COLLEGE GEOMETRY FROM AN ADVANCED HISTORICAL STANDPOINT FOR MATHEMATICS EDUCATION

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ABSTRACT

Following Felix Klein, an advanced historical standpoint is here presented for teaching college geometry for teachers. Three main ways of developing an advanced historical standpoint are discussed with classroom experiments. One is building connections among geometries by developing an inquiry into definitions of geometric objects such as rhombus, their extensibility with their family relationships across Euclidean and non-Euclidean geometries. Second is on the multiplicity and extensibility of transformations as represented by two historical approaches advocated by Klein and Usiskin. The third way to develop an advanced standpoint is by developing a critical look into a geometry practice tracing its change with the reforms in school geometry. The practice of constructions to connect geometry and algebra is impacted by two historical efforts. One is a supportive effort by Hilbert on the practice of constructions by Hilbert's Algebra of Segments dating back to 1902 to connect geometry and algebra. The other historical reform effort is by School Mathematics Study Groups (SMSG) during 1960s, which led to weakening the axiomatic foundations of the practice of constructability and exactness. The case of SMSG's angle construction axiom is criticized in their revision of axiomatic foundations of school geometry. Three approaches to develop an advanced standpoint informing research and practice of geometry teacher education towards a more historically connected stance.

1 Advanced Historical Standpoint on Geometry for Teacher Education

It is important for mathematics teachers to know some of the history of mathematics, but also the history of mathematics education. Felix Klein's *Elementary Mathematics from an Advanced or Higher Standpoint* (1908/2016) made a historical impact towards advancing mathematical preparation of teachers by analyzing elements of mathematics with its fundamental concepts informing school mathematics. Kilpatrick (2019) brought to the attention of history mathematics education researchers that Felix Klein's double discontinuity between university-to-school mathematics and triple approach to address it, by a unified approach to show how problems in branches of mathematics are connected (e.g., geometry, algebra), and how they are related to the problems of school mathematics. This approach focuses on improving teacher education and mathematical knowledge for teaching with a scholarship on practice by revising the mathematics content courses for teachers to help them gain a higher standpoint. Following Klein's epistemological approach, a higher stance for school geometry is targeted here with future teachers by focusing on the connections within sub-disciplines of mathematics, rather than treating them separately, and providing more unified view through common elementary constructions. Also integrating the practice of developing advance perspective as exemplified by Usiskin, Peressini, Marchisotto and Stanley (2003), the advanced perspective on geometry is pursued here by focusing on alternative definitions of familiar geometric objects, their extensions, and connections. This approach aligns with the recommendations made by Conference Board on the mathematical preparation of teachers suggesting future teachers to complete three courses with a focus on school mathematics from an advanced viewpoint (CBMS, 2012).

This paper with its three parts provides a contribution on the mathematical education of teachers by building a scholarship on teaching geometry with an advanced historical standpoint through classroom experiments. First, an inquiry-based approach is offered by revisiting familiar geometric objects in alternative geometries, with an attention to rhombus by exploring its definitions and connections within and across geometries. Instructional artifacts are given to describe this novel perspective in teacher education to develop knowledge of connections between different geometries. Second part focuses on the transformational geometry as advanced by Klein (1906) and Usiskin et al. (2003) for teaching geometry. The extensibility of transformation perspectives such as isometries and dilations across geometries are discussed. Third part is about developing a critical stance on the practice of construction as impacted by Hilbert's Algebra of Segments and the changes on axiomatic foundations of geometry.

2 Classroom Experiments in Building Advanced Perspectives

This work builds on author's scholarly mathematics teaching and learning research on geometry courses mainly for teachers integrating historical perspectives. Instructional materials incorporating higher standpoint on school geometry are experimented in undergraduate and graduate courses for teachers.

2.1 Building connections across geometries with familiar geometric objects towards developing an advanced standpoint

College Geometry course for teachers integrates historical perspectives by emphasizing alternative axiomatic foundations for geometry including Euclid's, Hilbert's, SMSG, and transformational geometry. The extensibility of the geometric objects such as rhombus or parabolas across Euclidean and non-Euclidean geometries are investigated to gain higher standpoint.

Preservice teachers explored the extensible definitions of the geometric objects exploring their alternative definitions in alternative geometries. Rhombus is a shared parent object for equilateral quadrilaterals subsuming squares in Euclidean and quasi-squares in spherical and hyperbolic geometries, providing a contrast to Saccheri squares building on perpendicular adjacent sides. The observed characteristics common to all three constructions of rhombus was that the diagonals are perpendicular and bisect each other in all three geometries. Students used this property as a defining characteristic of squares/quasi-squares in Euclidean and Non-Euclidean geometries. A square is redefined extensibly as a geometric object across alternative geometries as a rhombus with congruent diagonals (See Fig 1).

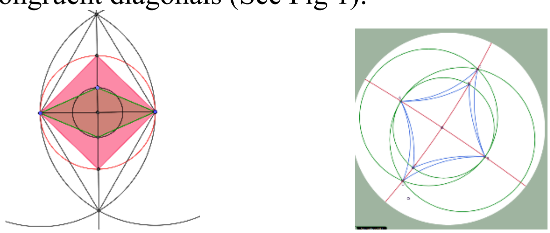

Figure 1. Quasi-Squares from rhombus as equilateral quadrilaterals with congruent diagonals extensible to Hyperbolic and Spherical Geometries

Higher perspective is gained by revising a familiar geometric object in alternative geometries and gaining a new sense by extending the geometric object into other geometries redefining it through its viable manifestations.

2.2 Advancing higher stance by historical perspectives on transformations for teaching school geometry

Elementarization of transformation perspective and groups were two main drivers of historical shifting efforts during the reform efforts in school geometry in USA (Schubring, 2019). Klein defines Euclidean geometry as a science that studies those properties of geometric figures that are not changed by similarity transformations. Historical perspectives on definitions of isometries were compared by students analyzing textbooks comparing the alternative definitions of isometries as defined by experts such as F. Klein (2004), and Z. Usiskin et al. (2003). Students generated their own definitions of reflection, rotation, and translation. Students worked on alternative definitions of reflection avoiding common circular definitions. Students examined the definition by the textbook defining reflection about line *l* as a transformation of the plane which, for every point P on the plane: P = P' (if P is on *l*) and, *l* is the perpendicular bisector of PP' (if P is not on l). Students criticized this definition since it used the reflected point P' as a part of reflection, which is clearly not an operative definition to construct a reflection of P but it helps us validate/refute a point P' if it is a reflection. To advance their perspective, students developed a non-circular definition of reflection for a given P and *l* based by constructing congruent triangles APB and AP'B for any two points A and B along *l*, forming a kite APBP' with perpendicular diagonals, which helped to build students' inquiry into relevant propositions to justify.

2.3 Advanced Stance by Studying Historical Changes in the Practice of Constructibility and Algebra of Segments in Geometry Education

D. Hilbert (1906) and School Mathematics Study Group (SMSG) during 1960s advocated two opposing historical perspectives related to the practice of constructability. SMSG developed a revised axiomatic system on school geometry during the New Math reforms in 1950s. Hilbert's *Foundations of Geometry* (1906) presented a revised Euclidean axiomatic system containing the Algebra of Segments that can be traced back to Euclid's Elements and Descartes' geometric method of constructing segments to solve polynomial equations (Bos, 2001). Building on Hilbert's Algebra of Segments, students were here given segments a and b to produce segment c corresponding to the addition, subtraction, division and multiplication. Students constructed geometric multiplication (Fig. 2). Given lengths were placed along axes. Parallel lines were constructed. Depicted by the circle, the congruency of x*y and y*x indicated the commutativity of geometric multiplication.

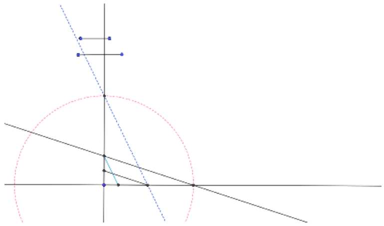

Figure 2. Geometric multiplication of given two segments

Students realized that constructing the division is a multiplication with an inverse. Division was built on finding inverse of a point with respect to a circle with a given radius. This path of development helped students understand how one can build segments corresponding to a polynomial equation as introduced by Descartes (Bos, 2001). Both Euclid's or Descartes' geometry produces the exact measure of length or area through geometric constructions (Bos, 2001). Students analyzed Descartes' method on using geometry to solve quadratic equations. A higher stance was gained by highlighting connections between geometry and algebra that was lost in school geometry. Next, students approached the angle related axioms with a critical stance comparing axiomatic systems. Hilbert's Congruence Axiom (Postulate III.4) states that if ∠ ABC is an angle and if B'C' is a ray, then there is exactly one ray B'A' on each "side" of line B'C' such that ∠ A'B'C'≅∠ABC. Half-century later, SMSG's Angle Construction Axiom suggests that for every r between 0 and 180 there is exactly one ray AP with P in H such that m∠PAB=r. This axiom is essentially about existence and uniqueness of a terminating side of an angle for a given measure. While it is not about its constructability, it was named as "Angle Construction Axiom" with consequences to trivialize the construction of angles for any measure. While the exactness had been an essential feature of geometry throughout history (Bos, 2001), the exact constructions of angles lost its sense with the SMSG axioms. The students' focus was on geometric construction of segments with irrational lengths with regular polygons and their connections to solving polynomials, connecting golden triangles, pentagons, and golden ratio and solving x^2 $-x$+1=0 with geometric approach.

3 *Discussion and Conclusion*

This presentation provides a contribution on the mathematical education of teachers by developing an advanced stance for geometry teachers by building connections across geometries, axioms, integrating algebra and geometry with historical perspectives. Algebra of Segments is a neglected historical component during the school mathematics reforms in 20th century. Elementarization and Klein's "historical shifting' are complementary processes in the transformations of school mathematics responding to advances in mathematics and mathematics education (Schubring, 2019). It is exemplified here that historical shifting process does not always yield the desired consequences.

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