HANDS ON EUCLIDEAN GEOMETRY

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ABSTRACT

The formation of rational thinking is one of the most important general objectives to be pursued in learning. The study of Euclidean geometry makes a valuable contribution in this regard, especially if students are able to grasp the meaning of propositions independently. For these reasons, we consider it essential to make students use teaching tools that simulate mathematical entities and can be manipulated directly by them. Various teaching activities are described in the paper with the overall aim of solving concrete problems using Euclidean propositions.

1 *Introduction*

The Italian National Guidelines emphasise that it is absolutely necessary to propose problems in school practice, whose search for a solution strategy stimulates productive rather than reproductive thinking. Furthermore, it is considered appropriate for students to develop a positive attitude towards mathematics through meaningful experiences. In fact, it is very important that they understand how mathematics is necessary to develop those processes that are useful for problem solving.

We believe that drawing inspiration from Euclid's Elements for the design of teaching activities enables us to achieve these objectives. The nature of the Elements and the contributions of neuroscience suggest that it is absolutely necessary to make activities truly concrete in order to develop and amplify geometric perceptual skills in students. The manipulation of concrete tools that simulate mathematical objects has many advantages including driving the autonomous discovery of geometric properties of figures. Furthermore, teaching tools built according to mathematical rules allow for self-correction during the discovery process. Hands, therefore, assume a fundamental role in learning mathematics. Such learning, which significantly contributes to the construction of rational thought, absolutely must precede a formalization process if the latter is to become a useful language for understanding mathematics and not an obstacle.

This article describes the main steps of a teaching activity during which students had to estimate how many people could actually occupy a town square. After an initial common part, the activity will be divided into two possible paths that differ according to the age of the target students, 12 - 13 years old (seventh grade) or 14 – 15 years old (tenth grade).

2 Theoretical background

The study of Euclidean geometry helps develop productive thinking (Di Martino, 2017), which is useful in the search for problem-solving strategies. The 'power' of Euclid's proofs lies in the explicit link between visual-spatial skills and verbal reasoning (Enriques et al., 2006). The continuous interaction between the two skills is realized (Russo et al., 2017) by constructing, through the practice of drawing, various geometric entities in order to ensure their existence (propositions called "problems") and establish the validity of their properties by reasoning on them (propositions called "theorems").

However, geometric design produces static figures, which do not contribute to imagining their movements as a whole, or of their parts. Such movements are, under certain circumstances, useful in determining a strategy for solving a problem. This can indeed be facilitated by the concrete construction of "artificial manipulatives" (Bartolini Bussi & Martignone, 2020) that simulate geometric figures. Our cognition would thus be amplified, as it is now known that manipulation and perception influence each other (Craighero et al., 1999, Rizzolatti & Luppino, 2001). Specially designed artificial manipulatives are used in the activities described in this document. An important effect of manipulatives is their ability to facilitate the discovery of invariant elements, which we call 'stable structures', that persist despite the movements performed. It is these stable structures that constitute the properties of geometric figures (Pasquazi, 2020).

Manipulatives are primarily designed for preadolescents. For complex topics more suitable for adolescents, it is usually more appropriate to use interactive geometry software. In this case, you lose the advantages of direct manipulation. However, one is facilitated to make more complex deductions due to the possibility of exploring many more cases than with manipulatives. The positive effects of using interactive geometry software were explored by Miragliotta & Baccaglini-Frank (2021) in the field of geometric prediction. The theoretical control provided by the software (as well as manipulatives) facilitates students' research and discovery of geometric properties.

It is hypothesised that the improvement in geometric perception skills due to the manipulatives and software also affects the ability to argue the discovered properties. Property deductions using manipulatives or software are not true Euclidean proofs. But they are preparatory to these. Both allow a continuous interaction between the object (concrete or virtual) and the deduction, between intuition and logic, preparing the groundwork for subsequent formalization.

3 Aims of the teaching activity

The history of mathematics offers many educational insights. An effective way to capture students' attention and interest is to start with concrete problems from which the first mathematical ideas developed. Among these is the problem of estimating very large surfaces. This is a very topical problem whose solution allows, for example, an estimate of how many people are contained in a town square. A concrete problem, therefore, whose resolution is by no means trivial. To solve it, we will work on a map of a town square that has an irregular shape and to calculate the area of these surfaces, we will use some Euclidean propositions. These propositions will be learned following axiomatic deductive logic by means of "figurative proofs" performed directly on manipulatives or on figures simulated by interactive geometry software. After having calculate the area of the town square represented on a map, using a scale factor, the area of the real square will be estimated. We speak of estimation because the accuracy of the calculation will naturally depend on how much the regular polygon chosen to approximate the square is. You will find that increasing the sides of the irregular polygon will give a better approximation of the town square.

4 Description of the main steps of the activity

Based on these assumptions, the mathematical prerequisites are the concept of area as the extension of a plane surface, the calculation of the area of triangles, common notions 2 and 3 of Euclid's Elements, the characteristics of parallelograms and proportions.

All activities were carried out in laboratory mode, proposing open problems to the students divided into groups. The first step in the activity was to pose a realistic problem: Piazza San Giovanni in Rome (where the schools in which the activity took place are located) has always been a venue for public events. By examining various sites that reported the number of participants at such events, the students found that the same square was judged to be full from a minimum of 150.000 people to a maximum of 1.000.000. The question posed was: what is the true capacity of the square? How can mathematics help us answer these questions?

To effectively define the problem, it is necessary:

1) a plan of the town square and its relative scale;

2) a count of how many people can be contained in one square metre;

3) a method for estimating areas that cannot be measured directly.

Looking for a plan of the town square and its scale is now a very easy problem to solve thanks to geographical applications. The students took a screenshot of the town square, including the scale, then turned it into an image to work with.

Knowing how many people can be contained in one square metre is also a very easy problem to solve. After creating a square on the floor with a one-metre side using paper tape, they counted how many people could fit into it. Generally, a minimum of four and a maximum of six people could fit.

Finally, the students had to learn a method to estimate the area of a very large surface. This is by no means an easy problem to solve.

4.1 Euclid's propositions.

The shape of Piazza San Giovanni is irregular. Therefore, there are no formulas to determine such an area. This forces students to work on geometric rather than arithmetic aspects. It is therefore necessary to transform the town square into an equivalent polygon for which it is easy to calculate the area. To solve this problem we used Proposition I.35 of Euclid's Elements which states: *parallelograms which are on the same base and in the same parallels are equal to one another (read equivalent).*

Students have to discover the previous property for themselves. To this end, they were provided with mathematical manipulatives (Figure 1) consisting of a trapezoid-shaped structure we call the *base*, which always remains fixed, and two triangles (indicated with “A” and “B” in figure), called *mobile figures*. These triangles must be inserted into the base and through their movement it is possible to discover the formation of parallelograms obtained from the difference between the trapezoid and all the mobile figures in it.

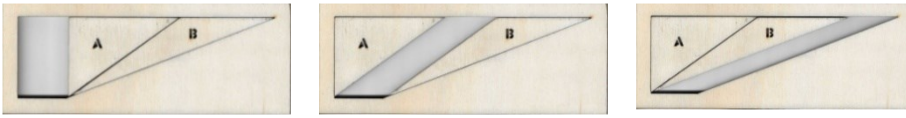
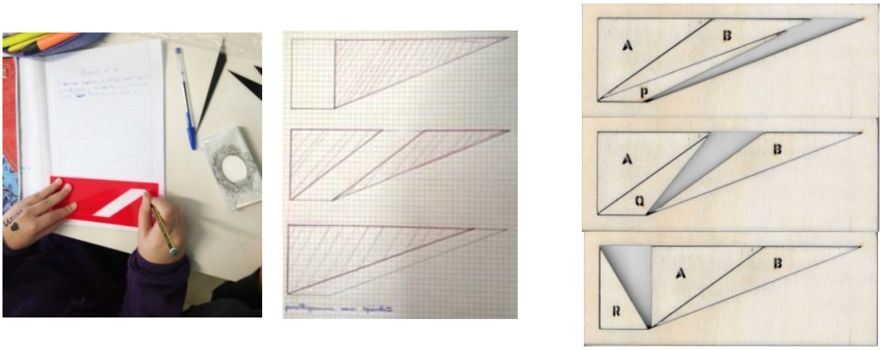


Figure 1. The equivalent parallelograms within the base

The students had to discover that the resulting parallelograms have three stable structures (Figure 2a and b). They have the same area (consequence of Common Notion 3). They also have base sides and are contained in the same parallels, i.e. they all have the same height. Learning is facilitated by the possibility of rotating the base. What was not observed according to a certain orientation of the base could be discovered with respect to another orientation.

Through appropriate observations, students discovered that the equivalence of parallelograms is a direct consequence of the equality of the base side and height of parallelograms (and not vice versa).



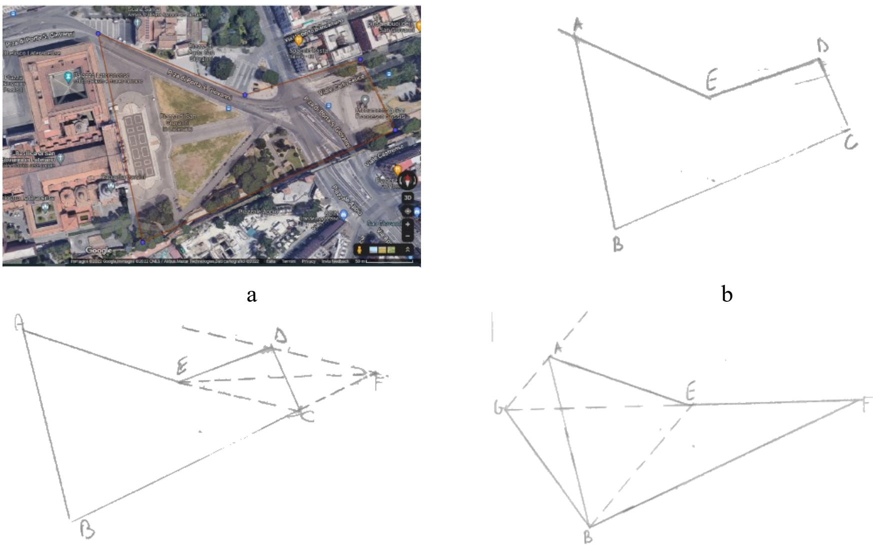
(a) (b) (c)

Figure 2. (a), (b) Students at work. (c) The equivalent triangles within the base

The following proposition I.37 of Euclid's Elements was the next goal to be achieved. This states: *triangles which are on the same base and in the same parallels are equal to one another* (read equivalent). The activity will be carried out in exactly the same way as the previous one. The difference will be that three triangles, will be added to the previously available manipulatives. It can be verified that each triangle inserted into the corresponding parallelogram will be its half (Figure 2c). Therefore, the remaining triangular empty spaces will always have the same area.

4.2 From a polygon to an equivalent triangle: a method for estimating an area that cannot be measured directly.

In the following activity, the map of Piazza San Giovanni to scale (1:50) and a tracing paper booklet were used. Initially, the last sheet of the booklet was placed on the map tracing the perimeter of the town square. We assume that the town square has been approximated by a non-regular pentagon ABCDE (Figure 3a and b) to be transformed into a quadrilateral with the same area.



a b

c d

Figure 3. (a), (b) Piazza San Giovanni drawn on the last sheet of glossy paper of the booklet; (c) the construction to transform the pentagon ABCDE into the equivalent quadrilateral ABFE; (d) the construction to transform the quadrilateral ABFE to equivalent triangle BFG.

To do this, draw the diagonal CE of the pentagon (Figure 3c); then, draw the parallel to the diagonal CE from point D. This parallel intersects the extension of BC at point F. It then joins point F with point E. It is recognised that triangle CEF has the same area as triangle CED, according to Proposition I.37. Thus, according to Common Notion 2, the initial pentagon ABCDE has the same area as the quadrilateral ABFE.

After redrawing the ABFE quadrilateral on the next sheet of tracing paper in the booklet, the same procedure must be repeated. Having drawn the diagonal BE, the parallel to the diagonal BE is drawn from A (Figure 3.d). The latter will intersect the extension of side FE at point G. Since triangle ABE and triangle GBE are equivalent according to Proposition I.37, triangle BFG is equivalent to quadrilateral ABFE. In conclusion, the final triangle BFG is equivalent to the initial pentagon ABCDE.

At this point, to determine the area of the triangle BFG the length of its side base and height were measured. The side base of the triangle measures $b = 18\,cm$ cm while the relative height measures $h = 6\,cm$. According to the scale used, 50 metres of the town square is equivalent to 2 cm of the map. Therefore, the effective dimensions correspond to $b_R = 450\,m$ and $h_R = 150\,m$. In conclusion, according to the approximation considered, the area of Piazza San Giovanni is $A = \frac{b_R \cdot h_R}{2} = 33.750\,m^2$. Taking into account this low approximation of the square's surface area and the fact that one square metre can hold a minimum of four and a maximum of six people, the students concluded that a minimum of 135,000 and a maximum of 202,500 people can be contained in the entire square.

A further step was to ask the students if and how the approximation of the calculated area could be improved. They had to note, in fact, that by choosing a pentagon to approximate the town square, many spaces in the latter were left outside the pentagon itself. They realized, therefore, that it was necessary to increase the number of sides of the polygon approximating the area of the town square in order to obtain a better estimate of its area.

5 Summary description of the activity in a 10th grade classroom.

We believe that the activities carried out up to section 4.1 are also suitable for 10th grade students. The subsequent activity for them is described in section 5.1. The objectives of this activity are: to learn Euclidean theory as a scientific theory by acquiring the general principles of the deductive axiomatic model. To this end, the students first worked on Geogebra and then proceeded to formalize the discovered propositions. As prerequisites, it was necessary to know and be able to apply the Pythagorean theorem (Proposition I.47).

5.1 Description of the main steps of the activity.

The students, following instructions on a worksheet, performed the required construction with Geogebra (Figure 4a). The first activity consisted of squaring a rectangle, i.e. constructing a square equivalent to a given rectangle. It is about proposition II.14 of the Elements.

The second activity consisted of constructing a square equivalent to a given triangle. Given triangle ABC (Fig. 4b) with base side AC and relative height BH, the segment DF = AM is constructed on the line DE (M is the midpoint of AC) and the segment FG = BH. Draw the semicircle of diameter DG, from point F draw the perpendicular FI and from vertex D draw the square of side FI, i.e. DKLJ. This square is equivalent to triangle ABC.

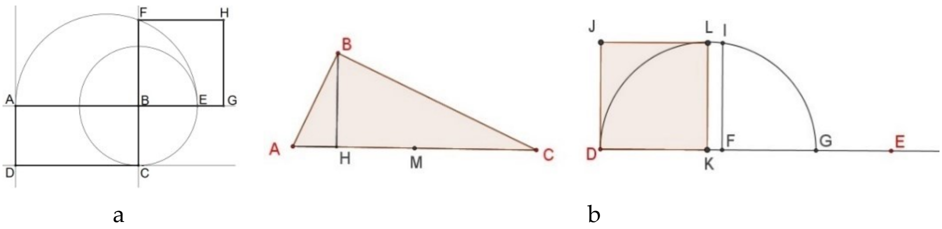


Figure 4. (a) The students' construction with Geogebra relating to proposition II.14; (b) The construction to transform triangle ABC into an equivalent square DKLJ.

The third activity consisted of transforming the irregular polygon approximating Piazza San Giovanni into an equivalent square. Students search the web for a map of the square with its scale, which is then uploaded to a Geogebra file (Figure 5). The polygon approximating the square, let us assume a pentagon, is outlined. Drawing its diagonals is divided into triangles. Knowing that each triangle is equivalent to the corresponding square, using the Pythagorean theorem the students found the area of the square FGHI as the sum of the areas of the different squares. Finally, the area of Piazza San Giovanni is determined based on the scale factor provided.

Using the software, the approximation soon becomes satisfactory. In fact, by approximating Piazza San Giovanni with a pentagon, it is established that it has an area of approximately 40,000 square metres. Again, by increasing the sides of the polygon approximating the town square, the approximation was greatly improved.

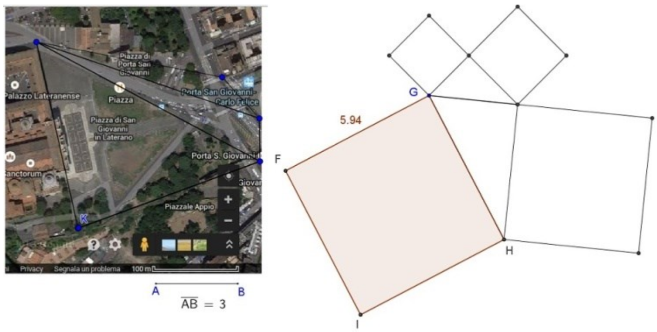


Figure 5. The algorithm to approximate the Piazza San Giovanni.

6 Conclusions and discussions

At the end of these activities, we have always noticed great satisfaction among the students. The reasons, in our opinion, are several. Firstly, students discover that the Euclidean geometry they study in school helps build their reasoning skills to the point where they are able to verify the validity of the information they learn from the media. They realise, therefore, that their critical thinking is greatly enhanced. Furthermore, they were actively involved in the activities, as protagonists. The discoveries they made were the result of their efforts. The students constructed their learning through own conjectures, subsequent verification and self-correction. Our belief in the effectiveness of manipulatives comes from studies of primates (Rizzolatti et al., 1998) which allow us to make inferences about humans by analogy. We know that manipulation of concrete objects allows for a 'pragmatic' mental representation of the object that facilitates the retrieval of associations established between movements performed on the object and related concepts (Jeannerod et al., 1995). Thus, the development of perceptual skills makes an important contribution to our cognition, because what is learnt during the use of mathematical manipulatives can generate the ability to imagine the same movements when observing drawn figures. Moreover, these skills appear to be long-lasting (Pasquazi, 2020). Therefore, knowledge and skills acquired during interaction with manipulatives can also be transferred to similar contexts. This transfer is all the more effective and persistent over time the more stable structures can be identified during the interaction with mathematical manipulatives. In conclusion, we believe that the learning of Euclidean geometry, which is fundamental for the development of rational thinking and is considered in some respects a scientific theory based on the concreteness of design, is certainly facilitated by the use of manipulatives. Therefore, we consider it necessary to further investigate the effectiveness of these methodologies.

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