GET IN TOUCH WITH CALCULUS

A new material device collecting a historical legacy

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ABSTRACT

We introduce an analog device designed for laboratory activities related to calculus. Such a device recollects the legacy of historical instruments to find the area by solving inverse tangent problems (*integraphs*), that analytically corresponds to the resolution of differential equations. We present an analysis of its exploration by a high-school mathematical teacher with experience in mathematical machines.

1 Introduction

In this paper, we introduce a new analog device that recollects the legacy of historical instruments to find the area by solving inverse tangent problems (*integraphs*). We present an analysis of it by a high-school mathematical teacher with experience in mathematical machines and analyze the exploration within the theoretical framework of the semiotic mediation in mathematics education from the perspective of its use in the classroom.

The starting geometrical problem is to construct a curve given the properties of its tangent, the so-called "inverse tangent problem" (cf. Bos, 1988; Tournès, 2009; Milici, 2015; Milici, 2020; Crippa & Milici, 2019). To mechanically solve an inverse tangent problem, we must constrain a point so that it moves along a direction. Considering a wheel rolling on a curve, the direction of the wheel is tangent to the curve. By guiding that direction, in the first half of the 18th century scholars like Perks, Poleni and Suardi were able to trace transcendental curves. In the late 19th century, similar technical ideas were independently rediscovered; for example, while moving a pointer along a traced curve, a machine was able to trace the integral of a previously traced curve. Machines of this kind were named integraphs. More historical details are available in this ESU9 volume (cf. Crippa & Milici, 2023).

2 *Our integraph*

We introduce an integraph invented by the second author and built by typical FabLab tools (laser cutting, 3D printing, CNC milling). Following the numbering on the left-hand side of Fig. 1, it is made up by the following components.

1. *The frame*. This stands on a sheet of paper and allows the plate [2] to slide.
2. *The plate*. This is a rectangular piece of transparent plexiglass with three guides carved out (the two little ones perpendicular to the big one).
3. *Two rods*. These act as linear guides to be put on the plate. They can be joint to form a T (the short rod contains the piece [6], the long one pieces [4] and [5]).
4. *The peg*. After fixing the peg in a point of the two little guides of the plate [2], it slides inside the long rod [3].
5. *The positional pointer*. This is a pen holder that can slide inside the long rod [3] and the big guide of the plate [2] (see also Fig. 1, top right).
6. *The directional pointer*. This is a pointer on which the bottom two parallel wheels can rotate at different speeds when touching the paper sheet. It has a cap to constrain the direction of the rod through which it passes to be parallel or perpendicular to the direction of the wheels (see also Fig. 1, bottom right).

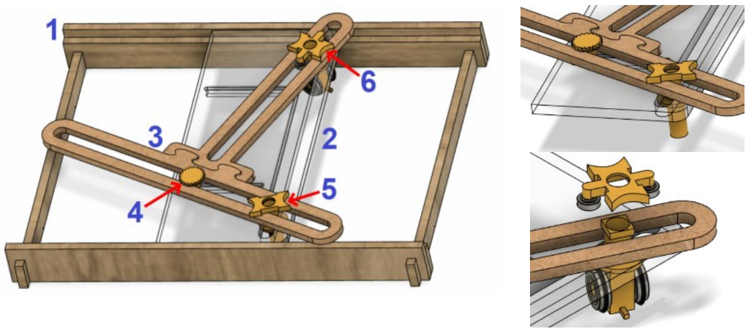

Figure 1. Left: our integraph and its components. Top right: the positional pointer and the peg (note that for the positional pointer the orientation of its top cap is irrelevant). Bottom right: the directional pointer (note that the square head allows the top cap to be right-angle rotated).

Pointers have a hole that can be used as a viewfinder (to move the pointer along a curve) but also to hold a marker (so that the pointer leaves a trace). To understand how this machine is related to calculus, let us introduce two reference frames such that abscissae correspond with the little guides on the plate [2 in Fig.1] and the two ordinate axes are superimposed. In Fig. 2, we represent in red the Cartesian axes related to the directional pointer D [6 in Fig.1], and in blue the ones for the positional pointer P=(x,y) [5 in Fig.1]. We also take as unit the distance between the peg [4 in Fig.1] and the big guide of the plane: thus the peg has coordinates (x-1,0) in the blue reference frame and the ordinate of P corresponds to the slope of its line through the peg. According to the configuration on the left of Fig. 1, the direction of D is set perpendicular to the short rod, that is perpendicular to the long rod: that implies that the line through P and the peg must be parallel to the direction of D. But if D follows the graph of a function *f*, its direction corresponds to the tangent to the curve, therefore the slope of the line through P and the peg is exactly the derivative f'. To sum up, if we move D along the graph of a function in the red reference frame (recall that, to move along a curve, we have to guide the direction of D), the point P can trace by a marker its derivative in the blue reference frame. Conversely, when P moves along the graph of a function in the blue reference frame, D traces one of its anti-derivatives (according to the initial position of D).

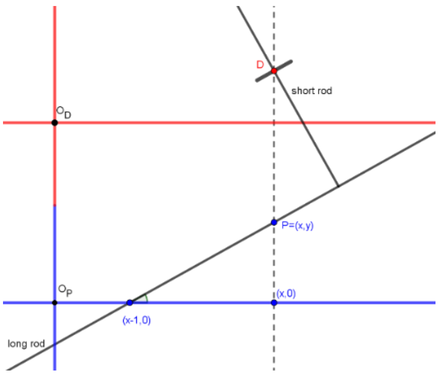

Figure 2. D is the directional pointer (the gray segment represents the direction of D) and its position is relative to the red axes; P is the positional pointer and its position is relative to the blue axes. Note that, in every configuration, the abscissae of the two pointers coincide.

3 Theoretical background and methodology

Our study of the use of the integraph in mathematics teaching and learning is based on the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008), in which an artifact is used for mediating mathematical meanings. The educational choice of an artifact is based on the analysis of its semiotic potential, i.e. how its use to accomplish a task is linked with emerging personal meanings and mathematical meanings embedded in the artifact. This analysis is essential for constructing tasks for students. It is mainly based on the exploration of the machine which is guided by four questions (Bartolini Bussi et al., 2011): How is the machine made? What does the machine make? Why does it make it? What could happen if ...?

Within these historical and educational references, we aim to answer to the following research questions:

1) What are the cognitive processes during the exploration of the machine?

2) Which, when and how do mathematical meanings emerge, in particular the idea of tangent line during the exploration of the machine?

For answering these questions, we proposed the exploration of the machine to a mathematics teacher (V) who is an expert in mathematical machines (she collaborates with the University of Modena e Reggio Emilia). This choice is based on three main reasons: the mathematical knowledge embedded in the machine is part of teacher's mathematical background; her expertise in mathematics laboratory with mathematical machines; sharing of the mathematics education theoretical framework. The teacher participated to the exploration of another inverse-tangent machine some years before (Maschietto, Milici & Tournès, 2019). Nevertheless, the two artefacts are very different.

The process of exploration was guided by one of the authors (P) of this paper. The analysis is based on the videotape of the session and V's drawings.

4 Analysis

In this paper, we analyze the first part of the exploration. For our analysis, we take into account four steps:

1. Start of the exploration and the emergence of the components;
2. Conjectures on movements of the machine without moving it;
3. Gestures performed during the manipulation;
4. The emergence of the idea of a tangent line.

1. The exploration starts with the description of the machine, corresponding to the answer to the question "How is the machine made?". At the beginning, V identified: «a plane [the frame, 1 in Fig.1], there are two rails on which I suppose this structure can shift…then there is a kind of set square … composed of two rods [3 in Fig.1] forming a right angle. Inside these rods, there are sliders [pointers 5 and 6, in Fig.1]. […]. Below, one of the two sliders [6 in Fig.1] has some small wheels, the other has not». The peg [4 in Fig.1] is mentioned by V only after P's question. After this, V paid attention to another characteristic of the machine: two pointers touch the plane, while the peg does not. About 5 minutes after the beginning of the exploration, V saw the holes into the pointers and highlighted the possibility to insert a pencil. After a certain time period, V asked about the role of the "transparent structure" [plate]. Even though V knew how the exploration of a machine should be carried out, the components emerged little by little through the interaction with P.

2. Concerning the movement of the machine, P asked how the machine could move and what were the constraints before moving the machine. First, V made conjectures about the relationships between the components. Then, V tested her conjectures directly by moving the plate. In particular, we have distinguished:

- Conjecture about movement: «I think that the structure on the two rails, in fact, moves vertically following the rails and ... we think ... if this structure moves […] this slider [the positional pointer] will move to the left and this one [the peg] will move upwards, I think».

- Conjectures about constraints: «The constraints are…, probably because there is another groove here, the machine can be turned, I think. You can turn it over by inserting this rod into this groove [the big guide with the directional pointer inside], I think. The constraints are given by ... these ... small grooves, I guess, the slider [pointer] can only shift in there and got to the bottom ... Then…».

3. During the exploration, we have identified three kinds of gestures, concerning how the different components are grasped and moved:

- Sliding the plate by moving fingers placed upon it. There are gestures of usage for discovering the possibility of movement of the machine.

- Following a drawn straight line. V was asked to follow a straight line drawn on the paper at the plane of the machine.

- Following a curve. After changing the structure of the machine by taking off the positional pointer and putting a single rod in the direction of the wheels of the directional pointer, P traced a curve on the paper sheet and asked V to follow it.

With respect to the previous gestures, V directly grasped the pointer and followed the curve.

4. Concerning the idea of tangent line, the task of following a drawn curve seems to be crucial. The controlled movement of following a curve seems to suggest to pay attention to the wheel and the relationship between the rod and the curve, but P's intervention is fundamental at this point.

P: «What does this rod represent? With respect to the drawn curve?».

V: «At each point, it is the tangent line… yes, at each point of the curve it is the tangent line at the curve in the point».

5 Conclusion

This article presents a step of our study of the machine from the perspective of its use in mathematics education. It considers the analysis of the semiotic potential, which is based here on the exploration of the machine by a high-school mathematical teacher.

The first research question concerns the processes activated during the use of the machine. The analysis highlights that the exploration of the machine is quite complex, because of its different and several components and their manipulations. Indeed, the components emerge little by little and at different steps; gestures depend on the configurations and constraints of the machine. The guidance of the researcher results essential during all the process, both to make the components emerge and to support the exploration. For instance, when the task of following a curve with the directional pointer is proposed, the focus on a particular component is necessary. It thus emerges that the rod evokes the idea of a tangent line in a strong way.

Our analysis aims to provide insights for constructing tasks for students and teachers who are not experts in exploring this kind of machines. It also suggested the construction of a new version of the machine, available at www.machines4math.com/; it can be made by downloading files for free from www.thingiverse.com/thing:5532958

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