SPATIAL THINKING IN ANCIENT GEOMETRY. THE CASE OF SUBCONTRARY SECTION

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ABSTRACT

Conics are not the exception into an ever-growing list of scientific findings with spatial thinking influence. Under this scenario, a proposition of Apollonius of Perga Conics treatise was analyzed for identifying and extracting processes of reasoning related to the space as a frame of reference, and representations of geometric objects; thus, explaining what the spatial thinking role in the construction of conic sections was.

1. Spatial thinking in the history of science and mathematics

The knowledge construction has been in a closed relation with spatial thinking because the scientific thinking nature is spatial, even non-spatial knowledge is communicated through diagrams, maps, and schemas (Newcombe, 2016). The history of science is full of examples where spatial thinking is the protagonist: the double helix of DNA was developed as a three-dimensional spatial model; the periodic table organizes into columns and rows the relations among elements (National Research Council, 2006; Newcombe, 2010; 2013); Dandelin spheres are a spatial mechanism to find foci and directrix of conic sections into the cone (Salinas and Pulido, 2017). Also, different disciplines, such as geoscience, engineering, and neuroscience have implemented spatial thinking to visualize the processes that affect the earth's formation, anticipate how a set of forces may affect the design of a structure, and visualize specific parts of the brain for surgical procedures, respectively (Newcome, 2010).

Other discoveries in history such as the atom, cell, the solar system, and the universe have different spatial representations which come from micro-space and macro-space, useful in the teaching and learning process, moreover, mathematics concepts are not the exception. "Geometry […] in particular is a mathematical area concerned with the space around us, with the shapes in the space, their properties, and different 'patterns' and 'thinking patterns' for which they serve as trigger and basis" (Hershkowitz, 2020, p. 774); therefore, the historical and epistemological construction of geometric concepts would have an implicit spatial component susceptible of being investigated.

1.1 Conics are not the exception, are an opportunity

We focused on conics because these notions are perennial in mathematics teaching as a result of their algebraic and geometric approaches, relations between plane and space geometry (Barbin, 2008), and different contexts related to the art, mathematics, astronomy, and architecture (Mancini and Menghini, 1984; Berger, 2010). However, this variety of contexts has not been considered totally in the mathematics curriculum and the geometric approach has disappeared or has been reduced as an illustrative tool, furthermore, spatial skills do not take a relevant place when these are fundamental in conics learning (Vargas-Zambrano and Montiel-Espinosa, 2020); although there is not an important open problem concerning the conics, there are still few circles remain in the curriculum, the other conic sections are gone, nevertheless, conics are an integral part of our lives (Berger, 2010).

Basically, there are at least three ways to think about the conics: via the cone, via quadratic equations, and via transformations; the first one keeps a spatial nature in conics' genesis which has mixed with different meanings throughout history (Bartolini Bussi, 2005). Although circles, parabolas, and sometimes ellipses appear in school as cuts of the cone, it seems that finding the foci and directrix of the conics as of the equation and vice-versa is enough in the teaching and learning processes; even the procedure for cutting a cone introduces the analytic treatment without any apparent relation between them (Salinas and Pulido, 2017), because foci and directrix are the shared geometric elements among the conics but irrelevant and skipped in their construction as cuts of the cone.

2. Research issue

Our research approach to the History of Mathematics and Education is epistemological: we recognize that mathematics are human activity product; history enriches the knowledge we impart in classrooms; interesting problems and original meanings have disappeared (Buendía and Montiel, 2011), and even ways of thinking too. For example, our literature review (see Vargas-Zambrano and Montiel-Espinosa, 2020) showed the procedure for cutting a cone has a high component of spatial thinking which has not taken advantage in the conics' teaching, the genesis of conics more exactly in Ancient Greece, considers essential spatial thinking to construct, communicate and understand these notions, consequently we pose two questions: what *processes of reasoning* allow the construction of conics as cuts of the cone? And what is the *spatial thinking* role? So, we will answer and discuss the circle as a conic section due to the classical discussion about the circle as *loci* or *solid loci* in Ancient Greece, specifically in *Conics* written by Apollonius of Perga.

3. Conceptual and methodological framework

"Spatial thinking concerns the locations of objects, their shapes, their relations to each other, and the paths they take as they move" (Newcombe, 2013, p. 28). Kind of thinking is based on a constructive amalgam of three concepts: *space, representation*, and *processes of reasoning* (National Research Council, 2006). Firstly, and in our case, *space* will be *Euclidean space;* it refers to a container made up of a network of positions where the geometric objects are located when they are mobile or stationary (Clements and Battista, 1992). Secondly, the *representation* corresponds to the set of *primitives* or geometrical objects (point, line, plane, 2D-figure, surface, 3D-figure) and their properties. Thirdly, *processes of reasoning* are the set of *dynamic relations, static relations*, and *transformations* between *primitives* (National Research Council, 2006). These conceptual elements can be visible when rationality is contextualized, owing to reasoning coming from human activity (Cantoral, 2020). Mathematical activity such as human activity organizes *actions:* direct interactions of the subject (individual, collective or historical) over the object, in a specific environment (Torres-Corrales and Montiel, 2019). In our case, direct interactions into *Euclidean space* between subject and geometric objects for social knowledge construction.

4. Results

The unit of analysis for this paper is proposition 5, book I of *Apollonius of Perga Conics.* We adapted from Torres-Corrales and Montiel (2019) and Cantoral (2020) the methodological questions "what does the subject do?" and "how does she/he do it?" for identifying *actions* and mathematical activities as will be seen below; therefore we recognize that propositions text from

Apollonius of Perga Conics —at least the first fourteen— are a clear description of an implicit diagram, which were developed through historical direct interactions between subjects and concrete material.

4.1. What did Apollonius do?

According to the proposition statement, Apollonius pretended to prove that section GHK is a circle; basically, circle GHK comes from a perpendicular section to the plane ABC (see figure 1, a). Also, Apollonius (ca. 200 B.C.E./2013) specified three necessary geometric objects —for his proof— that act as a set of *primitives* because these were defined or constructed in past definitions and propositions: oblique cone with vertex A, plane GHK and axial triangle ABC (Apol. I. 3). "The set of primitives is a way of capturing our encounters with a world full of objects (occurrences of phenomena): objects are the things that we are trying to understand" (Golledge, 1995; 2002 as cited in National Research Council, 2006, p. 36). Oblique cone appears in definition 3 of book 1, it is constructed by three *primitives:* points A and B; straight line AB and curved line BLC, these are related through a *dynamic relation (rotating)* because B moves in the circumference. On proposition 5 of book I, oblique cone be converted into a *primitive*, due to these geometrical objects, points, straight line and curved line keep intrinsic *static relations* like size, location, and orientation, because they are subparts with relations among them into a new object (Sinclair, Cirillo, and de Villiers, 2017; Newcombe, 2016).

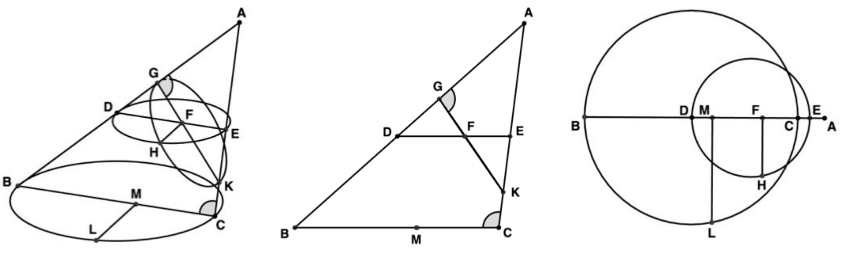

Figure 3. a) panoramic view, b) side view, and c) top view of proposition 5 of book I.

4.2. How did Apollonius do it?

Exposition of the proposition details the *primitives* and their properties. Into *Euclidean space* Apollonius got the triangle ABC, circle DHE, and circle GHK as sections, combining two *processes of reasoning: direction of move-*

ment and *cross-sectioning*. Apollonius cut an oblique cone with a plane respectively: through the axis and perpendicular to the circle BC; parallel to the circle BC; and subcontrariwise to the circle BC by side A of the axial triangle. Both *processes of reasoning* are *dynamic relations* between entities, their features are evaluated with respect to other entities or a frame of reference (National Research Council, 2006). Although *cross-sectioning* is a relation intrinsic and dynamic only (Sinclair et al., 2017; Newcombe, 2016), it depends on the orientation of the cutting plane and the geometrical structure (Cohen and Hegarty, 2012). For instance, when Apollonius generalized the procedure for cutting a cone, he used two geometrical structures: the right cone and the oblique cone, each one is cut by two orthogonal cutting planes to get the axial triangle and the circle, and one oblique cutting plane to get the conic section, however, the *orientation* as *static relation* is not enough, because in addition *direction of movement* underlies the geometric reasoning, planes that cut the oblique cone satisfy a condition with respect to other entities like the base of the cone or the axial triangle.

Exposition continues and Apollonius claimed: "[...] the triangle AKG similar to the triangle ABC and lying subcontrariwise, that is, so that the angle AKG is equal to the angle ABC. And let it make as a section on the surface, the line GHK" (Apollonius, ca. 200 B.C.E./2013, p. 9). After *cross-sectioning,* Apollonius recognized similar triangles on the plane ABC or axial triangle; his affirmation involves two *processes of reasoning: changing perspective* and *comparing shape and length* (figure 1, b). When Apollonius uses plane geometry in his proof, the point of view changes from a panoramic view (oblique cone) to a side view (axial triangle); this spatial *transformation* is elemental to scientific reasoning for comprehending and testing ideas (figure 1, a and b) (National Research Council, 2006). Consequently, Apollonius developed his ideas into the 2D-space, on circles BLC and DHE (Apol. I. 4) and axial triangle (figure 1, b and c). To illustrate: Apollonius (ca. 200 B.C.E./2013) specified that GHK is a circle again. Thus, in the plane DHE, he starts *comparing shapes,* $\text{rectangle DF, FE} = \text{square FH}$. This equality of areas depends on DE segment, which comes from *cross-sectioning,* exactly a common section between the axial triangle ABC and the circle DHE (Eucl. XI. Def. 4). Segment FH is not a common section, but it is parallel to LM (Eucl. XI. 6), then FH and LM are lines with *location* and *direction of movement;* furthermore, DHE is a right triangle, therefore $\text{DF} : \text{FH} :: \text{FH} : \text{FE}$

(Eucl. III. 31, VI. 8). If DF, FH and FE are proportional, then the rectangle contained by the extremes DF and FE equals the square on the mean FH (Eucl. VI. 17). Hence, mean proportional is linked to a *process of reasoning* into *spatial thinking: comparing length.*

The end of exposition focused on the side view and *comparing size,* angles AKG and ABC are equal, therefore AKG and ADE too. And the opposite angles at the point F in the plane ABC are also equal. Hence, *comparing shape and length,* triangles DFG and KFE are similar (figure 1, b). If EF, FK, GF and FD are proportional (Eucl. VI. 4), then the rectangle contained by the extremes EF, FD equals the rectangle contained by the means FK and GF (Eucl. VI. 16).

4.3. What was Apollonius doing this for?

Apollonius (ca. 200 B.C.E./2013) proved that there is another procedure for cutting an oblique cone and getting a circle. From the orthogonal cutting plane, he found the circle DHE and argued that $\text{square FH} = \text{rectangle EF, FD}$; therefore, from the oblique cutting is getting a circle because $\text{rectangle KF, FG} = \text{square FH}$.

5. *Conclusions*

We identified two *spatial thinking roles:* epistemological and communicative. The first of them refers to explicit *processes of reasoning* into *Euclidean space* in direct relation with the geometric objects' construction such as *rotating* for oblique cone*; direction of movement* and *cross-sectioning* for conic sections; and *comparing shape and length* for mean proportional. The second of them refers to implicit *processes of reasoning* into the *Euclidean space* which helps to understand and follow geometric reasonings, such as *changing perspective, location* and *orientation.*

REFERENCES

Apollonius. (2013). *Conics: Books I-IV* (R. C. Taliaferro & M. N. Fried, Trans.). Santa Fe, United States: Green Lion Press. (Original work published ca. 200 B.C.E.)

Barbin, É. (2008). Perennial notions and their teaching. In E. Barbin, N. Stehliková & C. Tzanakis (Eds.), *History and Epistemology in Mathematics Education: Proceedings of* *the fifth European Summer University* (pp. 157-161). Pilzen: Vydavatelský servis.

Bartolini Bussi, M. (2005). The meaning of conics: historical and didactical dimensions. In J. Kilpatrick, C. Hoyles, O. Skovsmose & P. Valero. (Eds.), *Meaning in Mathematics Education* (pp. 39-60). New York, United States: Springer. https://doi.org/10.1007/0-387-24040-3_4

Berger, M. (2010). Conics and quadrics. In M. Berger, (Ed.), *Geometry Revealed a Jacob's Ladder to Modern Higher Geometry* (pp. 181-247). New York: Springer. https://doi.org/10.1007/978-3-540-70997-8_4

Buendía, G., & Montiel, G. (2011). From History to Research in Mathematics Education: Socio-Epistemological elements for Trigonometric Functions. In V. Katz, & C. Tzanakis (Eds.), *Recent Developments on Introducing a Historical Dimension in Mathematics Education* (pp. 67-82). Washington, DC: The Mathematical Association of America. https://doi.org/10.5948/UPO9781614443001.008

Cantoral, R. (2020). Socioepistemology in Mathematics Education. In S. Lerman. (Ed), *Encyclopedia of Mathematics Education* (pp. 790-797). Springer, Cham. https://doi.org/10.1007/978-3-030-15789-0_100041

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420-464). New York: National Council of Teachers of Mathematics, Macmillan Publishing Co.

Cohen, C., & Hegarty, M. (2012). Inferring cross sections of 3D objects: A new spatial thinking test. *Learning and Individual Differences,* 22(6), 868-874. https://doi.org/10.1016/j.lindif.2012.05.007

Hershkowitz, R. (2020). *Shape and Space: Geometry Teaching and Learning.* In S. Lerman. (Ed), *Encyclopedia of Mathematics Education* (pp. 774-779). Springer, Cham. https://doi.org/10.1007/978-3-030-15789-0_138

Mancini, L., & Menghini, M. (1984). Conic sections in the sky and on the earth. *Educational Studies in Mathematics,* 15, 191-210. https://doi.org/10.1007/BF00305896

National Research Council. (2006). *Learning to Think Spatially.* Washington, DC: The National Academies Press. https://doi.org/10.17226/11019

Newcombe, N. (2010). Picture this: Increasing math and science learning by improving spatial thinking. *American Educator,* 34(2), 29-35.

Newcombe, N. (2013). Seeing relationships: Using spatial thinking to teach science, mathematics, and social studies. *American Educator,* 37(1), 26-31.

Newcombe, N. (2016). Thinking spatially in the science classroom. *Current Opinion in Behavioral Sciences,* 10, 1-6. https://doi.org/10.1016/j.cobeha.2016.04.010

Salinas, P., & Pulido, R. (2017). Understanding the Conics through Augmented Reality. *EURASIA Journal of Mathematics Science and Technology Education,* 13(2), 341-354. https://doi.org/10.12973/eurasia.2017.00620a

Sinclair, N., Cirillo, M., & de Villiers, M. (2017). The learning and teaching of Geometry. In J. Cai (Ed.), *Compendium for Research in Mathematics Education,* 457-489.

Torres-Corrales, D., & Montiel, G. (2019). Characterization of uses of trigonometric notions in Mechatronics Engineering from Mathematics Education. *ECORFAN Journal Spain,* 6(10), 9-21. http://doi:10.35429/EJS.2019.10.6.9.21

Vargas-Zambrano, L. C., & Montiel-Espinosa, G. (2020). A research approach on the role of space in the construction of conic sections. In A. I. Sacristán, J. C. Cortés-Zavala & P. M. Ruiz-Arias (Eds.), *Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 737-738). Mexico: PME-NA. https://doi.org/10.51272/pmena.42.2020-111