THE TANGENT LINE TO THE PARABOLA ACCORDING TO GREEK MATHEMATICS AND GALILEO GALILEI

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ABSTRACT

Our workshop aims to present the tangent line with the didactical use of historical sources from Greek mathematics and Galileo Galilei. The discussion starts with an analysis of the definition of the tangent line in Greek mathematics in the case of the circle, from Euclid's Elements. Then a method to draw the tangent line at a point on a parabola is presented. The description is taken from the Fourth Day of *Discorsi e dimostrazioni matematiche intorno a due nuove scienze* by Galileo Galilei (1638). Galilei's approach comes from Greek mathematics and is different from the algebraic one usually used at school. An application of this geometrical technique to high school exercises is carried out, with examples from textbooks.

1 *Using original sources in the history of mathematics*

In my approach the didactical use of original sources from the history of mathematics should possess the following characteristics: (a) the historical treatment should be neither too far nor too close to what is done in school today; (b) the sources, if possible, must be read in their original language (or at least in a "faithful" translation); (c) avoid the translation of mathematics into modern notation (some exceptions are possible but with some care, and always with warnings).

1.1 Our proposal

The first aim is to raise students' awareness that some geometrical objects could have different representations; parabola and tangent line in this case. Moreover, thanks to this approach, they should realize that mathematics is not a static science but evolves and changes over time. This will be achieved by reading a real mathematical book by G. Galilei *Discorsi e dimostrazioni matematiche intorno a due nuove scienze*, printed in Leiden in 1638. In this text, the author gave a geometrical construction for the tangent to the parabola that students will try to apply in some exercises taken from their textbooks.

The complete activity needs five or six hours and is addressed to an 11th-grade class of "Liceo scientifico" (however, it could be adapted to other schools). There are some prerequisites, especially for the geometrical proofs carried out by Galilei: properties of triangles and circles, and the intercept theorem. It is advisable (but not strictly needed) that students also know the parabola according to the usual algebraic approach used in high school. I proposed this activity in the school where I teach, and I thank my colleagues Sabrina Rossi, Maria Grazia Marzario, and Saverio Vignali for the opportunity they gave to me.

2 *The tangent line in Greek mathematics*

The activity starts recalling Euclid's definition of line tangent to a circle: (Heath, 1956, p. 1): "A straight line is said to **touch a circle** which, meeting the circle and being produced, does not cut the circle."

We need to pay attention to the reformulation that this definition could produce in the mind of students: "a line that intersect the circle only at one point is the tangent to the circle in that point". This *oversimplification* is correct for the circle, but it is not generally true. In this sense, in the workshop we read the answers to a questionnaire on the tangent I gave to my 13th grade; the questionnaire is adapted from (Maracci & Marzorati, 2019), (Biza, 2006) and (Biza, 2007). For example, the first question was:

> *Prova a spiegare, in parole semplici, a cosa pensi quando senti il termine "retta tangente". (Puoi scrivere liberamente quello che ti viene in mente, puoi fare un disegno, scrivere simboli o formule e usare qualsiasi mezzo più o meno formale per comunicare le tue idee).*
>
> *Try to explain, in simple words, what you think of when you hear the term "tangent line". (You can write freely whatever comes to your mind, you can draw a picture, write symbols or formulas, and use formal or informal means to describe your ideas).*

Among all the students' answers, we discussed the following one:

> *Una retta tangente è una retta che si interseca a una curva in un solo punto. In una circonferenza ad esempio la retta tangente in un punto è anche perpendicolare al raggio.*

A tangent line is a line that intersects a curve at only one point. In a circumference, for example, the tangent line at a point is also perpendicular to the radius.

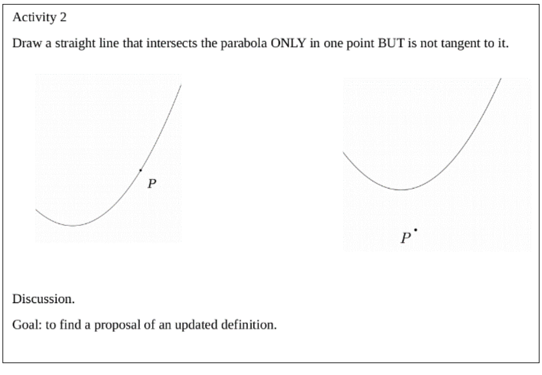
This statement is closely linked to the circumference. We shall see the case of the parabola gives an opportunity to broaden this point of view.

2.1 How to draw a tangent to a circle (Euclid III.16)

After the discussion on Euclid's definition, we proceed with the geometrical construction of the tangent to a circle, according again to Euclid (Heath, 1956, p. 37): "The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed." The proof is read thoroughly, because it is a good example of indirect proof (by contradiction) and it will be useful for the case of the parabola. In the workshop we discussed on the difficulties in proofs by contradiction (see for example, Antonini & Mariotti, 2008) and concluded that great care should be taken on this side.

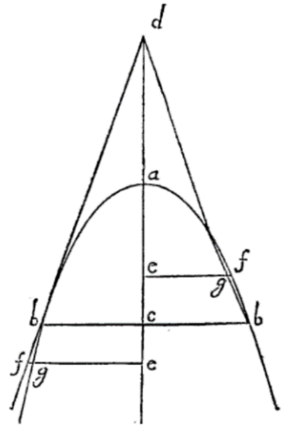
2.2 The tangent line to the parabola: definition

Having examined the case of the circumference, we introduce the tangent line to



the parabola with Activity 2 (see figure). The first step is to find a good definition of tangent line for the parabola. In the material used in classrooms two parabolas are given (and one point); students are asked to draw a straight line that intersects the parabola only at one point but it is not tangent to the parabola. The aim is to understand that in the case of the circle a line that intersects the circle only once is always a tangent line; for the parabola the same does not happen. In fact, the straight lines parallel to the axis intersect the parabola once, but they are not tangents. In the case of the parabola, a tangent line intersects the curve only once; but not all the lines that intersect the curve in only one point are tangents. Then, there are other curves (those in which the curvature changes sign) in which it is not even true that the tangent cuts the curve only once. This should make students aware that the definition of tangent line evolves along with the curves to which it is applied.

2.3 Galilei's construction

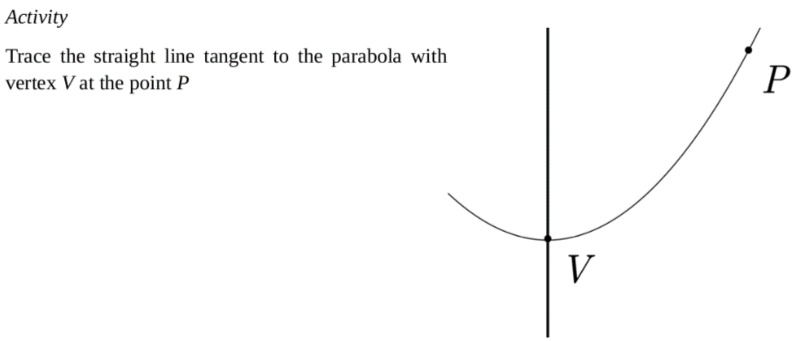


In this part, we show a geometrical construction to draw the tangent to a parabola. The text from *Discorsi e dimostrazioni matematiche intorno a due nuove scienze* (Galilei, 1638) is given, and the students are invited to read and discuss the following passage:

"Segniamo la Parabola, della quale sia prolungato fuori l'asse *ca* in *d*. E preso qualsivoglia punto *b*, per esso intendasi prodotta la linea *bc* parallela alla base di essa Parabola. E posta la *da* eguale alla parte dell'asse *ca*, dico, che la retta tirata per i punti *d, b*, non cade dentro la Parabola ma fuori, sì che solamente la tocca nell'istesso punto *b."* (Galilei, 1638 pp. 239-240).

“Draw a parabola with its axis CA extended to D, and take any point B [on the parabola], drawing through this the line BC parallel to the base of this parabola. Take DA equal to the part CA of the axis, and draw a straight line from D to B. I say that the line DB does not fall within the parabola, but touches it on the outside only, at point B.” (Drake, 1974, pp. 219-220). The students have to draw, with pencil and ruler, according to Galilei’s construction, the tangent line in the case presented in the following figure.

Figure



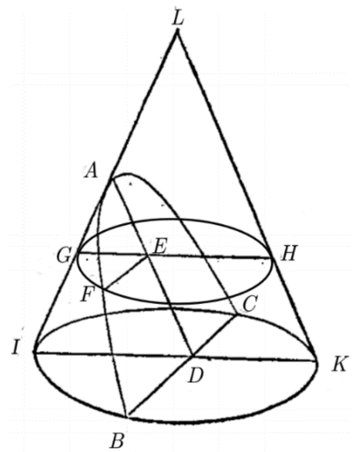
2.4 Application to textbook exercises

Then, I propose to solve two exercises with Galilei’s method: (a) find the tangent line to the parabola $y = 2x^2 - 4x$ at its point Q with abscissa 3; (b) find the tangent to the parabola with axis parallel to the x-axis and the point $V(1,1)$ as its vertex, passing through the point $Q(4,4)$ on the parabola. At the beginning, it is not clear how to draw the parabola following Galilei, since these exercises are stated from a Cartesian perspective. It is not hard to find the focus and directrix, given the points V and Q, and a horizontal axis; then it is possible to proceed in a Galilean manner, once the parabola is drawn. The construction can also be carried out with a dynamic software (GeoGebra, for example), following the steps of Galilei: (a) trace the perpendicular PT from the point P to the axis; (b) take T', the point symmetric of T with respect to V; (c) trace the line $T'P$: it is the tangent to the parabola at the point P. These examples give the opportunity to illustrate the differences between the Galilean and Cartesian approaches. For Galilei, following the Greek point of view, the parabola is a curve obtained by cutting the cone, while the Cartesian method identifies the parabola with its equation.

3 *Background to the proof of Galilei's construction*

A discussion concerning what the parabola was for Galilei is necessary before moving on to the proof of the tangent's construction. First, the teacher should make clear to the students why Galilei was interested in parabolas: a brief discussion on projectile motion (possibly already examined in physics) is carried out (Drake, 1974, p. 217): "When a projectile is carried in motion compounded from equable horizontal and from naturally accelerated downward [motions], it describes a semi-parabolic line in its movement."

But to understand Galilei's proof it should be clarified that the parabola at that time was not "our parabola": there was no equation, no analytic expression, like $y = 3x^2 - 2x + 1$ and not even a coordinate system. Following the



tradition of classical Greek mathematics, the parabola is the curve obtained by cutting a cone with a plane. So, there is no equation, but a characteristic property, expressed with a proportion between some geometric magnitudes $BD^2 : FE^2 = DA : AE$. In the material for the classes, I present a diagram of a cone cut by a plane in such a way as to obtain the parabola (see figure). Then, in the workshop we read and discuss the proof of this result (Galilei, 1638 pp. 238-239):

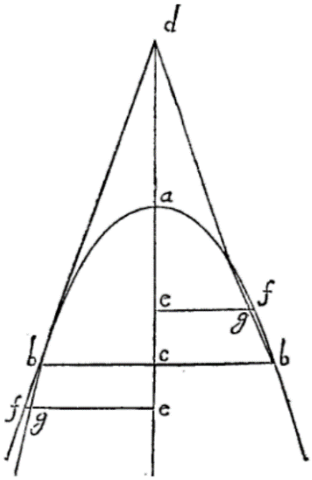
> Dico che il quadrato della *bd* al quadrato della *fe* ha la medesima proporzione che l'asse *da* alla parte *ae*.

Students also have at their disposal three-dimensional models of the cone and its sections (see figures below). These models were used in classes to follow and visualize every step of Galilei's proof. The models were provided by Monica Guadagni, a student writing her degree thesis about introducing conics at high school level with the use of models and mathematical machines. This representation is very important for students that can touch with their hands the cone and see the parabola as Galileo imagined it.

Figure

3.1 *Galilei's proof*

Galilei's proof is a proof by contradiction which assumes that the straight line constructed with the procedure described above (see section 2.3) cuts the parabola in two different point, and so it is not tangent to the parabola. This will lead to a statement that is known to be false (Galilei, 1638, pp. 239-240, Drake, 1974, p. 220):



> [la retta tangente] caschi dentro, segandola sopra, o, prolungata, segandola sotto, ed in

essa sia preso qualsivoglia punto *g*, per il quale passi la retta *fge*.

For if possible, let it [*DB*] fall within and cut [the parabola] above [*B*], or below when extended. Take in it [extended] some point *G*, through which draw the line *FGE*.

It is important to note that the figure contains both cases: on the left side the point *g* is below *b*, on the right side the point is above the point of tangency *b*.

Then Galilei, making use of the theory of proportion and of the property of the parabola previously proved (see section 3), shows that:

Maggior proporzione ha la *ea* alla *ac*, che'l quadrato *ge* al quadrato *bc*, cioè che'l quadrato *ed* al quadrato *dc*.

The last step is to prove this impossible. Galilei proceeds: "la linea *ea* alla *ac* ha la medesima proporzione che 4 rettangoli *ead* a 4 quadrati di *ad* cioè al quadrato *cd*. […] adunque 4 rettangoli *ead* saranno maggiori del quadrato *ed*". But this is impossible because the point *a*, which bisect the line *dc*, could not bisect also the line *de*.

There are no symbols in this proof (with the exception of letters that indicate geometrical points) therefore, students are "forced" to carefully read the text. Some expressions used by Galilei raised questions and discussion with the teacher and among students. For example, "il rettangolo *ead*" means the rectangle constructed on *ea* and *ad*. This way of expressing geometrical quantities is very different from what students are used to. They would have expected the product of two lines and not the construction of a rectangle, as Galilei stated.

4 *Conclusion*

The parabola is a well-known curve, studied at school. However, usually, in school it is defined as a set of all points that satisfy a specific condition (locus of points) or as the graph of a quadratic function $f(x) = ax^2 + bx + c$. These representations lead to the development of an algebraic method to find the tangent line to the curve: setting the discriminant of some equation equal to zero. Instead, in Greek mathematics – and still for Galileo – the parabola is the curve obtained intersecting a cone with a plane. Consequently, it is possible to construct geometrically the line tangent to the parabola, as in the case of the circumference. In our proposal, this analogy is carried out to make students aware of the different representations of mathematical objects.

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