ON VISUALIZATION IN HISTORY OF MATHEMATICS AND TEACHING

Signs, diagrams and graphics

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ABSTRACT

In this paper, we study types and roles of visualization in four historic key moments: Greek geometry with Euclid's *Elements* and Hérigone; Algebra in Renaissance with Jerome Cardan and Jacques Peletier du Mans; Cartesian geometry in 17th century with René Descartes and Bernard Lamy; Logic with Gottfried Leibniz and John Venn. For this purpose, we use the classification of signs and the design of diagrams of Charles Sanders Peirce. It leads us to a semiotic history which is rich in reflections on teaching practices with the aim, in particular, of enabling teachers to explain to students the role and meaning of signs, but also that of diagrams in mathematical practice.

1 Introduction

Our aim is to emphasize the role of visualization in teaching. This requires, not only to know when and how signs were introduced in history, but to present different meaning of signs, patterns and writings in teaching of mathematics such as to hold a discourse on figures, to give an operating status to objects, to generalize procedures, to represent relations between objects, or to transport practices and knowledge from one mathematical field into another. We begin by presenting the semiotic of Charles Sanders Peirce before using it to analyze four historic key moments.

2 The division of signs by Charles Sanders Peirce

Peirce (1839-1914) was an American logician and philosopher, son of the mathematician Benjamin Peirce. His numerous philosophical and scientific papers on semiotics or sign theory appeared in the *Collected Papers*. His division of signs in three kinds, which he called “icons”, “indices” and “symbols”, had been given in a paper of 1885. Peirce, 1998, 460-461).

"Icons" are used to represent the objects they resemble, as the "icon" for the square figure (Fig. 1, left), the "indices" represent objects to indicate objects, such as *A*, *B*, *C*, *D* for the vertices of a square (Fig. 1, center), "symbols" represent objects arbitrarily, as √2 for the magnitude of *BC* (Fig. 1, right).

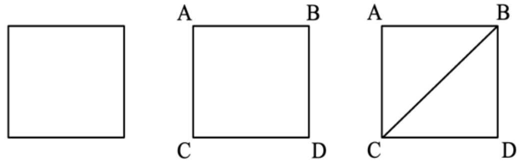


Figure 1. The three kinds of signs by Peirce

The roles of "icons" and "indices" are related to their uses. "Icons", which look like objects, can be used to observe them with a certain intention, such as the three equal triangles (Fig. 2, top). While the "indices" can indicate connections to be observed. "Indices" 1, 2 and 3 invite us to establish a link between the angles of the triangles, to see the three angles of the triangle joined together to form a flat angle (Fig. 2, bottom). They are used here to show a visual phenomenon. Peirce thus specified the function of an index (Peirce, 1885, p. 362): "I call such a sign an index, a pointing finger being the type of class. The index asserts nothing; it only says 'there!'. It takes hold of our eyes as it were, and forcibly directs them to a particular object, and there it stops.".

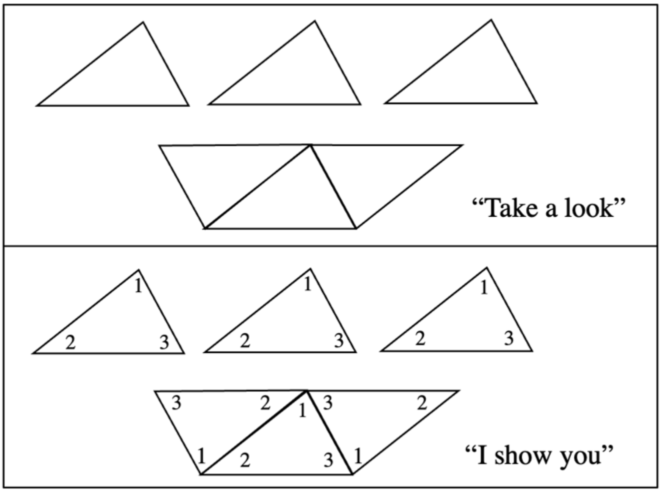


Figure 2. Roles of icons and indices

Peirce's "diagram" is also linked to the idea of resemblance, to represent not an object but a relation between objects. He wrote: "The pure diagram is designed to represent and to render intelligible, the form of relation merely. Consequently, diagrams are restricted to the representation of a certain class of relations; namely, those that are intelligible" (Peirce, 1906, p. 314).

2 Signs and diagrams in geometry: Euclid and Hérigone

We examine the role of "signs" and "diagrams" with the Proposition 32 of Book I in Euclid's *Elements* and then in a textbook of Pierre Hérigone (1639).

2.1 Signs and diagrams in Euclid's *Elements*

Euclid began with two statements. The first statement gives the proposition without letter to designate the elements of the figure, while the second statement uses "indices" that refer to a drawing (Euclid, 1956, p. 316):

> In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles. Let *ABC* be a triangle, and let one side of it *BC* be produced to *D.* I say that the exterior angle *ACD* equals the sum of the two interior and opposite angles *CAB* and *ABC,* and the sum of the three interior angles of the triangle *ABC, BCA,* and *CAB* equals two right angles.

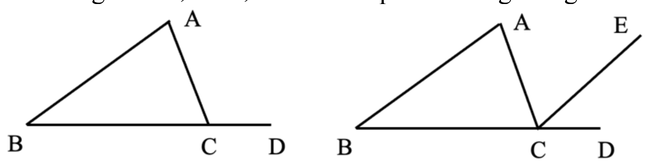


Figure 3. Proposition 32 of Book I

This statement introduces an "icon" for the triangle ABC, which "resembles" a triangle. But is not a triangle, since its segments are more or less thick lines, whereas a line is defined by Euclid as a length without width (definition 2). The "indices" *A*, *B*, *C*, and *D* allow us to follow and to understand the discourse of the first statement (Fig. 3, left). The proof begins after construction of *CE* parallel to *AB* (Fig. 3, right) (Euclid, 1956, p. 317):

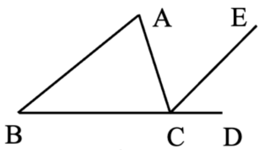
> Then, since *AB* is parallel to *CE,* and *AC* has fallen upon them, the alternate angles *BAC* and *ACE* are equal to one another (I. 29). Again, since *AB* is parallel to *CE,* and *BD* has fallen upon them, the exterior angle *ECD* is equal to the interior and opposite angle *ABC* (I. 29). But the angle *ACE* was also proved equal to the angle *BAC*; therefore, the whole angle *ACD* is equal to the two interior and opposite angles *BAC* and *ABC.*

The two first sentences have a common discursive pattern which forms a disposition to be observed and serves to associate them. This pattern is a "diagram" in the sense of Peirce, that makes intelligible a relationship between lines and angles (Peirce, 1885, p. 363-364):

> The truth, however, appears to be that all deductive reasoning, even simple syllogism, implies an element of observation. Indeed, deduction consists to construct an icon or a diagram such as relations between parts of this icon have to present a complete analogy with parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts.

2.2 Signs and diagrams in Hérigone's *Elements*

The French professor of mathematics and algebraist Pierre Hérigone (Massa Esteve, 2006) edited *The first six Books of the Elements of Euclid demonstrated by notes, with a very brief and intelligible method*. He wrote in his Preface (Hérigone, 1639, np.): "seeing that the greatest difficulties were in proofs, on whose intelligence depends the knowledge of all parts of mathematics: I invented a new method of making proofs, brief and intelligible, without the use of any language." We took his proposition 32 of Book I of Euclid (Fig. 4).



		< a + < b + < acb 2/2 2 ⌋
		Prepar.
	31.1	ce = ba. α
		Prove.
Hypoth.	α.29.1	< eca 2/2 < a,
abc is Δ,	α.29.1	< ecd 2/2 < b,
bcd est ____.	2.a.1	< a + < b + < acd. β
Requ. to prove.	13.1	< acb + < acd 2/2 2 ⌋,
< acd 2/2 < a + < b.	β.1.a.f	< a + <b + < acb 2/2 2 ⌋.

Figure 4. Proposition 32 (Hérigone, 1639, p. 74)

signs	types	things
Δ	icone	triangle
____	icone	line
<	icone	angle
abc	index	triangle
bcd	index	line
acd	index	angle
a, b	indices	angles
2/2	symbol	equality
=	symbol	parallel
⌋	symbol	right angle

Table 1. "Icons", "indices" and "symbols" in Hérigone

Hérigone used “icons”, “indices” and “symbols” (Barbin, 2011) in the sense of Peirce (Table 1). The use of “symbols”, that means arbitrary signs, permits to observe “diagrams” better and to render them to be more “intelligible”.

3 Symbols and diagrams in algebra: Cardan and Jacques Peletier

Most of the signs of algebra were introduced in the 15th and 16th centuries in books on arithmetic and algebra. In this period, each author has his own ways of writing arithmetic and algebraic signs. For our purpose, it is interesting to observe and compare some writings of the equation $2\ x^2 - 5x = 23$ (Plane, 2006), to find out some types of symbols and their roles (Table 2).

Stiffel (1525)	2 z equal 5 x + 23	Girard (1629)	2(2) – 5(1) = 23(0)
Gosselin (1577)	2Q M 5L equal 23	Viète (1600)	$2a_q$ – 5a æq. 23
Bombelli (1572)	$\overset{2}{2}$ m $\overset{1}{5}$ equal a 23	Harriot (1631)	2aa – 5a = 23
Viète (1580)	2Q – 5N equal 23	Hérigone (1634)	$2a_2 \sim 5a\ {}^{z}/_{z}\ 23$
Ramus (1586)	2q – 5 l equal 23	Descartes (1637)	2xx – 5x α 23
Buteo (1559)	2◊ M 5p = 23	All 18th century	2xx – 5x = 23

Table 2. Writings of an equation (Plane, 2006, p. 28)

For arithmetic operations plus and minus, Jerome Cardan used signs *p* and *m* in 1545, that are abbreviations. While the German algebraist used + and –, introduced by Johannes Widmann in 1489, that are “symbols” because they seem arbitrary. For the unknown of an algebraic problem and its power, the French Guillaume Gosselin also used abbreviations L (Ligne) for the unknown and Q (Quarré) for its square, like François Viète and Ramus, but Johannes Buteo (alias Jean Borel) used a geometric “icon” with the form of a lozenge for the square of the unknown. Rafael Bombelli and Albert Girard used numerical signs, that allowed them to make obvious, for example, the rule on the multiplication of powers of numbers. These last signs have to be considered as “indices”, that means signs to observe and for showing.

In 1559, Buteo adopted the sign for equality introduced by the Robert Recorde who wrote: “to avoid the tedious repetition of these words: is equal to: I will set a pair of parallels, 2 lines of one length thus = because no 2 thinks can be more equal” (Recorde, 1557, p. 235). His sign is both a geometric “icon” and a “symbol”, a kind of “metaphor”. For several decades, algebraists preferred to write in full the word “equal”, which designates the primary relation of algebra but which is also the most frequent word in algebra.

The introduction of symbolism in algebra made it possible to write algorithms for solving algebraic equations in symbolic form, thus giving rise to what Peirce called the “icons” of algebra. Indeed, he also used the term “icon” for the representation of formulas, because they are used for observing calculations and finding solutions (Peirce, 1885, p. 364):

> As for algebra, the very idea of the art is that it presents formulae which can be manipulated, and that by observing the effects of such manipulation we find properties not to be otherwise discerned. In such manipulation, we are guided by previous discoveries which are embodied in general formulae. These are patterns which we have the right to imitate in our procedure, and are the *icons par excellence* of algebra.

Thus, the solution of an equation can be given by a formula, as we are used today. Cardan in his *Ars Magna* (1545) wrote a solution of the equation of degree 3, $x^3 + 6 = 20$, as an “algebraic icon” (Fig. 5, left). This transition from arithmetic to algebra led him to write and observe an imaginary solution for an equation of degree 2. It seemed to him “sophisticated”, but Bombelli would admit imaginary solutions with profit to solve equations of degrees 3 and 4.

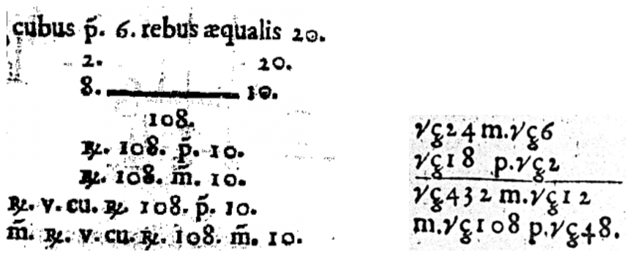


Figure 5. Formulas in algebra (Cardan, 1545, p. 250) (Peletier, 1554, p. 159)

As a result, too, the algebraists and geometers of the 17th century considered what they called, not numbers, but “irrational quantities” and “negative quantities”, because symbols made it possible to observe that arithmetic operations could be extended to these quantities (Barbin, 1995).

In the first algebra book in French, edited in 1554, Jacques Peletier considered that irrationals can be considered as real numbers because “they have their algorithm, their order and infallible rules, as well as the rational ones " (Peletier, 1554, p. 131-132). “Symbols” permit him to show the similarity of the rules and the operations for rational and for irrational numbers, for instance in a multiplication between (√24 – √6) by (√18 + √2) (Fig. 5, right).

4 Signs and diagrams in geometry: Descartes and Lamy

In *The geometry* of 1637, that is a part of his *Discourse of the method*, Descartes began by explaining how to use the operations of arithmetic in geometry. For example, the product of two segments is still a segment for Descartes, and not a rectangle, as in Euclid's *Elements*. That is why, he introduced a segment that he called "unit", by analogy with the unit of arithmetic and which he denoted by the symbol "1". This "symbol" is not, like for Peirce, an arbitrary sign. Taking *AB* the unit, the product of *BD* by *BC* is *BE*, thanks to Thales' theorem (Fig. 6).

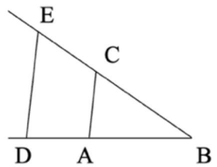


Figure 6. The product of two segments (Descartes, 1637, p. 334)

Descartes proposed a method that made it possible to solve "all geometric problems". It is based on an algebraic calculation of segments with the symbols of arithmetic operations and the square root. It proceeds in five steps: 1) the problem must be assumed to be solved; 2) designate each known and unknown segment by a "symbol", x, y, z, etc. for the unknown segments and a, b, c, etc. for known segments; 3) translate the problem into relationships between these letters; 4) obtain one or more equations; 5) solve the equation(s).

If we write, like Descartes, each segment by one italic letter only (Fig. 6), if $BD = a$, $BC = b$, $BE = d$, $AB = 1$, then $d = ab$. These letters are "symbols" that represent segments. The transition from the geometric problem to the algebraic one is therefore based on a new use of "symbols" in geometry, that means a symbolic arithmetization of geometry. This translation has the effect, as Descartes liked to remark, of no longer having to contemplate the figures of geometry, but to observe calculations. Therefore, there is a shift from "geometric icons" to the diagrams, that are "algebraic icons".

Let us examine the transition from "geometric icons" to "algebraic icons" by reading the Cartesian method in Bernard Lamy's textbook *Elements of geometry* (1734). Lamy took the proposition 4 of Book II of Euclid's *Elements*. It must be proved that, given any point *D* of a given segment *AB*, the area of the square constructed on *AB* is equal (in area) to (the sum of) those of the squares on *AD* and *DB* and twice the rectangle between *AD* and *DB* (Fig. 7).

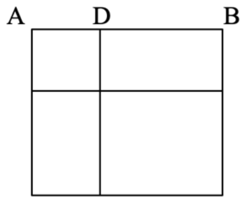


Figure 7. Proposition 4 of Euclid's Book II (Euclid, 1956, p. 379).

After the construction of the "geometric icons", Euclid's proof is based on the observation of "geometric icons". While, after having stated the Cartesian method, Lamy wrote (Lamy, 1734, p. 138):

> Let a line AB be cut in two parts by a point D. We have to prove that $AB^2 = AD^2 + 2\ AD \times DB + DB^2$.
>
> 
>
>
> Let $AD = b$, $DB = d$. Therefore $AB = b + d$. But the square *of* $b + d$ is $b^2 + 2\ bd + d^2$. But $AD^2 = bb$, $DB^2 = dd$, $2\ AD \times DB = 2bd$. Therefore $AB^2 = bb + 2\ bd + dd$.

It is noteworthy that Euclid's "geometric icons" have disappeared. Lamy only represented a segment with three points. The transition from geometry to algebra requires the Cartesian translation of the problem, namely:

$$AB^2 = AD^2 + 2\ AD \times DB + DB^2.$$

Its symbolic script, where each segment is represented by a letter, is:

$$(b + d)^2 = b^2 + 2\ bd + d^2.$$

Lamy concluded by observing this algebraic icon, "an icon par excellence of algebra", which is true as the result of an algebraic calculation. In the seventeenth century, evidence of the calculation seemed more enlightening than Euclidean discourse, which seemed more likely to convince (Barbin, 1992).

5 Graphics and diagrams in Logic: Leibniz and Venn

From Aristotle, logic appears as a means of verifying propositions by using schemas (Barbin, 2024). Peirce considered Aristotelian syllogisms as cases of what he called diagrams: "For instance, take the syllogistic formula,

All M is P
All S is M
(so) All S is P.

This is really a diagram of the relations of S, M and P" (Peirce, 1885, p. 364).

Leibniz introduced two geometric graphics to represent Aristotle's premises and syllogisms in a Latin manuscript of 1686, only known at the beginning of the 20th century, thanks to its edition by Louis Couturat. In the graphics, the affirmative universal premise “All B are C” is represented by two nested segments *B* and *C*. In the second, the premise is represented by nested circles. For the other ones, segments and circles are disjoint or overlapping (Fig. 8). Syllogisms are represented in the same way and, according to Leibniz's manuscript, they are thus “verified” (Fig. 9).

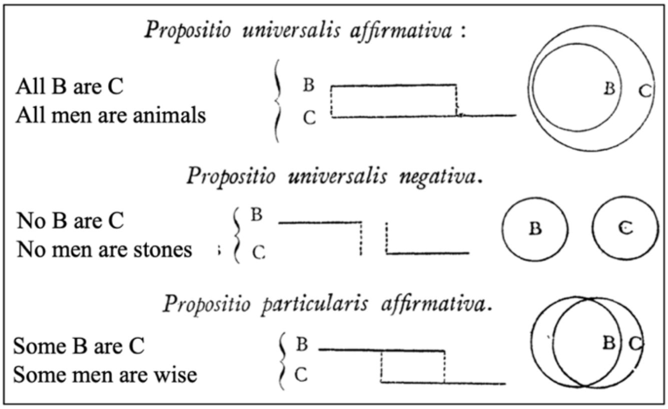


Figure 8. The premises in Leibniz (Leibniz, 1686, p. 292-293)

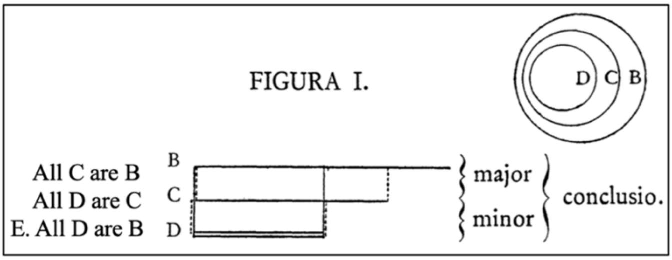


Figure 9. The syllogism “Barbara” in Leibniz (Leibniz, 1686, p. 294)

Representation of premises and syllogisms by means of circles appears in Leonhard Euler's *Letters to a Princess of Germany* (Euler, 1802, p. 397):

> These circles, or rather these spaces, for it is of no importance of what figure they are of, are extremely commodious for facilitating our reflections on this subject, and for unfolding all the boasted mysteries of logic, which that art finds it so difficult to explain; whereas, by means of these signs, the whole is rendered sensible to the eye.

He did not mention Leibniz and the success of his *Letters* led to the circles being called “Euler's circles”. Peirce studied them under the name of “Euler diagrams” in a paper of 1911. He explained how a syllogism is “illustrated by means of circles” on an example of the same type as Leibniz. He wrote that after Euler, there were several attempts to improve his system, but they were all failures until the publications of the logician John Venn.

In his 1881 *Symbolic Logic*, Venn associated syllogisms with “diagrams”: “This will set before the eye, at a glance, the whole import of the propositions collectively” (Venn, 1881, p. 123). He took care to distinguish the “Euler’s circles”, from his own diagrams. He took the syllogism “Celarent”:

No *y* is *z*
All *x* is *y*
(so) No *x* is z

With Euler, the circles are drawn one after the other. First, we must draw two disjoint circles *y* and *z* corresponding to the premise “No *y* is *z*”, then the circle *x* is inscribed in the circle *y*, to conclude that “No *x* is z” (fig. 10, left). On the contrary, with Venn diagrams, the circles are drawn from the beginning in a general position and then the areas corresponding to the premises are removed one by one. Here, we first remove (or hatch) all of the part of the circle *y* that is in the circle *z*. Then we hatch the part of *x* that is not in *y*. We can then see that all the remaining (unhatched) part of the *x*-circle is not in the rest of the *z*-circle (fig. 10, right). For Venn, diagrams were a “visual aid”: “One main source of aid which diagrams can afford is worth noticing here. It is that sort of visual aid which is their especial function” (Venn, 1881, p. 118).

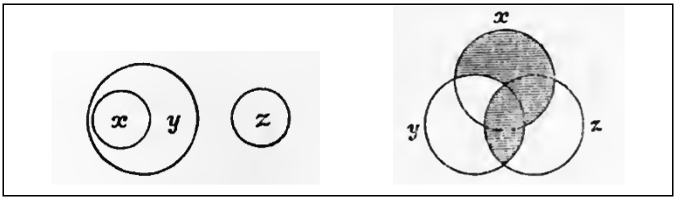


Figure 10. Euler’s circles and Venn’s diagram (Venn, 1881, p. 115-116)

Peirce wrote that Venn had made a marked improvement and he criticized Euler’s original proposal on the scope of “diagrams” (Peirce, 1911, p. 354):

> What is it, then, that these diagrams are supposed to accomplish? Is it to prove the validity of the syllogistic formula? That sounds rather ridiculous […] Suppose we ask ourselves *why* it is that, if a circle P wholly encloses a circle M which itself wholly encloses a circle S, the circle P necessarily wholly encloses the circle S.

Then, he made several improvements to the Euler’s diagrams.

6. What a semiotic history can teach us

The extreme attention of Peirce to the “philosophy” of notations, signs, and diagrams in mathematics and in logic explains why his work can be so valuable to mathematicians and teachers, because it linked to the meaning of the signs and graphics, to their usefulness in the practice of mathematics.

Semiotic history à la Peirce allows not to confine to a merely chronological history of signs, but to make a way into an epistemological history, as we show in this paper, that leads us: 1) to better see and understand the difficulties of the students with signs, especially in the introduction of algebraic symbols; 2) in general, to be careful with the students when introducing "symbols" by clarifying their arbitrary and their pronunciation; 3) to explicit them the roles and meanings of signs: signs are not simply notations, they permit to hold a discourse on figures in geometry, to represent relations between objects and to generalize procedures in arithmetic, to give an operating status to irrational and negative numbers; 4) to transport practices and knowledge from arithmetic to algebra and then to geometry, especially in the introduction of cartesian geometry ; 5) in general, to be careful with the students to the passage from one meaning to another for the same symbol.

Semiotic history also leads us to emphasize visualization in mathematics. Peirce broadened his discourse on signs in "An Essay toward Improving our reasoning in security and in uberty", by situated them in the general framework of the space, on the sheet of a paper (Peirce, 1998, p. 472):

> Reasoning is dependent on Graphical Signs. By "graphical" I mean capable of being written or drawn, so as to be spatially arranged […]. I do not believe one can go very deeply into any important and considerably large subject of discussion without using space as a field in which to arrange mental processes and images of objects.

Mathematics is writing, and what we see, and which everyone can recognize as mathematics, is writing. I like to write "I think with my pen" with Wittgenstein, and "I think with my inkwell" with Peirce. In the mathematics class, a specific time to write and to see what is thought in writing therefore seems necessary 6) to encourage students to visualize (not just to see); 7) to give students time to recognize patterns (diagrams) and to know how to use them with a full understanding; 8) to resort to useful diagrams to represent reasoning; 9) to encourage students to freely represent their ideas.

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