HISTORICAL PARALLELISM AND THE DIDACTICAL TRANSPOSITION OF HISTORICAL KNOWLEDGE

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ABSTRACT

The idea of *parallelism* between the way mathematics has evolved through the mathematicians' creative work and the way students learn mathematics (called the "*parallelism*" issue) is well-known and its naïve formulations have been rightly criticized as untenable oversimplifications. By means of a case study based on an example from Euler's *Algebra*, we provide more nuanced views, thus pointing to the complementary character of similarities and dissimilarities between past and present, its significance for appreciating the complementarity of the historians and mathematics educators' aims and commitments, and the need for and basic features of an effective didactical transposition of historical knowledge.

1 *Introduction: The parallelism issue*

The idea that teaching and learning mathematics should follow or/and correspond to the historical development, is a whole spectrum of views and methodological prompts or recipes appearing under various names since the late 19th century (Furinghetti & Radford, 2008). For brevity, we call it the "parallelism issue". In the last decades its naïve formulations have been rightly criticized as untenable oversimplifications, but appreciation of the inherent subtleties is implicit in early publications, albeit not discussed in detail (Freudenthal, 1973, pp. 101, 103; Memorandum, 1962, pp. 190–191; Vergnaud, 1990, p. 16). Among several relevant interesting ideas, Sierpinska (1990) pointed to a negative and a positive aspect of parallelism, corresponding respectively, to overcoming epistemological obstacles and understanding, as two complementary perspectives in Bohr's sense introduced via quantum physics as an epistemological principle to understand reality (Bohr, 1934, p. 10). This notion of complementarity can be useful for understanding deeper what so far has been considered as incompatibilities or clash of commitments between the history of mathematics (HM) and mathematics education (ME); i.e., manifestations of complementary perspectives not to be used simultaneously, but equally legitimate and necessary in order to teach and learn mathematics both as a structured corpus of human intellectual products and as a cultural endeavor leading to them (Thomaidis & Tzanakis, 2022, section 2). Here, this notion is considered only in relation to the similarities and dissimilarities between past and present.

2 *Complementarity between similarities and dissimilarities of the past and the present, and the didactical transposition of historical knowledge*

In the 1980s-1990s the HPM domain expanded, leading to an influential ICMI study (Fauvel & van Maanen, 2000). Theoretical issues became essential and the need to theorize from actual implementations went hand in hand with criticism as to the role of the HM in ME, including the parallelism issue. A characteristic example of such a criticism is Fried's (2007) pointing to the subtleties inherent in any use of the HM in ME that - for practical reasons - adopts an anachronistic perspective of the historical development in an educational context. Fried points to a "clash of commitments" between HM and ME, by stressing the distortions of historical knowledge caused by considering similarities between past and present without due attention to their dissimilarities.

This debate stresses the subtleties inherent in the attempt to serve "properly" the aims and commitments of both ME and the HM, hence the subtle balance that has to be achieved so that none of them gets distorted or/and ineffective. According to Fried the historical and the mathematical ways of knowing should be conceived as complementary epistemologies (Fried, 2007, p.204).

In a series of papers advocating a multiple-perspective approach to history, Kjeldsen went further, calling for a more nuanced view of history in ME, that (just as for school mathematics) requires a didactical transposition of scholarly historical knowledge in order to capture the variety of ways in which HM can be beneficial for learning of and about mathematics (Kjeldsen, 2012, pp. 333-334).

In the light of this critical discourse, further discussion of the parallelism issue raises several more subtle points: Are there similarities between past mathematicians' creative work and students' ways of learning mathematics? If so, how could they be beneficial both for ME and for understanding further the historical development? What are the limitations imposed by the differences between these two worlds? These questions complement the preceding critical account: History gets distorted by considering similarities between past and present without attention to their dissimilarities, and ME gets mis-focused by considering dissimilarities between past and present without attention to their similarities. Here, similarities refer to drawing possible parallels between, either obstacles, misconceptions, errors, difficulties, premature formulations met by students and also by past mathematicians (a kind of *negative parallelism*), or/and innovative, idiosyncratic ways to cope with questions, problems, etc., that cannot be adequately treated with the (available at the time) knowledge of the mathematicians' community, or in the students' classroom (a kind of *positive parallelism*; Thomaidis & Tzanakis, 2007, 2022). Dissimilarities concern taking account of the differences between the social, cognitive, cultural, and scientific conditions of the past mathematicians' world and that of students today. Thus, in the context of ME, *similarities* and *dissimilarities* between past and present are *complementary*. This is what could be understood as an appropriate *didactical transposition* of historical knowledge inspired and guided by similarities between past mathematicians' creative work and students' ways of learning, however, on the condition of both being properly contextualized, i.e., dissimilarities between past and present are carefully accounted for.

In addressing these issues systematically, historical analysis has to be compared with empirical data on how students conceive and use specific pieces of mathematical knowledge. Below we report on a case study, illustrating these ideas about complementarity and didactical transposition, and serving as an example of how historical research motivated by didactical problems can contribute to still open or debated historical issues.

3 *A case study: On Euler's "mistake" and its didactical significance*

Euler (1770), in his essentially didactic treatise on algebra, develops about one third of volume 1 without introducing the equality symbol, though it includes all basic operations, exponentiation and roots' extraction. In order to apply algebraic operations to the transformation of proportionality relations and the solution of equations, = is introduced in chapter 20 (ibid, Vol. 1, §206). To our surprise, this remarkable and peculiar – from a modern perspective – "delayed" introduction of = seems to have passed unnoticed by historians of mathematics.

Without =, standard algebraic rules, including the square roots of negative numbers, are *verbally* formulated in this part of the book; for instance, for the product of two square roots of positive numbers, or the square roots of negative numbers Euler writes:

> …if it is required to multiply $\sqrt{a}$ by $\sqrt{b}$, the product is $\sqrt{ab}$…the square root of the product *ab*…is found if the square root of *a*…is multiplied by the square root of *b*… (Euler, 1770, Vol. 1, §132 p. 56)

> Since –*a* is as much as +*a* multiplied by –1 and the square root of a product is found by multiplying together the roots of its factors, so the root of *a* multiplied by –1, that is √–*a* is as much as √*a* multiplied by √–1. But $\sqrt{a}$ is a possible number, therefore the impossibility appearing therein, always can be led to √–1. Consequently, because of this, √–4 is as much as √4 multiplied by √–1: But √4 is 2, hence √–4 is as much as 2√–1 and √–9 [is] as much as √9√–1, that is 3√–1... (Euler, 1770, Vol. 1, §147 p. 61)

Euler uses the expression "is as much as", ("ist so viel als"), not "is equal to" ("ist gleich"), indicating an operational meaning of the two factors giving the product as a result via multiplication. Moreover, the above two citations lead to *opposite* results for the product of the square roots of negatives (Thomaidis & Tzanakis, 2022, section 4.2). This is the reason for attributing to Euler grave elementary mistakes (Cajori, 1993, p. 607; Grattan-Guinness, 1997, §6.15, p. 334; Katz, 2009, §19.1.3, p. 670; Kline, 1980, p. 121) [43]. But Euler's formulation is not "relational", but "procedural", conveyed by standard expressions not containing the word "equal" or the equality symbol. Thus, if one wants to symbolize Euler's verbal expressions "is as much as", "is found", "can be led to", in principle a *different* symbol should be used, denoting the reduction of the left-hand side to the right-hand side (we call this the *reduction-conception* of equality); e.g., for the two citations above one can write respectively $\sqrt{(-2)}\times\sqrt{(-3)}\rightarrow\sqrt{6}$, $\sqrt{(-2)}\times\sqrt{(-3)}\rightarrow\sqrt{2}\times\sqrt{(-1)}\times\sqrt{3}\times\sqrt{(-1)} \rightarrow -\sqrt{6}$.

After introducing =, Euler deals with transforming equations in equivalent forms, and not just with algebraic or arithmetic operations that produce a result. Therefore, = concerns operations between equations: Equalities now become symbolic objects, hence = is necessary for their representation, with equality acquiring here its standard meaning of an *equivalence relation*.

Thus, there are two distinct equality conceptions: a *procedural* "*reduction*

[43] Martinez (2007) is the first to question this interpretation, however, following a different rationale.

conception" and a *relational* "*equivalence conception*", clearly separated by the introduction of = in Euler's book. This is crucial because it questions the established historical interpretation about Euler's elementary mistakes.

In fact, there are clear indications that Euler was not constrained by any single-valuedness of the square root either before, or after the introduction of the square root symbol (Euler 1770, Vol.1 §§122, 150). Therefore, since in evaluating the square root of the product of two numbers, Euler uses only the reduction-conception of equality, $\sqrt{(-2)} \times \sqrt{(-3)} \rightarrow \sqrt{6}$ and $\sqrt{(-2)} \times \sqrt{(-3)} \rightarrow -\sqrt{6}$ are equally valid *reductions* (not *equivalences*)! In Euler's *Algebra*, the two equality conceptions, though coexistent, are carefully separated with distinct operative roles, and used in a contradiction-free manner before and after the introduction of =. However, from didactical research it is well-known that though both conceptions are used by school students, they are often muddled. For instance, Kieran notes that

In elementary school the equal sign is used more to announce a result than to express a symmetric and transitive relation. In attempting to solve the problem

> *Daniel went to visit his grandmother, who gave him $1.50. Then he bought a book costing $3.20. If he has $2.30 left, how much money did he have before visiting his grandmother?*
>
> 6th graders will often write 2.30 + 3.20 = 5.50 – 1.50 = 4.00
>
> …the equal sign … is read as "it gives", that is, as a left-to-right directional signal. (Kieran, 1990, p.98)

Freudenthal notes that = is "…primordially read as a task, or a question… as unilaterally directed towards a 'reduction'." (Freudenthal, 1983, pp. 477, 481).

Although the *reduction-conception* of equality is evident in the above chain of "equalities", the pupils understood the problem and how to solve it. Their solution can be formulated without =: "adding 2.30 to 3.20 gives 5.50", then "subtracting 1.50 from 5.50 gives 4.00".

Bearing in mind the systematic use of this conception in Euler's *Algebra*, this is an example of "*positive parallelism*". That is, the reduction-conception of equality is used successfully – albeit in a symbolically idiosyncratic way – to tackle a problem by students who developed this conception apparently without having been taught it. But the inadequacy of this conception is also a case of "*negative parallelism*":

> [Many college students] …continue to view the equal sign as a separator symbol rather than as a sign for equivalence [as] seen in their shortcutting of steps in equation solving, and in their staggering of "adding the same thing to both sides": Solve for x: $2x + 3 = 5 + x$, $2x + 3 - 3 = 5 + x$, $2x = 5 + x - x - 3$, $2x - x = 5 - 3$, $x = 2$. (Kieran, 1990, pp. 100–101)

Students solving $2x + 3 = 5 + x$ in this way use the reduction-conception of equality in the successive transformation of each individual member, thus violating the very notion of an equation and the logical consistency of the solution's method. Hence, they meet insurmountable difficulties to proceed effectively in situations involving more elaborate algebraic manipulations.

Being aware of the primitive character of the *reduction-conception* of equality and its inadequacy for issues more elaborate than simple algebraic calculations, Euler went beyond it after introducing =, that he henceforth used to denote the deeper and effective *equivalence relation conception* of equality.

4 *Concluding remarks and comments on the didactical transposition of historical knowledge*

Since students hold and mix up the two equality conceptions, what could be done? In teaching mathematics today, it is impossible to ignore the established meaning of equality and to avoid setting as a principal aim its understanding by the students as an equivalence relation. But it is here that historical and educational research in cooperation could help from several perspectives that outline main points of the didactical transposition of historical knowledge in this case, namely: to appreciate the coexistence of the *reduction* and *equivalence conceptions* of equality; to point to the pitfalls and misinterpretations resulting when the two notions are muddled; to help teachers get aware of their existence in students' mathematical understanding. In view of this didactical problem, there are many alternatives for realizing this didactical transposition, adapted to the target population's characteristics and limitations (Thomaidis & Tzanakis, 2022, section 5).

With the example from Euler's *Algebra*, and the critical discussion that preceded, we emphasized some issues we consider important for understanding better the connections between HM and ME: the nuances related to the "parallelism issue" (in particular, the "positive-negative" aspects and their limitations); the complementary nature of similarities and dissimilarities between past and present and their educational significance; the fact that this complementarity can be a key idea for appreciating that the historians and mathematics educators' aims and commitments are complementary rather than in conflict; and the need of a constructive collaboration of these communities for the appropriate didactical transposition of historical knowledge.

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