A POSSIBLE TRANSITION FROM GEOMETRY TO SYMBOLIC ALGEBRA THROUGH THE HISTORY OF MATHEMATICS

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ABSTRACT

In this paper we present the experimentation of an educational path, designed for a secondary school class, aimed at highlighting a connection between geometric and algebraic knowledge starting from passages taken from the *Kitāb al-jabr wa al-muqābala* of Al-Khwārizmī and Abū Kāmil's *Kitāb fi al-jabr wa al-muqābala*. The motivation for such activity is twofold: the knowledge of the historical development of mathematics can increase its disciplinary knowledge; moreover, focusing attention on the role of the semiotic registers favors understanding of mathematics by the students. The main moments of the experience were: administration of a questionnaire to students, to detect any discontinuity in learning Euclidean geometry and algebra; linguistic-literary reading of passages chosen from the texts of Al-Khwārizmī and Abū Kāmil; identification of the mathematical elements in the chosen passages; identification of different semiotic registers, through which mathematics expresses itself, and of their role in the search for the solution of a problem and its parameterization; conclusions regarding the effects of the proposed educational path. Among the effects of the presented didactic path, we highlight the students' awareness of how the transition process from geometry to symbolic algebra improves understanding of the historical evolution of mathematics; the generalization of a problem and its solution through parameterization and the acquisition of operational skills in the modeling process.

1 *Introduction*

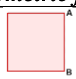
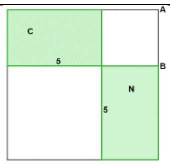
In this work we present the phases of a didactic experience carried out in Italy. The context of the experimentation, lasting 8 curricular hours, was a second class with 28 students of a Technical Institute in Genoa. The research question posed was to study the lack of connection, often perceived by students, between geometry and algebra. This experience was based on the didactic path described in Florio (2020) and Florio et al. (2020), in which students are shown, through the analysis of some problems, how algebraic language is grafted onto geometric language, allowing them to translate a problem from spoken algebra to symbolic algebra. The choice of particular propositions was made on the basis of the observation that they constitute a simple and effective environment for exploring the potential of some forms and registers with which mathematics expresses itself, to ascertain how the changes between these registers can happen in a way trustable and effective, and to evaluate the types of responses that such changes can produce in solving a problem. They also favored the development of students' ability to generalize a problem through the use of a parameter and to move towards mathematical modeling.

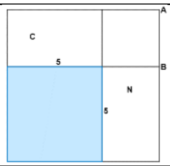
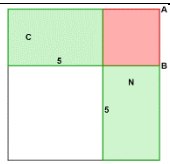
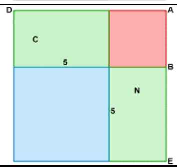
2 *Materials and methods*

The theoretical framework that supported the experimentation has taken into account the relationships between the built of mathematical knowledge and its historical construction (Menghini, 1994; Furinghetti, 2020). Furthermore, the semiotic and epistemological reflections, within which the proposed propositions can be seen as a valid support material for the teaching of mathematics (D'Amore & Radford, 2017; Duval, 2018), have been considered. In particular, Abū Kāmil's proposition comes at a crucial moment in the semiotic evolution of the language of mathematics and constitutes one of the first written testimonies. In it, a simple problem is presented and solved using spoken algebra, which gradually leads from the Euclidean narrative to the Cartesian one. In this step, the geometric constructions play an important role in promoting the recognition of the object we are talking about and which is fundamental for the realization of learning (Barbin, 2015). In the same passage, in which the register change takes place, it is possible to grasp essential prodromes to the modeling process and crucial for the development of problem solving skills. The conducting and results of the experiment found their main methodological tools following a "modular" approach (Jankvist, 2009) to mathematical concepts through the use of original sources or by considering "historical packages" (Tzanakis et al., 2002). After some historical notes on the two works by Al-Khwārizmī (Rashed, 2007) and Abū Kāmil (Rashed, 2012) under consideration, the students were given a questionnaire to test their opinions on the link between algebra and geometry and then, specifically, they were asked to solve an equation of second degree, to write the solution formula and to give a justification. We continued by presenting the geometric justification of a particular second degree equation taken from the work of Al-Khwārizmī, comparing it with the algebraic one present in their textbook. Work on the first proposition of Abū Kāmil's treatise followed. It was then proposed the reading of the Italian translation, made by us, of the original text and, subsequently, the translation into the language of spoken algebra and finally into the language of symbolic algebra was carried out. Finally, a questionnaire was proposed to the students to test the effects of the lived experience.

3 Results

The students responded to the first questionnaire by attributing greater complexity to algebra than geometry and greater learning difficulty as it does not allow a visual confirmation of the objects being treated. 57% of them did not see any link between algebra and geometry, 43% perceived some relationship. 86% of students solved the proposed second degree equation by correctly applying the solution formula, 79% were unable to give any justification for it. Starting from the fact that no student had been able to justify the solution formula, we gave them photocopies of the text, which we translated into Italian, taken from Al-Khwārizmī's *Kitāb al-jabr wa al-muqābala*, which shows the procedure indicated by the author to solve the equation $x^2 + 10x = 39$ and its geometric justification. After a careful and commented reading of the text, together with the students we built a first correspondence table between the passages described in words by Al-Khwārizmī and their translation into geometric form and a second correspondence table between the geometric passages and their translation into algebraic form. The two tables are summarized in the following:

Solution described by Al-Khwārizmī	***Geometric form***	***Algebraic form***
Let AB be a square, to which we add ten of its roots.		x^2
Let us divide ten into two halves and draw two surfaces C and N on both sides of AB. The length of each of the two surfaces will be one half of ten roots, and its width is the side of AB.		$10 : 2 = 5$ $5x$ $(5x + 5x = 10x)$

We complete the figure with a square with a side equal to 5, starting from one of the angles of AB, which is half of the ten roots we added to it.		$5 \cdot 5 = 25$ **25**
We therefore know that the first surface is the square, that the two surfaces on either side of this are ten roots and that the sum is thirty-nine		$\boldsymbol{x^2 + 10x = 39}$
and that, to complete the larger surface DE, we have to add the square of side 5 - that is 25, which we add to 39, obtaining 64;		$\boldsymbol{25 + 39 = 64}$
we take its root, which is eight and which is one of the sides of the larger surface;		$\sqrt{64} = 8$
if we subtract from it an amount equal to the one we added to it, that is 5, there remains 3, which is the side of the surface AB.		$x = 8 - 5 = 3$

To highlight the actuality of the procedure followed by Al-Khwārizmī and the fact that the solution formula they studied is obtained, in its algebraic form, with the same procedure, we have considered, instead of the particular equation $x^2 + 10x = 39$, the more general equation of the same type $ax^2 + bx = c$ and retraced the passages of Al-Khwārizmī, associating each algebraic expression with its corresponding geometric expression. We then built a table of correspondence between the algebraic and geometric passages that led to the determination and justification of the positive solution of the equation $x^2 + 10x = 39$. Some students were struck by the fact that a result presented to them only with formulas had been justified with geometric figures, thus making natural and sharable those passages that the formulas had made difficult and of which it was difficult to identify how they could have come to mind. At this point the students could have followed what Abū Kāmil did to solve the problem of finding the length of the side of a regular pentagon inscribed in a circle with a diameter equal to 10. We therefore proposed to them the reading of the translation made by us of the passage taken from the *Kitāb fi al-jabr* *wa al-muqābala* concerning the considered problem. Subsequently, we compiled together with the students Table 1 shown in Florio (2020), where in the left part is reported the problem expressed in words by Abū Kāmil and, in the right part, its translation into symbolic algebra. The students observed the use of the properties of the considered figure, described geometrically by Euclid in the Elements, to write the equation that solves the problem and then express it with the signs of our current mathematical culture. They were thus able to observe how Abū Kāmil started from those results obtained by Euclid (Elements IV, 11) which allowed him to think of the pentagon as already built. This made sense of his quest to determine its side. We pointed out to the students how the drawn figure and the beginning of Abū Kāmil's speech are in the Euclidean register, as can be seen from the terms introduced, from the use of capital letters to indicate points and segments and from the proposal to "construct" the line CLD. The subsequent observation concerned the change of register by Abū Kāmil. In fact, by placing ED equal to "a thing", he passed from the geometric register to the algebraic register. At this point we asked the students how they would solve the problem of finding the side of the regular pentagon inscribed in any circle of diameter $2r$. Some of them replied that it was possible to replace 10, that is the diameter of the particular circle considered by Abū Kāmil, with $2r$, that is, the diameter of any circle. We then invited them to retrace the steps contained in the right part of the previous table. We have thus come to determine the acceptable solution to the problem: $x = r\sqrt{\frac{5-\sqrt{5}}{2}}$. This new formulation has allowed us to observe that the ratio between the side x of the regular pentagon and the radius r of the circle circumscribed to it is constant and independent of the pentagon considered: $\frac{x}{r} = \sqrt{\frac{5-\sqrt{5}}{2}}$. In this last part of our path the students were able to observe and learn, on an elementary example, how the change of register generated the prodromes of a new process, the one related to parameterization and, therefore, to the subsequent *modeling* in mathematics. As a conclusion of the experience, students were given a second questionnaire to test which impressions and what effects had produced the collective experience we have had. The students declared: to have grasped an evident link between algebra and geometry and intuited how a synergistic use of these two mathematical fields can increase the possibility of producing ideas to solve problems, favoring imagination and personal creativity (93%); to have acquired the awareness of how the same mathematical object can be described using different languages (30%); to be able to use some mathematical tools related to previous years of study with greater certainty (21%); to have acquired a less abstract and algorithmic vision of mathematics and to feel more encouraged in learning (39%). With regard to the history of mathematics, the students felt that: it can in general facilitate the learning of mathematics (43%); having read it, some contents are current (29%); stimulate curiosity and interest in mathematics also for the human and social component often hidden in the presentation of the sentences of school texts (21%). For the students it was comforting to hear that a definition of a few lines or an agile calculation tool, presented "casually" in textbooks, sometimes took hundreds of years to be understood and formulated as we know it today.

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