STARTING FROM THE HISTORY OF MATHEMATICS IN EARLY MODERN ITALY: FROM PRIMARY SOURCES TO MATHEMATICAL CONCEPTS

Abacus Mathematics and Archimedean Tradition

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ABSTRACT

The proposed activity consists of the analysis of the Archimedean technique used by Piero della Francesca to determine the volume of the double vault, supported by figures drawn in dynamic geometry enivronment. The study of the problem, adequately contextualized from a historical point of view, provides interesting insights into the language of (Renaissance) mathematics and the evolution of the concept of demonstration.

1 *The proposed activity*

The following activity is related to two fundamental issues in medieval and early Renaissance mathematics: the abacus tradition [see Gamba and Montebelli 1987] and the Archimedean tradition [see Napolitani 1998 and 2010]. In the Middle Ages, these traditions were basically expressions of two culturally worlds apart – on one side the world of practitioners and on the other side the world of humanistic circles– but their fates began to intertwine in the second half of the fifteenth century by the work of figures such as Piero della Francesca (1412-1492) and Luca Pacioli (1445-1517), when they attempted to use Archimedean techniques for the determination of volumes of some solids. The case we examine is that of the volume of the double vault (or rib vault), described by Piero della Francesca in his *Libellus de quinque corporibus regularibus*.

The activity has not yet been tested with students, so this contribution is only a proposal: given its complexity, it is suitable for students in the final year of high school or for teachers in training. In particular, I believe it could be a suitable activity to be developed in the so-called "Liceo Matematico" (https://www.liceomatematico.it/), an experimental project spread throughout Italy that has been tried and tested for some years now, geared toward promoting interdisciplinary laboratory activities in high school. In this activity, in fact, mathematical, historical-mathematical and linguistic skills are involved as well as the ability to interact with dynamic geometry environments.

The path necessarily begins with the narration, given in lecture-style by the teacher, of the historical-mathematical context in which Piero's treatise is set. In the next paragraph I present a short *excursus* highlighting the most significant aspects of the historical framework, in order to define the relevance of the example proposed here.

The second phase of the activity consists of the analysis of Piero's text, to be conducted in groups of 4 or 5 students after a collective reading led by the teacher: students will become researchers and try to interpret a historical text by re-constructing its mathematical meaning. The various interpretations, resulting from the group work, will then be compared in a collective mathematical discussion, in which a shared interpretation of Piero's text will be negotiated. It is important that the teacher will coordinate the discussion in order to emphasize the critical points of the demonstration, as well as the linguistic-mathematical aspects, such as the lack of symbolic expressions.

The activity could then continue with a comparison between the determination of the volume of the double vault presented by Piero and by Archimedes in his *Method* and discussed in [Archimedes 2013] and [Napolitani & Saito 2013 and 2014], but we do not have the opportunity here to explore this in depth.

2 The Abacus tradition and the Archimedean tradition

The so-called "abacus mathematics" began to spread in Italy in the first half of the thirteenth century, following the economic revival and the restart of trade in the Mediterranean area. This economic and commercial development made an increasing need for a new mathematics, based on a system of numeration and computational algorithms more efficient than the Roman ones still in use in the Latin West.

One of the main instruments, but certainly not the only one, of this cultural revolution was the work of Leonardo Pisano, also known as Fibonacci. In particular, his *Liber Abbaci*, written in 1202 and revised in 1228, presented the positional decimal numbering system with Indo-Arabic numerals and new efficient algorithms for calculating. Besides, *Liber Abbaci* dealt with topics and problems typical of mercantile mathematics, and devoted the last chapter to the algebra of first and second degree equations. Topics in practical geometry, such as the area of plane figures and the volume of solid ones, were later treated by Leonardo in his *Practica Geometriae*. The dissemination of the *Liber Abbaci* and *Practica geometriae*, written in Latin, was made possible mainly by two factors: the birth of the "abacus schools" and the process of vulgarization of the texts. In the abacus schools students could receive either basic training, limited to elementary notions and knowledge of systems of weights and measures, or more advanced training, which opened up to the great mercantile trade and required complete mastery of mercantile mathematics. These schools also trained craftsmen, painters, sculptors, or what has been called the "middle cultural class" ("ceto culturale intermedio").

The approximately three hundred extant abacus treatises are collections of solved problems, generally written in vernacular language with a discursive style, rich in locutions borrowed from speech: they are very interesting both from a mathematical and linguistic point of view, but rarely known to high school students.

As for practical geometry, it did not always appear in abacus treatises, and when it did, it appeared in various forms: from short sections relating to plane geometry enriched by some notion of the volume of solids, to more extensive treatises. The writings on practical geometry also were a collection of solved geometrical problems: besides, they were not purely speculative problems as in Euclid's *Elements*, but they always contained numerical data and were solved using arithmetical or algebraic techniques.

At the time when abacus mathematics was beginning to spread, the Flemish Dominican William of Moerbeke arrived at the papal court of Viterbo, one of the most prestigious intellectual centers of the 13th-century Mediterranean. In Viterbo he found a circle of scholars – among them Campanus of Novara, John Peckam and Witelo – interested in science and translations. Even if Moerbeke is first known for his translations and revisions of Aristotle's complete work and commentaries on it, which became for about two centuries the standard text for university teaching, a very remarkable scientific achievement was his translation of Archimedes, Eutocius, and Ptolemy. In particular, he translated the Archimedean *corpus* from Greek into Latin using the so-called Codex A and Codex B drawn up around the 9th century in the Byzantine environment. Archimedes' work was particularly difficult to understand, both because of the content itself and because of the needed skills, which were not limited to Euclidean geometry alone but, for example, also included the theory of conics. Appropriating the entire Archimedean work therefore required not only tremendous intellectual effort but also the restoration of Greek mathematics as a whole. This project, however, was not on the horizon of 13th-century scholars, and thus William of Moerbeke's translation had little circulation in the scholarly community, so that Archimedes' geometrical knowledge and demonstrative techniques remained essentially unknown. Only a few results of *Measurement of a Circle* and *On the Sphere and Cylindre* - coming from the arabic tradition, filtered into the Latin world (and in the abacus schools) because they were useful in practical geometry.

The fifteenth century restoration of the Greek Classics, did not improve the situation: Moerbeke's translation continued to have limited circulation and at some point it apparently disappeared. It was rediscovered by the German philologist Valentin Rose only in 1881 in the Vatican Library in Rome (it is the codex Ott. Lat. 1850) and became a key witness for the Danish philologist Ludwig Heiberg who set up the critical edition of the Archimedean *corpus*. The fate of the Greek codices A and B was different: while Codex B disappeared in the 14th century, Codex A had better luck. The translation of Codex A was commissioned by Pope Nicholas V, the passionate bibliophile and humanist founder of the Vatican Library, to Iacopo di San Cassiano. All Renaissance editions of Archimedes' works, both manuscript and printed, including the *editio princeps*, published in 1544 in Basel, were based on Iacopo's translation.

Some documents of the Vatican Library Archives prove that the translation of Iacopo was loaned in 1458 to Francesco del Borgo, cousin of the famous painter Piero della Francesca, one of the greatest artists of the Italian Renaissance and also the author of mathematical works, such as an abacus treatise and the work on Platonic solids *Libellus de quinque corporibus regularibus*. Piero's works testify how abachistic mathematics began to broaden its horizons, expressing curiosity and interest in speculative and not just in practical works [Daly Davis 1977, Napolitani 2007]. The translation made by Iacopo was in fact lent by Francesco del Borgo to his cousin Piero, because he wanted to set up his own copy of the Archimedean *corpus*, correcting mistakes and remaking drawings by his own hand where necessary. Piero's redaction has been identified by James Banker in codex 106 in the Biblioteca Riccardiana in

Florence [Banker 2010] and was probably prepared by Piero while he was in Rome working on the Vatican Rooms (1458-1459). Piero then exploited his study of Archimedes to obtain some of the results illustrated in the *Libellus*, including the calculation of the volume of the double vault, that's *Casus X* of the fourth Chapter. The manuscript of the *Libellus*, now in the Vatican Library (Vat. Urb. Lat. 632), remained unpublished, but in 1509 Luca Pacioli published, without indicating the author, the entire *Libellus* translated into vernacular as the third part of his *Divina proportione*, under the title *Libellus in tres partiales tractatus divisus quinque corporum regularium et dependentium active perscrutationis*. The calculation of the volume of the double vault is *Casus 10* of this part.

3 The historical sources

Piero's calculation of the volume of the double vault presents multiple elements of interest.

From a historical point of view, this case represents one of the earliest evideces of a dialogue between the humanistic mathematics and the practical one. The most intriguing aspect, however, is the following one. As is well known, the so-called Codex C, discovered in 1906, contained the (unknown) *Method of Mechanical Theorems*. In this treatise, Archimedes explained to Eratosthenes a series of heuristic techniques he had used to find areas and volumes of various figures, and finally he showed a new technique to determine the volume of a cylindrical hoof and of the solid obtained as the intersection of two cylinders of equal radius at right angles, namely the double vault. However, Proposition 15 (which proves that the double vault is two-third of the cube circumscribed about the intersection of cylinders) is mutilated and its reconstruction is only conjectural [Napolitani & Saito 2013]. In light of this news, it is surprising and fascinating that Piero, without knowing of the existence of the *Method* (as far as we know), tackled the same problem faced by Archimedes and tried to solve it using Archimedean techniques.

From a linguistic point of view, "Case 10" is very lucky and fruitful, since we have at our disposal:

- the direct Latin source, namely the manuscript of Piero della Francesca's *Libellus*, digitized and available on the Vatican Library website (https://digi.vatlib.it/view/MSS_Urb.lat.632);

- the critical edition of the *Libellus*, which can replace the manuscript text (which is not easy to read) or serve as a guide to decipher it [Piero della Francesca 1995];
- Luca Pacioli's 16th-century vernacular translation published in *Divina proportione* (1509) and available at websites such as Gallica (https://gallica.bnf.fr/) or the digital library of the Museo Galileo in Florence (https://www.museogalileo.it/it/)

Together with to these primary sources, the English translation published by Marshall Clagett in his *Archimedes in the Middle Ages* [Clagett 1978] is available. It is useful for students who do not read Latin or Italian.

From a didactic point of view, this proposal provides the opportunity to study an original source: students will become researchers who have to interpret a historical text. The interpretation of this text will also require considerable ability to visualize three-dimensional objects: therefore they will construct and study, in small groups, the intersection at right angle of two cylinders of equal radius in a dynamic geometry environment (DGE) to observe from various points of view what the double vault looks like. It would also be interesting for the students exploring the use of the rib vault in architecture, starting with the porch of the Palazzo Ducale in Urbino, probably built at the time Piero was in town.

4 The analysis of the historical source

In this paper, I will necessarily use Clagett's translation of Piero's text (in italics) [Clagett 1978, pp.408-410]; of course in an Italian classroom I would use a direct source, i.e. Pacioli's vernacular translation and/or Piero's text.

As we have said, it is recommended that the first reading of the text will be guided by the teacher; later students, in groups, will carefully read the text aided by worksheets, trying to draw the suitable figures in a DGE in order to explore them.

Let us begin by reading the statement.

There is a certain cylinder whose diameter is 4 brachia – the diameter of each of its bases – and another cylinder of the same size pierces it orthogonally. We seek the quantity that is removed from the first cylinder by means of this hole.

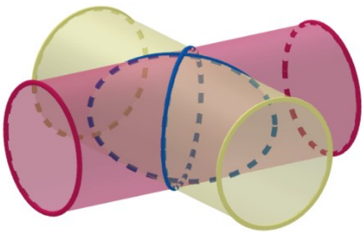


Figure 1. The intersection of two cylinders of equal radius at right angles

Piero seeks the volume of the intersection of the two cylinders using a very colloquial linguistic register, closer to the world of practitioners rather than the world of speculative geometry. As in all abacus treatises, the author deals with a specific object (the length of the diameter is given), thus turning the general problem into a generic example. In the following passage Piero describes "the cavity", that is, the double vault.

You ought to know that the perforated cylinder is perforated in a straight line both at the beginning and the end of the cavity, that is, where the hole begins and ends and the axis of the piercing cylinder crosses through the axes of the pierced cylinder at right angles in their cavity and the lines of these form a square [and, in fact, the intersecting lines in all the planes above and below and parallel with the plane of the axes form squares except] *at the top and the bottom* [where single lines only intersect] *and* [there] *they touch each other in two points, one at the top and one at the bottom.*

A description of the particular double vault – generated by two cylinders of diameter 4 "brachia" – and the procedure to determine its volume follow the general description. Note that Piero's style is completely prescriptive, as required for an abacus teacher.

Example. Let the pierced cylinder be H and the piercing cylinder be G and let the hole be ABCD, and let touching points in their cavity be E and F and we seek the volume of the hole. We have said that the width of each cylinder was 4 brachia. Therefore, the square ABCD, is 4 brachia on each side. These sides multiplied together make 16 and EF, which is the width of a cylinder, is 4, and when multiplied by the surface of the base, i.e. by 16, makes 64. This you divide by 3 and 21 1/3 is the result. This doubled becomes 42 2/3 and so *much is removed from cylinder H as the result* [of the formation of the] *said hole, i.e. 42 2/3.*

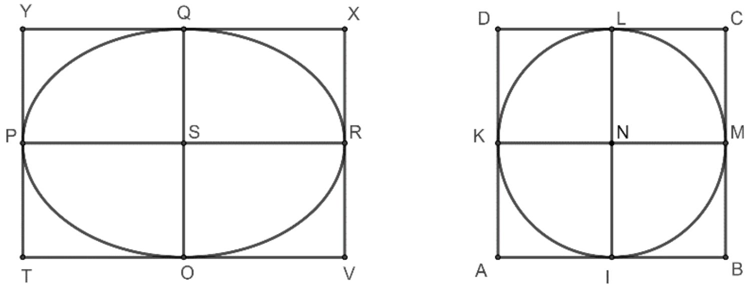


Figure 2. Plane sections: ellipse (inscribed in the rectangle) and circle (inscribed in the square)

From this point on, however, Piero abandons the abacus approach preferring to use an Archimedean technique, working first on plane sections and then on solids. He considers the square section ABCD and inscribes therein the circle IKLM, and considers the rectangular section passing through the diagonal of ABCD – represented by the rectangle TVXY, of side YT and TV, respectively equal to the square side and its diagonal – and inscribes therein an ellipse. He has now to determine what relationship holds between these figures.

This is proved as follows. You know that the said cylinders make a square in the hole, which square is ABCD. Therefore, you may draw a square hole of the same size which we let be ABCD and in it you inscribe circle IKLM with center N. Then you draw another [rectangular] *surface TVXY, each of whose opposite sides is equal to the diagonal AC of the said hole, while each of the other two sides is equal to AB. In this you describe a proportional circle* [i.e. an ellipse] *tangent to each side of the said rectangle in points O, P, Q and R. Let its center be S. I say that the ratio of square ABCD to rectangle TVXY is as circle IKLM to ellipse OPQR, and the ratio of circle IKLM is to its square ABCD as ellipse OPQR is to its rectangle TVXY as is demonstrated by the fifth* [proposition] *of the third* [work] *of Archimedes,* On Conoids.

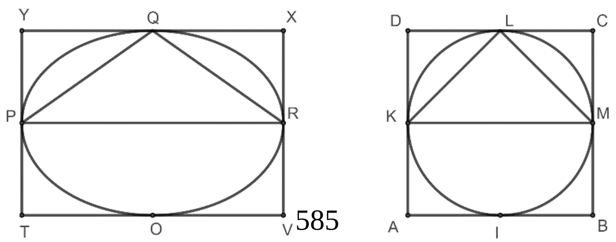


Figure 3. Triangles inscribed in the half-ellipse and in the semicircle

Piero recalls the fifth proposition of the third book of Archimedes' *Conoids* but in fact the *Conoids* was handed down in only one book. Piero's mistake is probably due to the fact that the *Conoids* is the third work transcribed in his manuscript. Proposition 5 states: "If AA', BB' be the major and minor axis of an ellipse respectively, and if be the diameter of any circle, then (area of ellipse) : (area of circle) = $AA' \, x \, BB' : \, d^2$" [Heath 2002, 115]. After establishing a proportion between the square, the inscribed circle, the rectangle and the inscribed ellipse, Piero moves on to consider the triangles inscribed in the semicircle and the half-ellipse.

Now you divide square ABCD into equal parts by line KM. Then you draw lines KL and LM and △KLM will be formed; and you divide rectangle TVXY into equal parts by line PR. Then you draw lines PQ and QR, forming △PQR. I say that

△ KLM : △PQR = square ABCD : rect TVXY

And

△KLM : square ABCD = △PQR : rect TVXY

And it was said above that

Circle IKLM : square ABCD = ellipse OPQR : rect. TVXY

And so it follows from common knowledge [viz. The axiom: quantities equal to the same quantity are equal to each other] *that*

△KLM : circle IKLM = △PQR : ellipse OPQR

After some arithmetic manipulation, Piero comes to determine that the ratio of the circle to the isosceles triangle inscribed in the semicircle is equal to the ratio of the ellipse to the isosceles triangle inscribed in the half-ellipse. Having determined this relationship between the sections, it is time to return to the solids

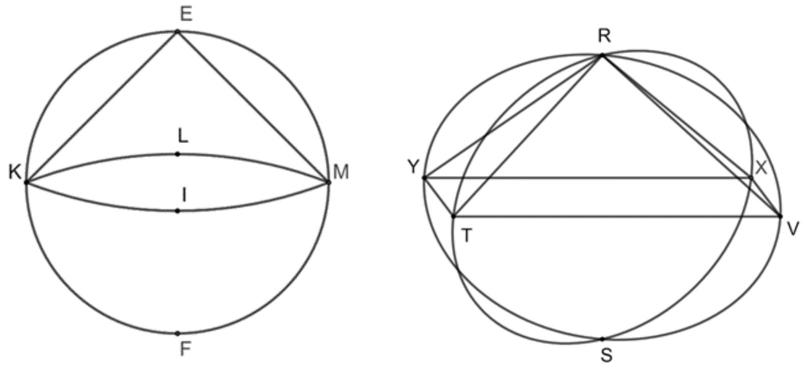
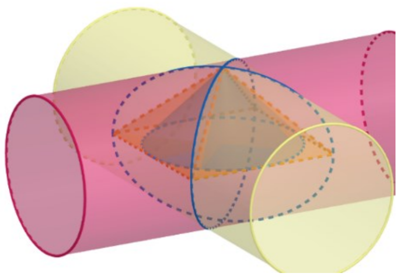


Figure 4. The cone inscribed in a sphere and the square-based pyramid inscribed in the vault

And with this understood, let us make solid figures. The first will be spherical and designated EKMF with axis EF and the other which encloses square TVXY[82] *by means of two ellipses. One is TRXS and the other is YRVS and they intersect each other in point R and in point S. In each of these two* [solid] *figures I shall produce a pyramid. In the sphere EKMF I shall delineate EM circularly. Then I shall draw lines KE and EM and produce pyramid KLMI on the round base* [i.e. cone KLMI]. *Then I shall produce another pyramid in the other corporeal figure, which will be TR, YR, XR, VR.*

The previous figures are remakes of Piero's figures, but the three-dimensional view below – which students can draw independently by trying to "translate" the text into a suitable graphical representation – can better illustrate how the cone and square-based pyramid described above are constructed.



[82] It will be useful to remark that the square TVXY is the the square before named ABCD.

Figure 5. The square-based pyramid and the cone inscribed in the intersection of two cylinders

These pyramids [i.e. the cone and the pyramid] *are in the same ratio as their parents, i.e. as the corporeal figures in which they are constructed, as is demostrated above in the plane figures, since circle TRXS is equal to circle OPQR*[83] *in surface TVXY and the sides of the pyramid TR, RX are equal* [respectively] *to the two sides of △PQR, i.e. PQ and QR. And the sides KE and EM of the cone in the sphere are equal* [respectively] *to the sides KL and LM of △KLM of circle IKLM. Let us conclude then that the ratio of the pyramid TR, YR, XR, VR to its* [parent] *solid TRXS* [i.e. to the common segment of the two cylinders] *is as the ratio of cone KEM whose base circle is IKLM to its* [parent] *spherical solid KEMF.*
In modern terms, Piero proved

$$\triangle \text{KLM} : \text{circle IKLM} = \triangle \text{PQR} : \text{ellipse OPQR}$$

or

$$\triangle \text{PQR} : \text{ellipse OPQR} = \triangle \text{KLM} : \text{circle IKLM}$$

and from that proportion he deduced that
Volume (pyramid) : volume (double vault) = volume (cone) : volume (sphere)
This is a very crucial passage but it is not well justified in Piero's text. Indeed, the reconstruction of Piero's whole argumentation is an open problem from the historical viewpoint, that could led to a very interesting discussion among the students, invited to formulate conjectures and suggestions. Students can also discuss the conjecture expressed in [Gamba, Montebelli, Piccinetti 2006].

To conclude the proof, it is easy to note that the previous proportion allows to find the volume of the double vault, since the ratio of the sphere to the inscribed cone with base the maximum circle is known thanks to Archimedes (*On Sphere and Cylinder*) and the volume of the pyramid is also easily determined.

[83] Questions for the students: are TRXS and OPQR really circles? Why?

Therefore by I.33 of On the Sphere and the Cone (!) *of Archimedes, where he says that any sphere is quadruple the cone whose base is equal to a greater circle of the sphere and whose axis is equal to the radius* [of the sphere]*, sphere KEMF is quadruple cone KEM and thus the parent solid TRXS* [which is the common segment of the two cylinders] *is quadruple pyramid TR, YR, XR, VR. And so you take the base TVXY which is 4 brachia on each side; multiply the sides together and the result is 16. This you multiply by the axis which is 2 and the result is 32. This you divide by 3 and 10 2/3 is the result* [as the volume of the pyramid]. *Its* [parent] *solid TRXS* [i.e. the common segment of the cylinders] *is 4 times as great. Therefore, mutiply 10 2/3 by 4 and the result is 42 2/3 as was said before. And thus you have what is removed from cylinder H by that hole* [namely] *42 2/3 brachia.*

To conclude, besides all the various interesting aspects mentioned above, the study of Piero's text could offer also a frutiful chance to reflect on the meaning of the term "demonstration" in practical geometry, where it usually means "to show by means of a good example": discussing the epistemological value of this approach also allows students' beliefs about the concept of demonstration in mathematics to emerge.

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