PEDRO NÚÑEZ: ALGEBRA IS ALSO FOR GEOMETRICAL PROBLEM SOLVING

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ABSTRACT

In this workshop, we will show how a 16th century historical text can be used to learn mathematics. We will present some problems from the *Libro de Algebra en Arithmetica y Geometria* by Pedro Núñez, about squares, rectangles, and triangles and how to use them in the classroom. Specifically, they can be used in the curriculum of Catalonia in the third or fourth year of compulsory secondary education (14 to16 year-old students), when the subject of study is the resolution of squares, rectangles, and triangles[56].

1 *Introduction*

One of the undertakings of the ABEAM[57] history group, to which we belong, is to design math activities to implement in the classroom, for students to learn mathematical concepts or procedures from original sources. Since it is about students learning mathematics through history, and not just the history of mathematics, not all historical episodes or sources are appropriate for designing activities to implement in the classroom (Romero-Vallhonesta & Massa-Esteve, 2016). In this sense, we consider the *Libro de Algebra en Arithmetica y Geometria* (Book of Algebra in Arithmetic and Geometry) by Pedro Núñez, a relevant text to implement some mathematical concepts in the class. We presented it to the students in the original Spanish version that they can read and understand without great difficulty.

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2 Pedro Núñez and his Libro de Algebra en Arithmetica y Geometria

First of all, we presented the author, Pedro Núñez, a mathematician considered by some historians as the most important figure in Portuguese science, focusing on his *Algebra*, his main mathematical treatise (Leitão, 2010).

Pedro Núñez (1502-1578) was born in Alcácer do Sal (Portugal). It is for this reason that he added Salaciense to his name in most of his printed texts. He studied at the University of Salamanca (ca. 1521-1522) and at the University of Lisbon where he obtained a degree in medicine (1525). He also learned astronomy and mathematics, and is probably best known for his contributions to the nautical sciences, which he approached, for the first time, in a mathematical way.

By 1534, he may have begun writing the *Libro de Algebra en Arithmetica y Geometria* probably during his stay in Salamanca, but which was not published until 1567 (Núñez, 1567). This work contains the problems that we will analyse in this session. It is one of the first works with algebraic content to be published in the Iberian Peninsula.

This discipline was developed from the so-called practical arithmetic, where equations begin to be written with abbreviations and symbols, in contrast to the totally rhetorical way of expressing them in most previous texts. An example is the *Suma de la art de arismetica* by Francesc Santcliment, a work printed in Barcelona in 1482.

The *Libro de Algebra* by Pedro Núñez is divided into three parts, which the author calls main parts and consists of 710 pages. The first part deals with the purpose of algebra, its rules and *conjugations*, which is how the author refers to the equations. Also it demonstrates the rules of simple and compound *conjugations* and particular cases. In the second main part, Núñez declares the unknowns which he calls *dignities* and explains how to operate with them. There are 12 chapters dedicated to operations with roots. The author devotes 15 chapters to the theory of proportions, the last of which to the roots of binomials.

The third main part of the work is the most extensive and develops the solution of equations that he had addressed in the first part, and it does so in a much more complete way. It consists of 7 chapters, 5 have a rather theoretical character and 2 of which exemplify the application of algebra to arithmetic and geometry. The work ends with a letter from the author to the readers in which Núñez cites authors who have written algebra texts such as Pacioli, Cardano, and Tartaglia.

3. Two problems implemented with students (16 years old)

During the 2021-2022 academic year we gave two problems from the *Libro de Algebra en Arithmetica y Geometria* (Massa-Esteve, 2010) to a group of 16-year-old students. The first one was Problem 19 of Chapter 7, of the Third Part of the book, and the second was the Demonstration of the Third Rule of Compounds, First Part of the book (p.10v).

In this section we present Problem 19 of Chapter 7, as it was introduced to students, and an explanation of how they solved it. We divided the original text into different parts and asked the students questions to guide them in understanding the text. They were given an explanation of some keywords.

We started by telling the students that solving a problem generally requires several stages as follows:

a) Interpret the problem presented and understand what is required
b) Apply known solution strategies or explore new ones
c) Answer what the problem asks
d) Argue the justification of the solutions obtained
e) Ask new questions related to a problem solved.

Dealing with the first step, interpreting the problem presented and understanding what is required, can be complicated because we are starting from a formulation written in 16th century Spanish. In addition, at the time of the author, symbolic language was quite different from what we use nowadays. The relationship between known quantities and unknown quantities was written using some abbreviations.

These are some of the keywords and their abbreviations:
The unknown was called the *cosa* (thing), abbreviated to *co*, nowadays known as x. The unknown multiplied by itself, what we now call the unknown squared (x^2)was called the *census,* abbreviated as *ce.*

To make it easier for students to interpret the problem presented and understand what is required, we suggest reading the text, sentence by sentence, to make the corresponding interpretation as well as drawing and writing what is needed to solve the problem. After each part of the problem, different steps are proposed to the students, eight in total.

This is the first part of the problem given to the students:

> 19. Si la area fuere conoſcida,y la sũma de los dos lados tambien fuere conoſcida,cadavno de los lados por ſi ſera conoſcido.

That means: If the area were known, and the sum of the two sides were also known, each side would be known.

First step. How would you interpret this formulation? What figure is the author referring to?

Second step. Can you reformulate it in your own words?

Third step. Draw the figure.

> Sea la area del rectangulo de 12. braças quadradas,y la sũma de los dos lados ſea vna linea de 8 braças.

The students had no problem with these three questions, everybody could do them, although some didn't answer the second question.

That means: If the area of a rectangle is 12 *fathoms squared* and the sum of the two sides is a line of 8 *fathoms*.

Fourth step. Construct your drawing according to these conditions, with the symbols you think are appropriate.

They placed the measurements in the drawing, *x* & *8-x*, and had no problem using the sign "-". They mainly used *x*. One student used *b* and *h* (base and height). Two students referred to the unknown sides as *a* and *b*.

Fifth step. What do you think a *fathom* is? Do you need to know exactly what it is to solve the problem?

They all deduced that this is a unit of measurement, and they didn't need to know the equivalence with the measures we use nowadays.

> Pornemos pues que vno de los lados ſea 1 co. y ſera luego el otro 8.m̃.1 co. y multiplicãdo vno por otro,haremos 8 co.m̃.1 ce.que ſeran yguales a 12. porque la area ſe haze por la multiplicacion de vn lado por el otro,que con el haze angulo recto.

That means: Let’s say one of the sides be 1 *co* and the other 8 *m*∼ 1*co* and multiplying one by the other we get 8 *co m*∼ *1ce which* will be equal to 12, because the area is made by multiplying one side by the other, that produces a right angle with it.

Remember that *co* is the abbreviation of *cosa* and *ce* is the abbreviation of *census* that Núñez used for the unknown and its square respectively.

Sixth step. Write this part of the problem symbolically and place the data in your drawn representation. What do you think *m* ∼ represents?

Most of the students wrote x(8-x); 8x - x^2 =12; 8x = x^2+12 or the same with a, instead of x.

In some cases, they managed with xy=12 and y=8-x

Or even ab=12; a+b=8. In general, the sign “-“ was not an obstacle for them.

Ygualando hallaremos que 8 co. ſon ygua-
les a 1 ce. p̃.12. que es la tercera delas compueſtas,

That means: Equalling, it will be found 8 *co* as equal to *1 ce p* 12, which is the third of the compounds.

Seventh step. Write this equality (it will be found that 8 *co* equals *1 ce p* 12) symbolically, that is, the corresponding equation, in current notation.

In general, they didn’t separate the 6th and 7th steps, they did it all on step 6 and after writing the final equation again in the 7th step.

Perhaps it would be better to present the problem using a sheet with two columns, with the wording of the different parts on the left and the right column left blank because some students do not need such structured guidance.

y obrando por ella, hallaremos que vno de los
lados es 6. y el otro es 2.

That means: Applying it (that means following the rule for the third case of the compounds) we will find that one of the sides is 6 and the other is 2.

Eighth step. Using your chosen method, find the values of the sides and explain how you have achieved the solution.

The students mainly solved the problem but with no explanation.

Three of them factorized the equation x^2-8x+12=0, into $(x - 2)(x - 6) = 0$ but only one of them arrived at the end concluding that $x = 2$ and 6.

We explained that Núñez avoids writing expressions with negative quantities. Negative numbers were not used at that time. To solve the equation, the author refers to “the third of the compounds” which is one of the cases that Núñez considers at the beginning of his book.

Núñez classifies the equations (conjugations) into simple and compound. The simple ones have only two terms (regardless of their degree) and the compound ones, three terms. One formula is enough for us to solve the three compound cases because we accept negative coefficients, but at that time, negative coefficients made no sense, so they had to consider three cases.

The classification of Núñez:

Cõjugaciones ſimples:	1. Cenſos yguales a Coſas.
	2. Cenſos yguales a Numero.
	3. Coſas yguales a Numero.
Cõjugaciones compueſtas:	4. Cenſo y coſas yguales a numero.
	5. Coſas y numero yguales a cenſo.
	6. Cenſo y numero yguales a coſas.

The problem we have shown led to one equation corresponding to the third of the compounds which, when Núñez refers to all conjugations, in general he calls it the sixth of the conjugations.

The “third of the compounds” which Núñez refers to in this problem, is the case in which the linear term equals the square plus the independent term and that we currently write as $x^2 + c = bx$, being b and c positive.

After the explanation in class and because of the students' interests, we proposed that they demonstrate the Third Rule of Compounds. We work in the same manner as with Problem 19. We divided the original text and asked the students questions to guide them in understanding every fragment and completing the demonstration.

4. Two more problems to discuss with the attendees of the workshop

As we also find in Euclid's *Elements* (Heath, 1956), Núñez sometimes solves problems, the results of which he will use later. One of them is problem 358[3] that we gave to the attendees so that they could understand Núñez's meticu-

[58] Núñez, 1567, 281^{v}-284^{r}.

lous way of working that leaves no unanswered details and often checks what he has done using other methods. In addition, it is a good example in which the power of algebra is shown to solve the problem in a simpler way.

The wording is as follows:

If we know the length of the sides of a triangle, the height (cathetus in Núñez's words) that falls on the side adjacent to two acute angles is known, and also the parts it divides the side on which it falls.

First of all, Núñez says that it has to be taken into account that if the triangle is acute (oxygonio in Núñez's words) there will be three heights within it (that is, the three heights will fall inside the triangle), but in the right triangle and in the obtuse triangle there will be no more than one height within.

The author considers the cases of equilateral and isosceles triangles. In the first, any height bisects the side on which it falls. In the second case, if it is considered the height that starts from the angle that forms the two equal sides, it also divides the opposite side in half. Both cases are proved by applying the Pythagorean theorem.

Then, Núñez considers the case of a scalene triangle and in the example he gives, of sides 13, 14 and 15, he finds the height that falls on the longest side and the segments into which it divides, applying Euclid (II,13)[59]. Since the square of the longest side is smaller than the sum of the squares of the other two sides, he deduces that it is an acute triangle and can apply the same procedure to all three sides.

The author adds that it is not necessary, however, to repeat the whole procedure because once one perpendicular is known, the others can be easily calculated, since the proportion of the sides is reciprocal to the proportion of the perpendiculars that fall on the sides. He calculates the perpendicular that falls on the side measuring 13 from the one already calculated on the side 15, using the reciprocal proportion. Núñez also directly used the formula of Euclid (II, 13), and checked that it gave $12\frac{12}{13}$, the same result[60].

The author uses an identical method when one of the perpendiculars falls inwards and the other outwards, as in the case of obtuse triangles. Also, if one

[59] In Euclid (II, 13) there is the procedure for finding the square of the opposite side at an acute angle.

[60] Núñez gives the results in the form of a mixed number, which we would currently express as 156/13.

perpendicular is known, the others can be calculated by the rule of three, according to Núñez. That means, using the reciprocal proportion he mentioned before. However, the perpendicular that falls outside also has its own rule for Euclid (II, 12)[61]. He says that it can also be found by algebra considering 1*co*[62]. one of the parts in which the side is divided. Applying the Pythagorean theorem, he obtains an equation of 1st degree (simple conjugation) and solves it. Finally, he says that the "oxygonium" case can also be solved by algebra.

After having commented on what Núñez intends with problem 33, problem 61 was given to the attendees to solve in their own way and think about how they would implement it in their classrooms and what guidelines they would give to their students, along the lines of those we had shown with problem 19. Then Núñez's way of solving it was discussed.

The wording of the problem 61[63] is as follows:

If the side of the equilateral triangle is known, the side of the largest square that fits in it will also be known.

Núñez begins the problem by supposing that the side of the equilateral triangle is 10 and that in the attached figure, *edfg* is the largest square that can be inscribed in the triangle.

Núñez then gives the following rule to solve the problem:

i) multiply the side of the triangle by itself obtaining 100

ii) multiply this 100 by 12 and it will be 1200

iii) from the root of 1200 remove the triple from the side of the triangle which is 30.

iv) the reminder is the side of the square, that is, in current notation:

The author, then, justifies the rule geometrically.

First, he proves that the triangle *aef* is equilateral and, therefore, its side is the side of the square.

The proof of this, is based on the parallelism of *ef* and *dg* (Núñez says that they are *equidistant*), which makes the angles *aef* and *abc* equal as are the angles *afe* and *acb*. Since the angle in "a" is common, the two triangles *aef* and

[61] In Euclid (II, 12) there is the procedure for finding the square of the opposite side at an obtuse angle.

[62] x in current language.

[63] Núñez, 1567, 242^v-247^r.

abc have the same angles and therefore their sides must be proportional. Since *abc* is equilateral, so is *aef*.

After that, Núñez calculates the angle *bed*, which is half of *ebd* and it is deduced that *bd* is half of *be*. To find the angle *bed*, it is known that *aef* is 2/3 of a right angle (it is an angle of an equilateral triangle). Since the angle in *fed* is right, *bed* is the difference to reach a flat angle, which will be 1/3 of a right angle. An alternative method is applied also, as is usual in Núnez's explanations. The author considers the triangle *ebd* which has a right angle and an angle which is 2/3 of a right angle. The third angle, therefore, must be 1/3 of the right angle.

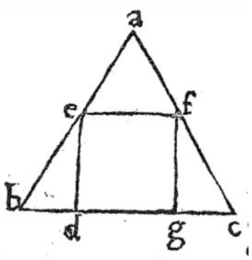
Next, the author supposes that *eb* is 2 and applies Pythagoras to triangle *ebd*. As *bd* is 1, *ed* will be $\sqrt{3}$. Then $ab = 2 + \sqrt{3}$.

But at the beginning of the resolution, Núñez had supposed that *ab* was 10, then:

$$\frac{2+\sqrt{3}}{3} = \frac{10}{?} \Rightarrow ae = \frac{10+\sqrt{3}}{2+\sqrt{3}} = \sqrt{1200} - 30$$

Here the author explains step by step how to rationalize (i.e., multiply by the conjugate, which the author expresses as *residue* or *reciso*). Finally, it is not necessary to divide because the denominator is the unity.

Núñez now summarizes the procedure and continues with an explanation of the operations he has done, of multiplying by the *residue* and summarizing again. In the midst of these explanations, he also says that if the side of the triangle were 20, this 20 would be multiplied by itself equalling 400 and then by 12 equalling 4800, and the root of this number, $\sqrt{4800}$, is the first part of the *residue*. The second part is 60, the triple of 20. He concludes that the side of the square is $\sqrt{4800} - 60$.



He now says that he will solve the problem by algebra and that he will use the same figure:

Let the side of the square be 1 *co*. (we will use x). The sum of *bd* and *gc* is 10 - x.

And since *ae* equals *ef*, and since *aef* is equilateral, it will be that *be* also equals 10 - x and so *bd* equals $5 - \frac{1}{2}$x

Now the author applies Pythagoras to the triangle *bed*:

$$\frac{3}{4}x^2 - 15x + 75 = (de)^2$$

$$100 - 2 \quad = (de)^2$$

$$(10-x)^2-\left(5-\frac{1}{2}x\right)^2=(de)^2$$

But *de* = *dg* = x and, therefore,

$$\frac{3}{4}x^2-15x+75=x^2 \Longrightarrow 300=x^2+60x$$

This equation is of the first type of *compound conjugations* and the author says that the value of the thing is $\sqrt{1200}-30$.

As we have already said in the introduction, our aim with the implementation in the classroom of this type of activity designed from relevant historical texts, is the learning of mathematical techniques and concepts in their context.

The history of mathematics is important, not only for the memory of our heritage, but also for educators to return to sources, and therefore a knowledge of these is very useful (Romero-Vallhonesta & Massa-Esteve, 2016).

The workshop was very interesting because not everyone chose the algebraic method. Some sought a more creative solution based on the symmetry of the figure. The importance of discussing alternative methods was recognised, and a creative approach to solve the problems was highlighted. In the workshop it was also noted that the Catalan students have the opportunity to read a text from the original 16th century version.

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