tack—to apply without modification—historical study onto didactical study. It is from this origin, too, that we must draw arguments in order to choose a genesis of a concept suitable for use in schools and to construct or 'invent' teaching situations that will provide this genesis.

An important element is that historical studies may not be applied directly in a didactical situation; a modification must take place. Although it is not direct ontogenesis-phylogenesis that Brousseau argues for, it still contains central elements of evolutionary argumentation. Here, it may be relevant to also notice Brousseau's concept of "*genèse fictive*" (fictive genesis), related to the "true functioning of science," in the context of his distinction between savoir (knowledge as a body of content) and *connaissance* (personal understanding) (Brousseau, 1986).

3.3 Historical-epistemological factors of students' proof schemes

According to Guershon Harel and Larry Sowder (2007, p. 809), "A person's (or a community's) proof scheme consists of what constitutes ascertaining and persuading for that person (or community)". They categorize a proof scheme into one of three main classes, each containing various subclasses. The first class is what they refer to as 'external conviction proof' schemes. These can appear as an authoritarian proof scheme, where something is believed to be true because an authority figure or textbook says so. Their second class is the 'empirical proof' schemes, which includes inductive proof schemes, where one is convinced by specific empirical examples or a "crucial" generic example. Their third class is deductive proof schemes from mathematics, based on deduction from a set of premises.

Upon describing their construct of proof schemes, Harel and Sowder turn to a discussion of mathematical and historical-epistemological factors in relation to proof schemes. They provide an analysis of the proof scheme constructed across three historical periods of mathematics: "Greek mathematics, post-Greek mathematics (approximately from the 16th to the 19th century), and modern mathematics" (p. 811). The reasoning that they provide is that proof schemes are used to validate assertions within specific contexts. Therefore, it is essential to consider the nature of these (historical) contexts when discussing proof schemes. Additionally, they argue, the motivation or intellectual need driving conceptual changes over time is important, hence their discussion focuses on three interconnected aspects of historical and epistemological development: (a) the context of proving, (b) the methods of proving (proof schemes), and (c) the motivation behind conceptual changes. An understanding of these elements can provide insights into key aspects of learning and teaching proof, they assert, and this also with reference to evolutionary arguments:

> It is still an open question whether the development of a mathematical concept within an individual student or a community of students parallels the development of that concept in the history of mathematics, though cases of parallel developments have been documented (e.g., Sfard, 1995). If this is the case, one would expect that the path of development would vary from culture to culture. (p. 816)

Harel and Soweder are interested in what extent the history of mathematics may reveal the motivation for the shift from more empirical proof schemes (and even external proof schemes) to deductive proof schemes:

> To what extent did the practice of mathematics in the 16th and 17th centuries reflect global epistemological positions that can be traced back to Aristotle's specifications for perfect science? These are important questions, if we are to draw a parallel between the individual's epistemology of mathematics and that of the community. (p. 818)

Although several of the references to evolutionary arguments that Harel and Sowder provide, are phrased as questions, e.g., as in the quotes above, their very endeavor of trying to find parallels between students' learning of mathematics and the coming into being of mathematical constructs reflects the authors' conception of the potential role of the history of mathematics in mathematics education.

4 Discussion

To be sure, the recapitulation thesis has been subject to criticism from the various domains of history, psychology and biology. Here we will only mention one aspect of the discussions within the field of history and didactics of mathematics, namely the underlying theory of history. As has also been pointed out by Schubring (2011) and by Furinghetti and Radford (2002), taken strictly, the recapitulation theory in the learning of mathematics leaves no room for a genuine history of mathematics. It presupposes a view of the history of mathematics as a subject that in essence only has a history in the sense that definitions and theorems are articulated and written down in historical contexts. What is missing is sensitivity to the contexts in which the development of mathematics took place by people who lived and acted under specific historical circumstances.

Harel and Sowder draw no conclusions regarding the evolutionary nature of their investigation. Although they do not argue for or against the ontogenesis-phylogenesis thesis, it is still saying that when they refer to the history of mathematics, it is this thesis that underlies and permeates their investigation. An investigation that at its heart is "onto-phylo" since it searches for parallels between the historical, mathematical communities' proof schemes and that of modern-day individuals. They are not blind, though, to the critiques (Harel & Sowder, 2007, p. 816):

> Are there common elements or phases to different paths of development across cultures? Did the development of the concept of proof in, for example, China and India follow a similar path to that of the Western world or was there a leap in time from using perceptual proof schemes to modern axiomatic proof schemes?

Brousseau suggested that history serves a role in the teaching of mathematics due to the inevitability of epistemological obstacles, although it is not to serve in a direct manner. Rather it must be considered from a didactical point of view and adapted to the teaching and learning situation of today. In that sense, it is perhaps not too different from the way Freudenthal saw history as a "guide" to guided reinvention (see Jahnke et al., 2022). Brousseau (1997, p. 101) concluded that "historical arguments can intervene in choices of teaching under the surveillance of a Theory of Didactical Situations".

Sfard wrote against what she called a structural way of teaching new concepts to students, which she found was in custom in high school teaching at that time. She is using historical observations to construct a model for the formation of mathematical concepts. She uses the model to understand students' cognitive behavior, and this is where her second thesis comes into play:

> what has been said (about the formation of concepts learned from history [in the 1992 paper with the function concept as case, p. 65]) applies also to cognitive development of mathematics students.

She wants to make a case for another way of introducing mathematical concepts to students. She uses her model to say that we should begin operationally—and she uses traits from the history of mathematics to show instances where former mathematicians, according to her, have worked in that way. If we look at the practice of teaching, she is advocating for history of mathematics as part of pre-service teacher education, so they can teach with history of mathematics, which, as we interpret her writings, may have similarities to the distinction of teaching *with* and *about* the nature of science in science teaching from Abd-El-Khalick's (2013) framework (see also Kjeldsen, 2014).

Our three cases are influential theories in didactics of mathematics, and they are examples of theory building that are informed by history of mathematics—and for all three, it is in a form, which is related to evolutionary arguments. On the one hand, in the huge amount of research done in the HPM community over the past decades (see e.g., Chorlay et al., 2022), we find inspiration and opportunities for research where the history of mathematics has potential to inform theory building in general mathematics education without reference to evolutionary arguments. On the other hand, there are other theories in didactics of mathematics where history is a coherent domain to look for in order to inform further theory building, and there are (at least a few) 'voices' in mathematics education research outside the HPM-group, pointing towards history of mathematics having such a role.

One such theory is Anna Sfard's (2008) theory of commognition She did not herself make the connection, but within that theory it is possible to make a theoretical argument for using history of mathematics to make students aware of meta-discursive rules in mathematics and make them explicit objects of students' reflections (Kjeldsen & Blomhøj, 2012; Kjeldsen & Petersen, 2014). This has been further investigated in teaching practice within the impressive TRIUMPHS project (e.g., Barnett, 2022), see also Bernardes and Roque (2015).

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