MATHEMATICAL OBJECTS
WITHIN A TRANSITORY EPISTEMOLOGY

Miglena ASENOVA
Free University of Bolzano, Italy
miglena.asenova@unibz.it

ABSTRACT

In this paper we present a category-theoretical characterization of mathematical objects as synthetic objects within a transitory epistemology. This allows us to take into account pragmatic and dynamic issues as in the definitions of mathematical object in mathematics education research and in the historical evolution of mathematical practice. We discuss the relations of such objects to mathematical objects considered from an objective, set-theoretical perspective, as well as the implications of the category-theoretical characterization for teaching and learning mathematics.

1 Introduction

In Mathematics Education (ME) research there are a lot of definitions of mathematical object (MO) (e.g. Chevallard, 1991; D'Amore, 2001; Duval, 2009; Font et al., 2013; Lavie et al., 2019; Radford, 2008). These definitions highlight pragmatic aspects (e.g., reference to human activity), epistemic constraints (e.g., reference to individuals that display routines), recourse to semiotic resources (e.g., objects seen as invariants behind semiotic transformations), and dynamicity (e.g., evolution of MOs over time). These definitions are far away from the ones used in mathematics that shape the definitions used in textbooks and taught in math classroom (Asenova, 2021).

How these two different kinds of definitions of MOs can be linked to each other is an important ontological issue in ME research (Asenova, submitted; Asenova et al., submitted), but it has also interesting implications for the teaching-learning process in the classroom. We suggest that starting from a suitable epistemology of mathematical practice that fits the historical emergence of new MOs, it is possible to frame MOs theoretically in a way that is close to the 'pragmatical' needs of the classroom but that also consider the relation to their 'objective' mathematical definition.

2 Theoretical framework

Charaterizing the main issues of the philosophy of mathematical practice, Giardino (2017) sums up four aspects on which this current of thought focuses: (1) the dynamicity of MOs in mathematical practice; (2) the importance of semiotics; (3) the emphasis on epistemological aspects that go beyond construction of formal systems; (4) the emphasis on pragmatic issues, in reference to the use of objects and tools in context. The phylosophy of mathematical practice expresses the more recent developments in the history of phylosophy of mathematics and could be useful for better frame the way MOs should be introduced and could be conceptualized in mathematics classroom.

Indeed, the four aspects mentioned by Giardino are very close to the concerns that characterize the definitions of MOs in MER presented in the introduction. Indeed, teaching-learning phenomena: (1) Are strictly related to cognitive processes and require a dynamic approach to mathematics; (2) Are rooted in the web of semiotic transformations that underly mathematical thinking; (3) Are necessarily involved in epistemic constrains that deal more with knowing why and how rather than with systhematysing; (4) Are related to the use one is able to make of MOs, rather than to their abstract meaning the discipline as a whole. In this sense, the phylosophy of mathematical practice allows to connect the viewpoint on mathematics as a discipline to the viewpoint on ME as prexeology in the classroom.

In order to operationalise the very general ideas rooted in the phylosophy of mathematocal practice, we need to choose a specific philosophy of mathematical practice that is able to provide a suitable epistemology for characterizing MOs. In this sense, in the following we refer to the Synthetic Philosophy of Contemporary Mathematics (SPhoCM) (Zalamea, 2012).

3 MOs within a transitory epistemology

From the viewpoint of the SPhoCM, mathematical practice claims for a transitory epistemology that considers knowledge pragmatically, in evolution over time, and MOs as synthetic „(quasi-)objects“ (Zalamea, 2012, p. 323), that means dynamic objects defined by their relations with other objects, rather then as analytic objects, considered in themselves. The idea of synthetic vs. analytic object is expressed in Figure 1.

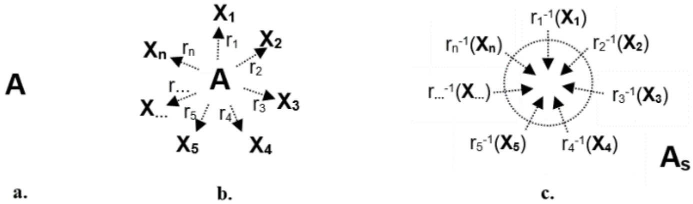

Figure 1a. The object *A* from analytic viewpoint (as object in itself); **1b**. The context in which *A* is immersed (composed by the objects $X_1, \dots, X_n$); **1c**. *A* as synthetic object (A_s).

In Figure 1a the MO *A* is represented from analytic viewpoint: It is an object in itself, without relations to other objects. In Figure 1b, the object *A* is condidered as immersed in a context, where $X_1, \dots, X_n$ represent the objects related to it. In this sense, *A* is not considered ‚in itself', but by its relations to the objects belonging to its context ($r_1, \dots, r_n$). By considering the counter-images ($r_i^{-1}(X_i)$) and the relations between them (dotted circle), the object *A* 'disappears' and is seen from synthetic viewpoint (A_s), as mirrored by its context (Figure 1c).

According to Zalamea (2012), the transitory epistemology of contemporary mathematics can be framed by a conceptual use of mathematical tools. This use does not refer to the formal aspects of the mathematical tools but to their characteristics as means of universal thought: For example, a category is interpreted as a suitabel context, focusing on relations between objects that can be composed in an associative manner. Asenova (2021) transposes this way to use category-theoretical tools in ME research and characterizes it as *metaphorical*, according to the idea of structural metaphor, based on an analogy (Pimm, 1981). In this sense, an anlogy between the category-theoretical tools and the concepts in the discursive language of ME research is established (Asenova, 2021): A category is a coceptual context, in the sense of a web of objects or concepts; an arrow is a relation; a functor is a way of meaning-making; a natural transfomation between functors is a translation between two diferent ways of meaning-giving to the same context.

In this way, the category-theoretical model used to frame MOs from synthetic viewpoint can be immersed in the category of the ME research-practice by an immersion-functor. This functor transferes the relations of the category-theoretical model into the discursive language of ME research and justifies the metaphorical use of category-theoretcal terms (e.g., we cen stalk about a functor that ‚gives meaning to *A*').

According to this metaphorical use of the category-theoretical tools, to know an object belonging to a context, means to give meaning to its relations to all the other objects belonging to the context. This brings us back to the idea of MO from synthetic viewpoint. To better explain this point, I use the idea of representable functor. Let us consider the context of the object *A* as a category (the category ***C***), in the way it is represented in Figure 1b. Let us immerse ***C*** in the category ***Set***[41] in the following way: each object of ***C*** is represented by the set of relations (arrows) that have *A* as domain-object and the object itself as codomain-object; each relation (arrow) in ***C*** is represented by the functions that map between those sets. The functor that creates this translation of ***C*** 'from the viewpoint of *A*' is a functor representable by the object *A* and it creates a copy of ***C*** in ***Set***. A special kind of representable functor, the hom-fuctor, is a contravariant functor that inverts the directions of the relations (arrows) with respect to the ones present in ***C***. To know the object *A* means to give meaning to the representations of ***C*** from the different viewpoints of the objects belonging to the context expressed by ***C*** and to the ways they can be translated to each other. Since a functor is a way of meaning-making and different functors express different ways to give meaning to a context, a functor also expresses the way meaning is given to a single object belonging to that context. Coming back to Figure 1, we can state that if we consider the MO *A* as an object in itself, we can return to its characterization as analytical object (1a). From the other hand, if we consider the feedbacks turned back by the context, the object can be interpreted as evolving in a temporal sequence, according to the indices of the relations r_i, and we are able to recover also the dynamicity of evolution over time. As in this paper the focus is on ME as praxeology of the classroom and not on the epistemology of ME as research domain, we focus only on the idea of MO from synthetic viewpoint because it is the most fruitful for the purpose to explain how the transitory epistemology of mathematical practice can support teaching and learning

[41] ***Set*** is the category whose objects are sets and whose arrows are functions.

mathematics in the classroom. For the complete definition of MO specific to MER, the reader can refer to Asenova et al. (submitted).

4 Discussion

The characterization of a MO from synthetic viewpoint, based on the transitory epistemology of the SPhoCM, mirrors the way MOs emerge in the history of mathematics: They first arise as a tool to solve problems within mathematical practice in a context composed by other objects, and than gradually acquire meaning from the “feedbacks” send back from the context they arise from: “The new discourse started emerging when people realized that a number of routines displayed the same pattern” (Lavie et al., 2019, p. 164). The transitory epistemology of the SPhoCM fits well this idea of emergence of new MOs from historical and epistemic perspective and the idea of MO from synthetic viewpoint represents a model of this way to conceive MOs. Furthermore, the knowledge of a MO from synthetic viewpoint is a potentially complete knowledge (Asenova, 2021) and this supports the idea that a new MO should be introduced in the classroom as a solution of suitable mathematical problems that defines it in a synthetic way by mirroring its characteristics by all the other objects involved in the problem and the relations between them. In this way, the objects emerge from the practice, from the routines and by the semiotic transformations carried out by the students. This, conversely, fits the way new MOs emerge from the mathematical practice seen as a historical process. The usual (Bourbaki-style) definition acquires meaning only as the end-point of such a gradual emergence from social problem-solving-practices that require to carry out semiotic transformations.

5 Conclusions

The characterization of MOs from synthetic viewpoint seems to fit well the features of MOs emerging from mathematical practice from a historical perspective but it also fits the way which is usually considered ‘sustainable’ in ME while introducing new mathematical objects in the classroom. It is particularly interesting to see that what is usually stressed by scholars in MER and by philosophers of the mathematical practice can be suitably modelled by mathematical tools, provided we consider them as universal means of thought, rather than as formal objects. This seems to be a promising way to find technical tools to model an analyse aspects related to ME research (Asenova et al., submitted), but it also can be seen as a backing of what is well known from research practice: From an epistemic viewpoint, meaning making is something very pragmatic and is, at least at the beginning, far away from the idea of set of elements that satisfies a certain property.

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