HISTORICAL JOURNEY OF THE CONCEPT OF PERIODICITY AND ITS DIDACTIC IMPLICATIONS IN TRIGONOMETRY

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ABSTRACT

The teaching innovation proposal about the study of one of the most important but sometimes undervalued topics in High School Mathematics consists of a multidisciplinary approach, specifically based on the historical evolution of the concept of periodicity and in multiple applications. This article summarizes the results of a study that evaluated the effectiveness of the historical and epistemological approach in teaching the concept of periodicity in Mathematics, which students often perceive as a hard topic. The study was carried out with 62 high school students aged 16-18 years in a public scientific high school in Poggiomarino, Naples (Italy). After observing the results collected from different running kinds of tests on the students, we concluded that the approach based on these advanced perspectives and methodologies was significantly helpful in making trigonometry easier and digging deeper into the concept of periodicity.

1 *Introduction*

The concept of periodicity is rather familiar to young people from an early age. Periodicity is profoundly inherent in the cycles of many phenomena of nature, inside our body, such as heartbeat and breathing, or outside us, such as seasons and day cycles. But repetition is also typical of patterns and scheduling conceived by the human mind and widely spread everywhere, not only in the works of all arts of any historical age but also in many organizing solutions of everyday life. There are many examples of the human trend to conceive and think in terms of periodicity and to exploit periodic schemes.

Calendars have been the common tool to rationalize the flowing of time for ancient and recent civilizations worldwide, even though different kinds of arrangements have been adopted (Zerubavel, 1985). Observing so many periodicity examples in nature and arts is also very interesting. “Many different artworks mimic the statistical properties of natural fractal patterns – in particular, self-similarity at different scales” (Balmages A. et al., 2021). Repetition of colors, shapes, and forms is the fundamental principle of painters like William Morris, Piet Mondrian, and Andy Warhol. In (Neeman S. & Maharshak, A., 2006), there is a specific reference to the beneficial psychological effect of repetition and variation in arts: “When we are introduced with an irregularity, a piece of information that is out of place, we try to make sense of it, and when we succeed we have a feeling of tension reduction, which gives us pleasure”. According to this assumption, our perception of repeated similar shapes in space or recurrent events in time transfers to the brain signals, processed to seek organized and regular patterns or a generalization: as in the Gestalt theory (Wertheimer & Riezler, 1944), our mind finds a sort of fulfillment in finding regularities. It is not by chance that Islamic patterns, inspired by vegetal patterns, in Granada’s Alhambra elicit a sense of beauty, perfection, and wonder and have inspired the artistic compositions of artists like Escher. Nevertheless, regarding the music, “Large- and small-scale patterns are the essential building blocks for a musical composition, be it improvised or slowly conceived and notated […]. Perhaps the sense of order suggested by patterns in a given composition brings a certain satisfaction to the listener, while a temporary disruption of those patterns promotes a sense of drama” (Wilson, D., 1989). As in many historiographers' conjectures, periodicity is also recurrent in historical interpretations of the events. From prehistoric times onwards, many ancient cultures had some idea that the natural and human worlds moved in cycles of origin → growth → prosperity → decline → fall, with various repeating patterns. In Greece, Thucydides, who lived in the 2nd half of the V century b.C., most typically embodies the concept of cyclic human history. After this vision, he believes that the collection of past speeches and actions will have a profitable use in the future. He says: “If my work is judged useful by any who shall wish to have a clear view both of the events which have happened and of those which will someday, according to the human condition, happen again in such and such-like ways, it will suffice for me [...] But those who want to look into the truth of what was done in the past-which, given the human condition, will recur in the future, either in the same fashion or nearly, so those readers will find this History valuable enough, as this was composed to be a lasting possession and not to be heard for a prize at the moment of a contest" (Thucydides). Also, even though it is not a science, astrology is based on eternal repetition. "Socrates recommended above all to learn astrology, in order to know the time of the night, of the month or the year, in case of travel, navigation or service, or for everything that is done at night, in the month or the year; it is about having benchmarks to distinguish the moments of these different times, but it is easy to learn them from night hunters, sailors and many other people who have an interest in knowing them". (Thucydides)[42] Due to the previous considerations, periodicity can be considered a common factor in many disciplines. Thus, it is suitable to be the topic for a transversal learning unit.

Therefore, it makes sense to propose a deeper analysis and reflection on periodicity to students of a scientific high school. To put it another way, practices related to the construction of the concept of periodicity and the introduction of analytic instruments to deal with it along the history of mathematics and human thought can make students aware of a model and a category of the human spirit. This is why, we proposed to a group of students aged 16 to 18, supervised by teachers and experts while studying trigonometric functions, research the origin and development of the theory of periodic functions in the past centuries.

2 *The mathematical prototypes of periodic functions are trigonometric functions.*

The origin of trigonometry goes back to astronomy studies at the geometric school of Alexandria (Barbin et al., 2015). These studies were motivated by the requirements to build a *quantitative* astronomy that could be used to predict the motions and positions of celestial bodies and to aid in the determination of time, the compilation of calendars, navigation, and geography (Heath, 1921; Rogers, 2010). This explains why spherical trigonometry historically precedes plane trigonometry against the natural scale of difficulties. The founder of trigonometry was probably Hipparchus of Nicaea (II century BC). Fundamental contributions to spherical trigonometry are also due to Theodosius of Tripoli (1st century BC) and Menelaus of Alexandria (1st-2nd century AD). But most of the information on the Alexandrian trigonometric methods

[42] Freely translated by the authors.

comes from the *Almagest* by Ptolemy (II century AD), who laid the foundations of the astronomical theory, which dominated the scientific scene until the seventeenth century. The key difference between Greek and modern trigonometry is that Alexandrian trigonometry used the chords of a circle instead of sines.

From the 9th century, the natural successors of the Greek geometers were the Arab mathematicians. They quickly assimilated most of the studies known at that time, unifying them in an original method, which was transmitted a few centuries later to European scholars. The first innovation with respect to Alexandrian trigonometry was the use of the sine instead of the chord and a systematic study of circular functions, so defined because radian measures of angles are determined by the lengths of arcs of circles. Trigonometric functions, a special type of circular functions, are defined using the unit circle. For the Arabs, spherical trigonometry was particularly important also for religious reasons: the direction of Mecca, marked on all public sundials, was determined by solving the spherical triangle, which has the position of the beholder, Mecca, and the north pole as vertices. As we said, trigonometry reached the West mainly by translating Arabic sources into Latin. Medieval development was slow, and European scholars made no interesting contributions before the fifteenth century. A further boost to the development of trigonometry comes from topography, which, unlike astronomy, is based on rectilinear trigonometry. The first formalization of the plane and spherical trigonometry is contained in *De triangulis omnimodis* by Regiomantanus, written around 1464 but printed only in 1533. Numerous treatrises followed it; among these, we mention that of Nicolò Copernicus contained in his famous work *De revolutionibus orbium caelestium* published by G. J. Rhaeticus (1542).

The introduction of logarithms by Napier gave a strong boost to the development of trigonometric techniques: trigonometric calculations could be greatly simplified by combining the tables of circular and logarithmic functions. Rhaeticus himself prepared a series of tables of the six circular functions. Until the middle of the seventeenth century, sines, cosines, tangents, etc., were numbers given by tables, which provided for each value of the angle the value of the sine, or later its logarithm.

Around 1650, a different point of view began to spread: the functional one, or rather, since the concept of function was not yet well defined, the geometric one. Thus, the curves of sines, cosines, tangents, and others were studied.

With the creation of infinitesimal calculus and the formulation of the concept of function, the mathematics of trigonometric functions was also systematized. Studying the irregularities of the motions of Jupiter and Saturn, Euler gave a systematic and complete treatment of the trigonometric functions: their periodicity is clearly expressed in his *Introductio in analysin infinitorum* (1748), in which the measure of angles in radians is also introduced. To keep up with achievements in navigation, astronomy, and geography, greater precision was needed in the interpolated values of trigonometric, logarithmic, and nautical tables: this problem led to the series expansion of functions (Kline, 1972).

3 Periodic functions by a didactic point of view

The first and most suitable question about a periodic function is what is actually "periodic"? The most common answer could be that it consists of or contains a series of repeated stages, processes, or digits at regular intervals, as the ones depicted in figure 1a) below. In order to avoid confusion in the students' learning cognitive operation, it is important to specify how it is possible to find the "regular interval", i.e., the "period" of the function: it is the least repetition pattern's length that can be detected as can be seen in figure 1b). So the prevailing challenges and criticality in learning periodicity, on the one hand, can be to detect confidently which is the period of the function and how long it takes for the pattern to be repeated (Inan, 2013; Kamber & Takaci, 2018). But on the other hand, periodicity hides the extraordinary potential of future forecasting: once detected the period, it is possible to predict the trend of the function at any upcoming point.

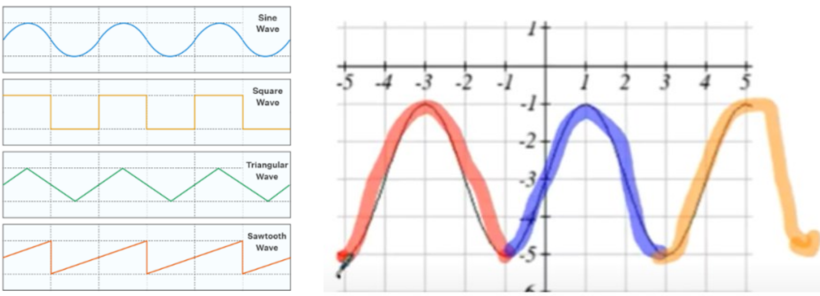

Figure 1. a) Examples of periodic signals b) The periods of the function are highlighted using different colors

To better delimit the set of periodic functions, two limit cases are shown: the constant function (periodic of period t, with t any real number) and a non-periodic function, such, for example, an exponential function of the type e^x, which turns out to be periodic of the infinite period. This consideration recalls students' attention to the real significance of periodicity, besides eliciting their curiosity about other special cases of periodic functions, for instance, quasi-periodic functions, depicted in figure 2, and almost periodic functions, which understanding needs a deeper knowledge, most likely out of high school student's outlook. For the sake of congruence with the concepts and definitions known to the students to whom the experimentation was directed, the concept of quasi-periodic functions was taught in a intuitive and graphical way.

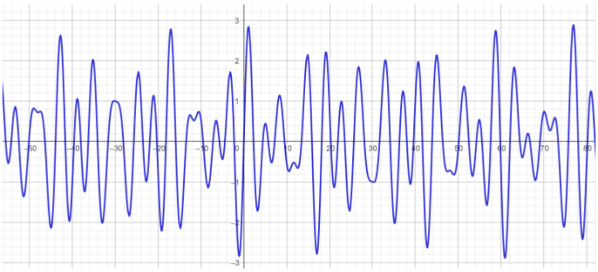

Figure 2. Graphic of the function $\boldsymbol{f(x) = \sin x + \sin(\sqrt{2}x) + \sin(\sqrt{3}x)}$

For personal experiences of the authors and as reported in Maor (2013), a concept that students in the last years of high school or at university usually accept with high interest is that almost any periodic function can be represented by a sine and cosine convergent series, such as:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right)$$

Which is known as the Fourier series. It means that as more and more components of the series or "harmonics" from the series are summed, each successive partial Fourier series sum will better approximate the function and will equal the function with a potentially infinite number of harmonics.

4 Experiment with students

Our experiment was carried out with a total of 62 students of three similar class groups of 21, 21, and 20 members in their fourth year of a Scientific High School in a small town in Poggiomarino, near Naples (Italy). The first group of students was taught with a traditional approach, frontal lessons, guided exercises in the classroom, and individual homework followed by a collective correction. The second group of students was taught using a flipped classroom methodology, with large historical enhancements and many lab applications. The third group received lectures based on an experimental historical background, with frontal lessons and study material given by the teacher. The experimentation started in October 2020 and has been continued since April 2021. In order to establish their starting conditions and to assess their prerequisites, an entrance test was given to all the students. The groups were formed based on the test results, with maximum homogeneity and similarity criteria.

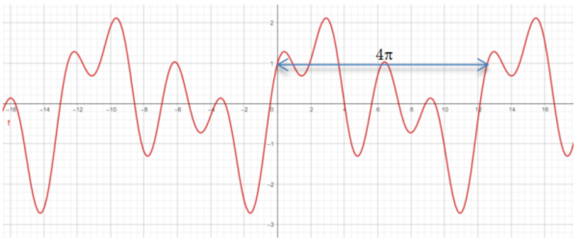

Figure 3. Example of a periodic function

For the second and third groups, the first question asked to students was about their idea of periodicity: what does the adjective 'periodic' mean? They could have an introductory immersion in the topic with a brainstorming session.
The first group, according to a traditional teaching, followed a different approach.
Students of the second and third groups were then urged to draw what they figured out to be a periodic function by hand or use a dynamic geometry tool, such as *Geogebra*. Then teachers showed them many kinds of function graphs (see figure 3 above for an example) and asked them to classify the functions, possibly detect the period, and link the periodicity to their previous experiences. Then, with the established methodologies, each group followed its specific path, as previously specified.

4.1 Students' misconceptions

As specified above, the preliminary activities highlighted the students' ideas about periodicity. In particular, it seems interesting to focus on their naive, most common conceptual mistakes about what periodic means.

- Misconception 1: Periodic functions are exclusively trigonometric functions.
- Misconception 2: If the expression of a function is a combination of sine and cosine functions, the function is necessarily periodic.
- Misconception 3: association with a periodic table. The periodic table is called periodic because of the regular variation of one or more parameters, such as atomic radius and electronegativity.

4.2 Activities and Results

Among others, the most important activities performed with students in our experiment on an action-research modality were the application of the periodicity concept to Physics, specifically to electromagnetic radiation of oscillation in space and in time and to wave packet; the creation with a *3D* printer and the finding of 3-dimensional periodic artifact, as in figure 4 a) below, and informatics labs, where CORDIC algorithm to compute in an elementary way the trigonometric functions were experimented by students and some machine learning algorithms to detect the periodicity of a function was tested as shown in figure 4 b).

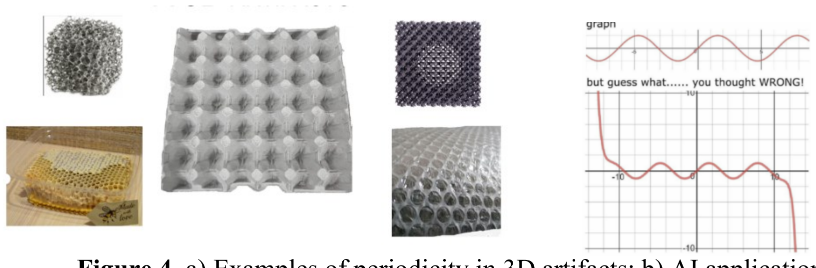

Figure 4. a) Examples of periodicity in 3D artifacts; b) AI application

The results, collected using written and oral tests, simulations, and a challenge between groups, are resumed in the table below, wherein the third column EMR stands for Electromagnetic Radiation, whose representation is a set of periodic functions.

	Periodic functions	Goniometric equations	EMR
Group 1	68 % sufficient or more	51 % sufficient or more	62 % sufficient or more
Group 2	85% sufficient or more	54 % sufficient or more	72% sufficient or more
Group 3	76% suffucient or more	53% suffucient or more	70% suffucient or more

Table 1. Outline of the results of the experiment

Observing this table, there is evidence that the most part of students of any of the groups showed good results by the testing phase after the activities about periodic functions and the applications on electromagnetic radiation. Less positive was the impact of experimental teaching had on the solving strategies for goniometric equations.

5 Conclusions and future developments

In our study, we tested the effectiveness of the historical and epistemological approach in teaching the concept of periodicity in Mathematics and trigonometry, which students often perceive as a hard topic. We organized the activities to explore the evolution of the concept of periodicity, starting from the history of trigonometry. The study was carried out on 62 students of a scientific high and was quantitative. From the observations of the results of the tests, we concluded that the approach based on history and epistemology, with the observation of cognitive values expressed by mathematics, went out to help make learning trigonometry easier. As shown earlier, we collected proofs that after the experiment, students' awareness about periodic functions and periodicity in general improved by a significant amount.

In the future, we plan to widen the experiment to a larger number of students, to focus our attention on the periodic but not trigonometric functions, and to better characterize students' difficulty with periodicity in Mathematics.

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