THE HISTORICAL FAGNANO'S PROBLEM: TEACHING MATERIALS AS ARTIFACTS TO EXPERIMENT MATHEMATICAL AND PHYSICAL TASKS IN ITALIAN HIGH SCHOOL

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ABSTRACT

The starting point of this research is Fagnano's problem: "*For a given acute triangle, determine the inscribed triangle of the minimal perimeter.*" This problem has been investigated using methodologies both from mathematical analysis and synthetic geometry, and many demonstrative strategies have been provided. The *orthic triangle* is the problem's solution. This problem has been recently extended to convex quadrilaterals. In particular, Fagnano's problem is relevant in billiard physics: the orthic triangle is the minimum periodic orbit of an acute triangular billiard. An open question still remains referring to quadrilaterals: "Are the Orthic Quadrilaterals the minimum periodic orbits in a real billiard?" This communication aims to describe experimental teaching in an Italian high school, focused on Mathematical and Physical tasks, from a historical perspective. Starting from a historical and epistemological context, an interdisciplinary learning path has been planned, which has been experimented with about sixty fifteen-year-old students. The applicability of some geometric theorems to different contexts of reality has been tested through the realization and use of specific artifacts as teaching materials.

1 *Rationale*

This work draws on other works by the same authors on the history of geometry teaching in Italy in the early 20th century (Furinghetti, 1997; Clark, 2012; Adesso et al. 2018; Adesso et al. 2019a; Adesso et al., 2019b; Adesso et al., 2020). In the previous works, the history of geometry teaching in Italy, in the period from the final years of the 19th century to the first half of the 20th century, was analyzed, taking into account the influence of both school reforms and the "New geometry of the triangle", first introduced in France in 1873. In the last works, we referred to some theorems about Cevian and orthic triangles, which may be included in the "New geometry of the triangle", although they were discovered in Italy before 1873. Some Italian booklets and textbooks have been analyzed to show the influence of these factors on geometric teaching.

2 Fagnano's problem: a historical overview

The starting point of this research is Fagnano's problem: *For a given acute triangle, determine the inscribed triangle of the minimal perimeter*.

This problem has been investigated using methodologies both for mathematical analysis and synthetic geometry, and many demonstrative strategies have been provided.

Coxeter & Greitzer (1967) quoted that this problem "*was proposed in 1775 by Fagnano, who solved it by calculus*." Nevertheless, Coxeter showed a geometrical proof. "The proof shown here is due to H. A. Schwarz." The Schwarz proof was based on triangle reflections.

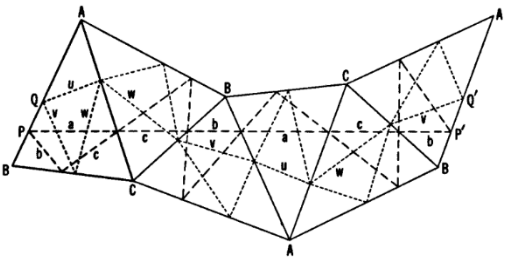


Figure 1. Schwartz proof to the Fagnano problem, as figure 4.5A in Coxeter-Greitzer

Fagnano's problem was first published by Giovanni Francesco Fagnano dei Toschi (1715-1797)in Nova Acta Eruditorum, 1775, p. 281-303, «*Problemata quaedam ad methodum maximorum et minimorum spectantia*». Here, Fagnano's calculus solution was not shown (and it seems unpublished), whereas the relevance of geometry as a means to both elegantly and simply solve certain problems was outlined.

PROBLEMATA QVAEDAM AD METHODVM maximorum et minimorum ſpectantia : Auctore Archidiacono IOHANNE FRANCISCO de TVSCHIS a FAGNANO, ex S. Honorii Marchionibus, Patricio Romano et Senogallienſi. Articulus VIII. Tomi I. Eruditorum Diarii, quod Mutinae editur, occaſionem praebuit ſequentia publicandi Problemata, quae ſi communi infinitorum methodo tractarentur, vix ſine ambagibus expediri poſſent. Placuit quoque ſolutiones ex ſimplici Geometria depromptas adiungere, vt videant in ſublimiori Analyſi initiati, non eſſe illam omnino negligendam; aliquando enim euenit, vt illius ope elegantius et facilius quaedam ſoluantur problemata, quae alteri impervia credas.	*Article VIII of a tome I of the Journal of Scholars, which was published in Modena, offered the opportunity to publish the following problems, which, if treated with the calculus method, could hardly be solved without uncertainty. It seemed opportune also to add solutions derived from simple geometry so that those initiated to higher analysis may realize that this (i.e., geometry) is not to be neglected/despised altogether; indeed, it sometimes happens that thanks to it, one can more easily and elegantly solve certain problems that one might otherwise consider impervious.*

Figure 2. The Geometry relevance to solve some problems, in Problemata quaedam ad methodum maximorum et minimorum spectantia

Problem IV was the well-known Fagnano problem, and he solved it by using circle properties.

PROBLEMA IV. (Fig. XVII.)

In Triangulo oxygono B A D inſcribere Triangulum C F S, cuius laterum Summa ſit minima.

Monitum.

Solutio indicabit problematis limitationem.

In the BAD triangle, inscribe CFS triangle, having the addition of the minimum side.

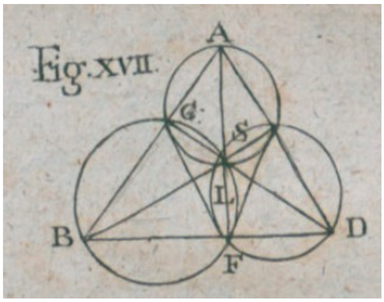


Figure 3. Fagnano's problem in Problemata quaedam ad methodum maximorum et minimorum spectantia

Nevertheless, it seems that Giulio Carlo Fagnano dei Toschi (1682-1766), Giovanni's father, first introduced this problem, giving a part of the solution: In the book When Least is Best (Nahin, 2021), it is shown that "*This problem has its origin with the Italian mathematician Giulio Carlo Toschi di Fagnano, who showed the existence part, and his priest-mathematician son Giovanni Francesco Fagnano, who completed the minimization argument in 1775. The* *father's contribution was to show, given any acute-angled triangle ABC and any given point U on one of the sides, how to construct the inscribed triangle of the minimum perimeter with a vertex on BC side.*"

The orthic triangle (Fagnano, 1775) is the problem's solution. An orthic triangle of a given ABC triangle is defined as the figure obtained by drawing the three segments joining the feet of the three heights of the ABC triangle. In the acutangle triangle, the orthic triangle is inside the ABC triangle (fig. 4a), in the right-angled triangle, the orthic triangle is the height relative to the hypotenuse, whereas, in the octusangle triangle, it is outside the ABC triangle (fig. 4b).

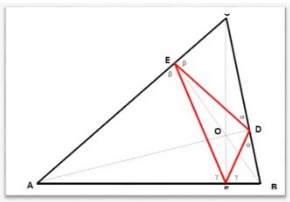


Fig. 4a. The orthic triangle in an acutangle triangle

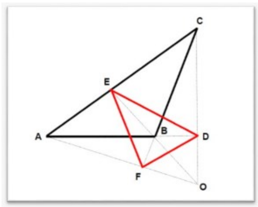


Fig. 4b. The orthic triangle in an the octusangle triangle

This problem has been recently extended to convex quadrilaterals (Mammana et al., 2010). Specifically, they proved the following theorem:

If Q is cyclic and orthodiagonal, the orthotic quadrilaterals of Q inscribed in Q have the same perimeter. They also have the minimum perimeter with respect to each quadrilateral inscribed in Q.

The given proof was similar to the Schwarz ones, starting from reflection properties.

3 *Activity design*

Fagnano's problem is also relevant in billiard Physics: the orthic triangle is the minimum periodic orbit of an acute triangular billiard (Gutkin, 1997). Nevertheless, in Gutkin's work, there is an inaccuracy. Instead of the term, "orthic triangle" is used "pedal triangle", actually the orthic triangle is a particular pedal triangle, it is the pedal triangle relative to the orthocentre. Our activities focused about the following question:

"Are the Orthic Quadrilaterals the minimum periodic orbits in a quadrilateral billiard?". Actually, no proofs of this question could be found in the literature. Nevertheless, the authors verified this hypothesis by using some artifacts. An interdisciplinary unit has been planned focused on the historical and epistemological context of Fagnano's problem: starting from its geometrical proofs to the Physics applications. The interdisciplinary learning unit included History, Latin, Literature, Maths, Physics (Capone, 2022). This learning unit has been experimented with about sixty students (three classrooms), attending the second year of a Scientific High School in the South of Italy. Three Mathematics and Physics teachers carried out the laboratorial activities, planned with two researchers, one in Mathematics education and other one in Physics education. In each classroom, the students were grouped in five groups (about four students in each group). The activities have been planned in the following phases:

1) Discovering the triangle with minimum perimeter must be inscribed in a triangle using GeoGebra, a geometrical dynamical software. Our students still don't know calculus, so they followed the Giulio-Giovanni Fagnano proof.
2) Discovering quadrilaterals with minimum perimeter: here, we just planned to verify that the orthic quadrilateral was the one with minimum perimeter to be inscribed in a quadrilateral, always by using GeoGebra.
3) Creating an orthic triangle or quadrilateral on real triangular and quadrilateral billiards, to verify the previous phases.
4) In the real world, the Physics billiard is affected by friction, so an innovative artifact was built to verify that the orthic triangle and the orthic quadrilateral are the minimum orbits in triangular and quadrilateral billiards: authors defined them as "light billiards", because the laser light was used instead of a typical billiard ball.

4 Activity development

4.1 Phases 1 – 2 Historical and geometrical analysis of Fagnano's problem

The historical context was first analyzed. Students were involved in a translation from the medieval latin of the *«Problemata quaedam ad methodum maximorum et minimorum spectantia».* The students also analyzed all the historical documents about Fagnano's problem, as we summarised in the previous paragraphs (Coxeter & Greitzer, 1967), and they organized them in a text.

The Maths teacher guided them through analyzing and reproducing (by using GeoGebra) different geometrical proofs of the Fagnano problem (L. Fej´er and While H. A. Schwarz). Group 3 shows the existence (similar to Giulio's proof), and group 4 shows the unicity (similar to Fe'ier proof).

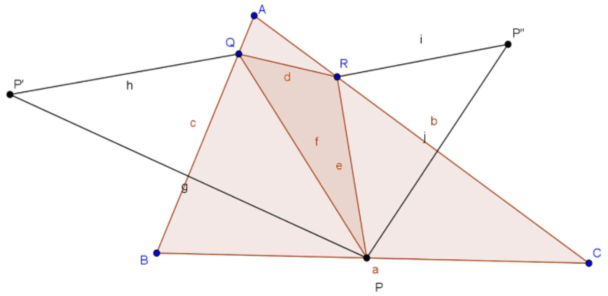


Figure 5. The students (Classroom 1, group 3) proof that exist a triangle, inscribed in another one, which has a minimum perimeter.

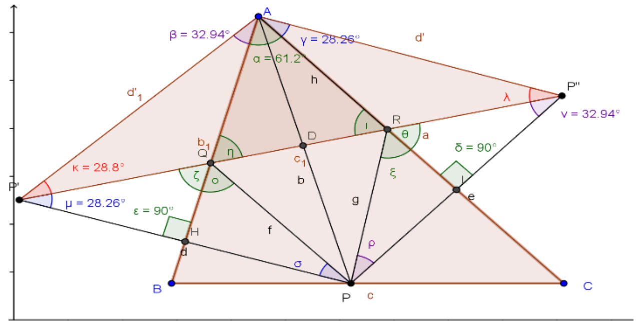


Figure 6. The students (Classroom 1, group 4) verification of the unicity, following Fe'ier proof.

Here we also show the students translated proof text about the existence: *P' is the symmetric point of P with respect to AB side and P'' with respect to AC side. P'Q is congruent to PQ and P''R is congruent to QR. So, the perimeter of PQR triangle is the same of P'Q+QR+RP''. It will be minimum if this broken line is a single line. For each position of P on BC there exists and is unique a triangle of minimum perimeter corresponding to this condition. The solution to Fagnano's problem is therefore to be found in the family of triangles with this characteristic, obtained by varying P on BC.*

4.2 Phase 3 Orthic quadrilateral on billiard.

In the next activity phase, students learned the orthic quadrilateral; in Figure 7. Group 2 in Classroom 3 showed different orthic quadrilaterals inscribed in a quadrilateral Q, also including the principal orthic quadrilateral, with respect to the Varignon parallelogram (1731).

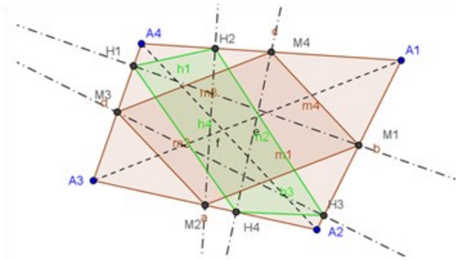


Figure 7. Main orthic quadrilateral, with respect to the Varignon parallelogram, MiHi = M-altitudes as shown by group 2 in Classroom 3.

Students were involved in the creation of an orthic quadrilateral by using coloured ribbons on a rectangular billiard (see Fig. 8):

1) Particular case: a) Orange ribbons: diagonal of the billiard; Blue ribbons: Varignon parallelogram (joining the midpoints of each sides), which is also the main orthic quadrilateral in the rectangular billiard
2) General case: a) Orange ribbons: diagonals (fig. 8a); b) Yellow ribbons: starting from a point P, students draw parallel to the diagonals, such as obtaining the parallelogram $V_1V_2V_3V_4$. (fig. 8b) c) Blue ribbons: from each vertex of the parallelograms, a perpendicular to the opposite side starts: the V-altitudes H_1, H_2, H_3, H_4. (fig. 8c) d) Red ribbon: joining the foots of the V-altitudes we have the orthic quadrilateral (fig. 8d).

Students also built small modelling of real quadrilateral billiard with different shapes (see for example Fig. 9, built by group 4 in Classroom 1)

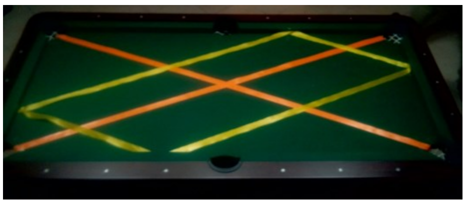
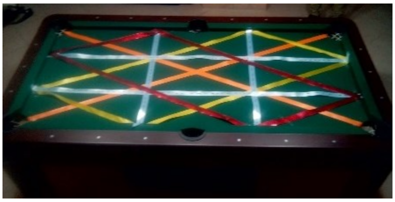


Figure 8a. Orange ribbons: diagonals

Figure 8b. Yellow ribbons: parallelogram $V_1V_2V_3V_4$



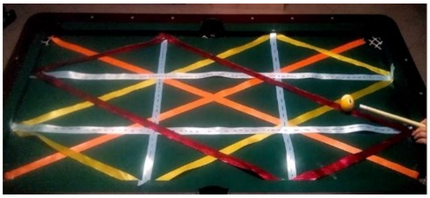


Figure 8c. Blue ribbons: V-altitudes H_1, H_2, H_3, H_4.

Figure 8d. Red ribbon: orthic quadrilateral on a rectangular billard

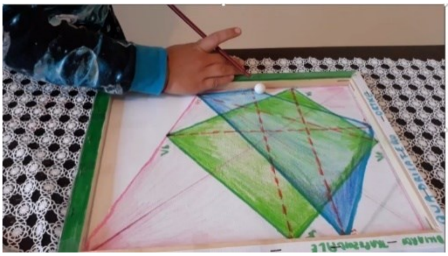


Figure 9. Trapezoidal billiard and an orthic quadrilateral

After they built the orthic quadrilateral on quadrilateral billiards, they tried to verify that the ball's trajectory was the same as the orthic quadrilateral, thus showing that the family of the orthic quadrilateral (having minimum perimeter) was also the closed periodic orbit for the main quadrilateral.

The results were that, in each billiard, the trajectory followed the perimeter of the orthic quadrilateral, but the orbit was never definitely closed.

4.3 "Light Billiard"

The trajectory of the billiard ball was analyzed by using the Tracker software. It was observed that the Snell law was not completely satisfied: as it is shown in Fig. 10 the motion was uniform but the velocity before and after the collision with the billiard side was different. The friction caused it.

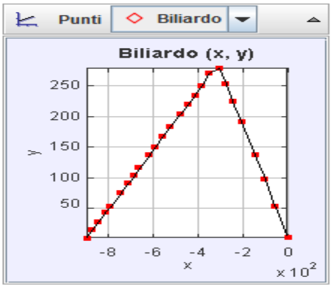


Figure 10. Tracker analysis: the friction caused a motion with different velocities, contrarily at the Snell law.

In order to avoid friction and to verify that the orthic quadrilaterals was the closed orbit with a minimum perimeter, a "light billiard" was built by the students, supported by the Maths and Physics teacher. In Fig. 11 a triangular and a quadrilateral light billiard are shown, where the laser light was the trajectory of the orthic quadrilateral, as supposed.

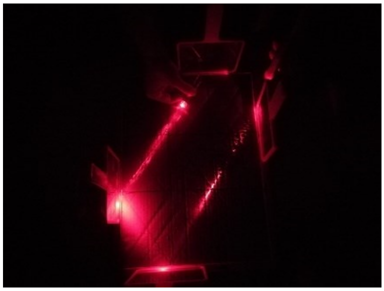


Figure 11. Triangular and quadrilateral light billiards

5 *Conclusions*

About sixty students were involved in an interdisciplinary learning unit including History, Latin, Literature, Maths, Physics. They analyzed and translated original Fagnano paper and critically studied the different geometrical proofs to it, using dynamical geometrical software to verify the different approachs. Nevertheless, some artifacts have been realized to verify a «new idea»: a correlation between the billiard Physics, the Geometric Optic and the Fagnano's problem. The purpose of these activities carried out with the students had several educational aims: to highlight the importance of contextualising a problem historically; to motivate students to study mathematics and physics; to show how interdisciplinary learning can lead to a broader view of a problem. This approach allows students to learn mathematics in their own way and develop mathematical ideas without the textbook and beyond the classroom. The collaborative learning framework involving heterogeneous group members provides an opportunity to learn mathematics concepts, even at a higher level. It is concluded that students' knowledge constructability requires an apprenticeship in culturally specific cognitive and social practices. Furthermore, the history of mathematics makes up an important component of learning mathematics. Its integration into mathematics curricula helps the students to understand that mathematics is "...a discipline that has undergone an evolution and not something that has arisen out of thin air." (Jankvist, 2009, p.239)

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