FIRST SIGNS OF EXTENSION FROM THE PLANE TO THE SPACE: CONTRIBUTIONS BY JOHANN HUDDE AND PHILIPPE DE LA HIRE TO 3-DIMENSIONAL ANALYTICAL GEOMETRY

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ABSTRACT

The goal of this paper is to present a historical approach to the first indications of several variables' functions space graphical representations from the generalization of plane representations, which may be key to facilitate the teaching of this type of curves. The two representations found and mentioned in this paper were developed by Hudde and La Hire. We show the way they introduced this representation as a transition from a geometrical to an algebraical interpretacion.

1 *By way of introduction*

The study of curves in the field of geometry is one of the axes of the current development of the concept of the function of several variables. It is for this reason that the present study presents the first efforts made by mathematicians to establish graphical representations of a type of curve in space with a certain type of equation. It is worth noting that this initial connection between geometry and algebra coincides with the birth period of an area of mathematics known as Analytic Geometry.

Descartes (1596-1650) points out that his method for dealing with curves associated with an algebraic equation could be extended from the plane to space although according to Anfossi (2004) "he mentioned three-dimensional geometry but wrote nothing about it" (p. 25). It is important to note that the work of Fermat and Descartes during the seventeenth century caused a real revolution in the field of geometry but this new discipline, as they conceived it, was from a didactic point of view not very effective and difficult to understand.

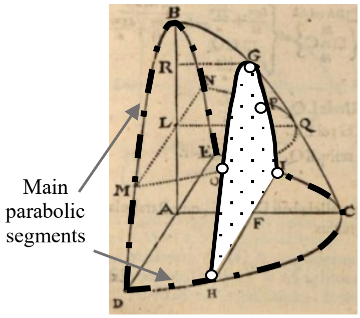
2 *First representation as plane sections of a surface*

Hudde (1628-1704), presented a foretaste of the use of spatial coordinates in his work on plane sections of a surface. Although he did not develop his own notation for 3 dimensions, he emphasized the novelty of manipulating curves of degree greater than two as plane sections of a surface (Boyer, 1967). Hudde manages to generate a network of curves in space that were expressed by means of analytical equations of degree greater than 2, taking as bases curves of degree 2. Although the method is not sophisticated, he manages to surprise that they are generated precisely in space and not in the plane as they had been presented since Descartes (figure 1).

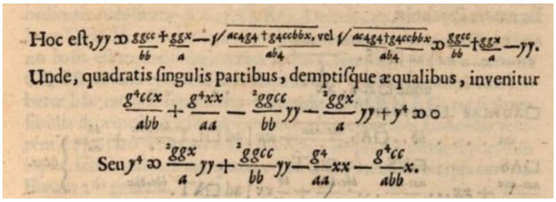
"Exercitationes mathematicae libri quinque". Frans Van Schooten (1657).
Fifth book; thirty miscellaneous sections.

Representation of the sections	**Notation used**

i)



ii)


Hoc eſt, $yy \propto \frac{ggcc}{bb} + \frac{ggx}{a} - \sqrt{\frac{acg^4 + g^4ccbbx}{ab^4}}$, vel $\sqrt{\frac{acg^4+g^4ccbbx}{ab^4}} \propto \frac{ggcc}{bb} + \frac{ggx}{a} - yy$.
Unde, quadratis ſingulis partibus, demptiſque æqualibus, invenitur
$\frac{g^4ccx}{abb} + \frac{g^4xx}{aa} - \frac{2ggcc}{bb}yy - \frac{2ggx}{a}yy + y^4 \propto 0$
Seu $y^4 \propto \frac{2ggx}{a}yy + \frac{2ggcc}{bb}yy - \frac{g^4}{aa}xx - \frac{g^4cc}{abb}x$.

Equation for the curve defined by the HOGPI points

$$\left\{ y^4 = \frac{2ggx}{a}yy + \frac{2ggcc}{bb}yy - \frac{g^4}{aa}xx - \frac{g^4cc}{abb}x. \right.$$

- Note that it expresses the equations of degree 4 that define the curves of the sections of the parabola in space.
- But there is no proper notation for the 3 dimensions of the generated surface.

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Figure 1. Representation of flat sections of a fourth-degree surface proposed by Johann Hudde

As shown in Figure 1, Hudde, starting from a parabolic segment in the plane, constructs a new parabolic segment perpendicular to it. From these principal curves, other segments perpendicular to each other are generated. Forming a surface in space; where each section of the surface is a power curve of fourth, octave, and so on. As can be intuited in the description and the visualization of figure 1 this network of curves generated in the space, Hudde expresses them in analytical equations of degree higher than 2; taking as base curves of degree 2 that he initially defines in the perpendicular planes. It is surprising how he generates these equations in space, and not in the plane as they had been presented.

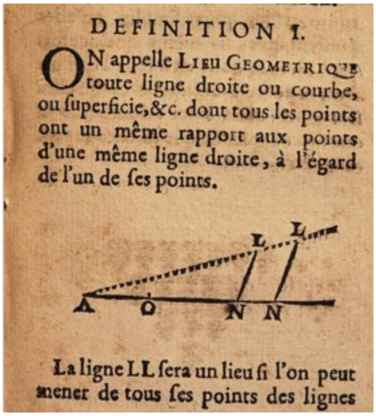
This anticipation of using spatial coordinates presented in a paper on plane sections of a surface in the year 1657, does not generate a specific notation for curves in space but innovates in the method for manipulating curves of degree greater than two, as plane sections of a surface. This fact shows us how complex it can be to approach a new creative experience of mathematical knowledge starting from the basis of a known or existing model.

3 First analytical extension of the plane to space

La Hire (1640-1719), simultaneously, presented important and significant advances to this new discipline. His work published in 1679 entitled: "Nouveaux Éléments des Sections Coniques: Les Lieux Géométriques: Les Constructions ou Effections des equation" deserves special attention. The author presented the use of Cartesian methods to solve geometric problems and their translation into equations, established conditions for a geometric locus, as well as the equations for the construction of those locus. It is precisely through this exceptional mastery of analytic geometry that he presented the first hint of how this condition can be extrapolated to three dimensions. La Hire defined the equations of the geometric locus for the plane and described, with its respective figure, how it should be approached in the case of a surface.

"Chapitre II: De la nature des lieux et de la réduction des équations pour la construction des lieux". Philippe de La Hire (1679). Second book; The Geometric Places.
For the two-dimensional case

i)



DEFINITION I.

ON appelle Lieu Geometrique toute ligne droite ou courbe, ou ſuperficie,&c. dont tous les points ont un même rapport aux points d'une même ligne droite, à l'égard de l'un de ſes points.

La ligne LL ſera un lieu ſi l'on peut mener de tous ſes points des lignes

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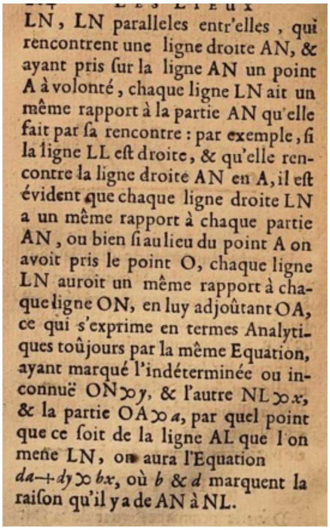
Note that it defines what a locus is and proposes in analytical terms equations that correspond to the type of place.

ii)



LN, LN paralleles entr'elles, qui rencontrent une ligne droite AN, & ayant pris ſur la ligne AN un point A à volonté, chaque ligne LN ait un même rapport à la partie AN qu'elle fait par ſa rencontre : par exemple, ſi la ligne LL eſt droite, & qu'elle rencontre la ligne droite AN en A, il eſt évident que chaque ligne droite LN a un même rapport à chaque partie AN, ou bien ſi au lieu du point A on avoit pris le point O, chaque ligne LN auroit un même rapport à chaque ligne ON, en luy adjoûtant OA, ce qui s'exprime en termes Analytiques toûjours par la même Equation, ayant marqué l'indéterminée ou inconnuë $ON \propto y$, & l'autre $NL \propto x$, & la partie $OA \propto a$, par quel point que ce ſoit de la ligne AL que l'on mene LN, on aura l'Equation $da+dy \propto bx$, où b & d marquent la raiſon qu'il y a de AN à NL.

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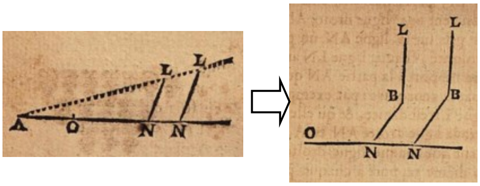
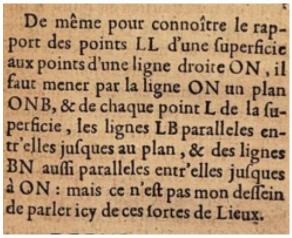
Extends the analytic description of a place with three dimensions





De même pour connoître le rapport des points LL d'une ſuperficie aux points d'une ligne droite ON, il faut mener par la ligne ON un plan ONB, & de chaque point L de la ſuperficie, les lignes LB paralleles entr'elles juſques au plan, & des lignes BN auſſi paralleles entr'elles juſques à ON : mais ce n'eſt pas mon deſſein de parler icy de ces ſortes de Lieux.

- Note that although he extends the analytic description to a locus of a surface in space, he expressly states that it is not his intention to speak openly of such loci.
- Although in an enunciative way it is the first time that it is alluded to directly.

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ii.- In: http://reader.digitale-sammlungen.de/en/fs1/object/display/bsb10525757_00493.html
https://books.google.be/books?id=_13EahsLFVQC&printsec=frontcover&hl=es&source=gbs_ge_summary_r&cad=0#v=onepage&q&f=false

Figure 2. Analytical descriptive extension on a 3-dimensional place proposed by Philippe de La Hire

As shown in figure 2; La Hire in chapter 2 of his second book, describes how to extend or pass from a condition of a geometric place of 2 dimensions to 3 dimensions, and although he points out that it is not his intention to speak openly of this type of places, he describes and represents explicitly how to approach a geometric place of a surface in space. Thus showing the first example of how to express a surface analytically through an equation with three unknowns.

4 *In summary*

These are the first two signs of the transitional stage that occurred at the end of the 17th century between a geometrical interpretation and its corresponding algebraical translation. Mathematicians, although they do not develop their own systematic notation for surfaces, manifested the need to use three-dimensional auxiliary constructions to perform plane demonstrations.

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