WRITING A BOOK FOR TEACHERS:
an introduction to history of mathematics and its connection to education

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ABSTRACT

I am in the process of writing a book that will to be an introduction to history of mathematics for teachers in primary and lower secondary school. In this workshop, I outlined a series of objectives in writing the book, and then invited participants to discuss (in groups) concrete examples of sections of the book.

1 Introduction to the book project

On a rainy day in Salzburg, after an inspiring ESU6 in Vienna, I opened my laptop and started writing an outline of a book on history of mathematics (HM) for teachers. Nine years later, in 2019, I visited Crete and Costas Tzanakis, where I talked about my ideas and got valuable feedback and additional references. In 2022, as another attempt to jumpstart the project, I presented ideas from the book at the ESU9 conference in Salerno, Italy. At the current speed, the book will be key task for me when I retire in some 20 years' time.

The book is supposed to be a textbook for teachers in primary and secondary school in Norway and be in Norwegian. I have several objectives in writing the book:

- I want to give teachers rich examples of how HM can be included in mathematics teaching (see for instance Smestad (2011) for a number of such "hows").
- I want examples that are a fertile ground for discussing important reasons for including HM in mathematics teaching (which many publications in the HPM group, for instance Clark et al. (2019); Grattan-Guinness (2004); Jankvist (2009), have discussed).
- I want to show how the development of mathematics as a science, the use of mathematics in different cultures and ages, and the develop-

ment of mathematics as a school subject (or more general, as an object of learning), are all fertile areas for inclusion in mathematics teaching.

- I assume that the reader has little previous knowledge of HM, and that the book will provide a first introduction.
- Of course, one major objective is to be able to write in a "popular" way, without simplifying too much.

It is important to note that it is not a goal to be completely original. If one idea has worked wonderfully in the hands of Adriano Dematte, Peter Ransom or Frederic Metin – or other inventive educators who have shared their ideas – that makes the ideas more, not less, relevant for Norwegian teachers. Of course, it is essential to refer to where the ideas came from when possible.

The chapters will be arranged according to different “hows” of including HM. The outline of chapters is like this:

1. Introduction (incl. “hows”, reasons, three perspectives, an overview of HM)
2. Using old notations and solution strategies
3. Working with manipulatives
4. Drama/theatre as a method
5. Original sources: Pictures
6. Tasks based on history of mathematics
7. Etymologies
8. Original sources: Mathematical texts
9. Cross curricular work
10. Original sources: Textbooks
11. Biographical information
12. Project work
13. Texts on history of mathematics
14. HM games
15. The road ahead

The design of the book is a bit ambitious, as in addition to this arrangement according to “hows”, I want to discuss the reasons for including HM in teaching throughout the book, I want to illustrate three aspects of HM throughout (mathematics as science, mathematics in use, and mathematics as a school subject). Finally, to reduce confusion, I want the main examples used to be in chronological order. These ambitions perhaps partly explain why the book has been in the making for 12 years so far. All of the chapters that discuss a way of including HM in teaching, have a similar structure: 1) Why use this way of including HM?; 2) Historical introduction to a period; 3) Main HM example (including suggested teaching unit and comments from a mathematics education perspective); 4) Perhaps another main HM example; 5) Other (short) examples (not necessarily from the same period).

In the workshop, the participants formed groups and selected chapters were distributed to the groups. The goal of the groupwork was to elicit insightful discussions of importance for the HPM group. To newcomers, it might serve as a good introduction to established theory in the field, discussed in concrete cases. Moreover, the discussions might provide valuable input to me. The concrete tasks for the groups were:

For each of the four examples (in early drafts, of course), discuss:

- In what way does the example help in reaching the goals I set?
- Suggestions for improvements to reach the goals in a better way?
- Suggestions for other parts of history to include?

And finally, please comment on the overall ideas of the book.

2 *Example*

The four examples were:

- Chapter 6 Tasks based on history of mathematics. Main example: The Pascal-Fermat correspondence, in the form of tasks based on their correspondence.
- Chapter 8 Original sources 2: Mathematical texts. Main example: The Pascal-Fermat correspondence, in the form of longer extracts of original text from the correspondence.
- Chapter 10 Original sources 3: Textbooks. Main example: Regula de tri
- Chapter 11 Biographical information. Main example: Niels Henrik Abel (and the quintic equation)

The examples handed out at the workshop totalled 18 pages. For issues of space, I will include just two of the examples here – the text on using textbooks as original sources and the text on including biographical information. I need to point out that the chapters are written in Norwegian, and that the translations into English have been done for the purpose of this workshop. This translation may be inaccurate in some details.

Chapter 10 Original sources 3: Textbooks

10.1 Why work on old textbooks?

A text on the "whys" – based partly on what Jan van Maanen has written. Also a reminder about depaysement.

10.2 Historical introduction: Mathematics education through the times

A historical introduction will go here.

10.3 Main example 1: Regula de tri

10.3.1 Introduction

Regula de tri has been a common part of mathematics education for centuries, in many different cultures. For instance, it has been treated in Bhaskara's *Lilivati* from India and in Fibonacci's *Liber abaci* from Italy. It was also considered part of standard education in Norway. For example: from 1817, to be accepted for *Sjøkadettinstituttet* (Sea cadet institute) you needed to be a boy no older than 15 years of age, be able to read and write and to calculate with the "4 species" (addition, subtraction, multiplication, and division) and regula de tri (Botten, 2009).

The text we will be looking at, is from *Arithmetica Danica* by Tyge Hanssøn (not to be confused with other books of the same name). This was the first textbook in mathematics written in Norway, published in 1645, written for students of Trondheim's cathedral school. It includes topics such as addition, subtraction, division, multiplication, regula de tri, square roots and cube roots. It also includes motivational poems and some parts written especially for girls.

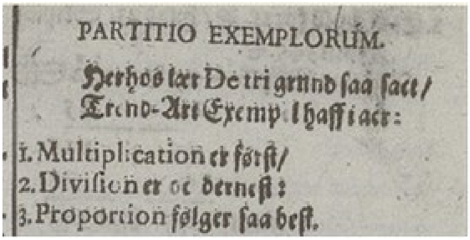
Using such a source in teaching may very well challenge the teacher's usual role in the classroom. While mathematics teachers are often prepared for many of the questions students will come up with, in working on an original source, there is a multitude of questions that may come up that just experts who are well versed in such sources can answer. While the teacher needs to prepare for the obvious questions, the task probably needs to be framed as an exploration of the past where the teacher takes part in the exploration.

10.3.2 Teaching unit

Calculation	Grade 7	1-2h	1600s	School subject	Mathematics Nature Depaysement

Below is a text with excerpts from *Arithmetica Danica* by Tyge Hanssøn from 1645. When students see this text, much will seem difficult or incomprehensible. Students are asked to work in groups. First, they are asked to make a list of what they are curious of about this excerpt. Thereafter, they are asked to find out as much as they can about these excerpts, both concerning the content and concerning the context. At the end of the lesson, students present their findings and remaining questions.

In the first Norwegian textbook in mathematics, *Arithmetica Danica* by Tyge Hanssøn from 1645, *regula de tri* was treated over more than fifty pages. The start was like this:


PARTITIO EXEMPLORUM.
Herhos lær De tri grund saa sact/
Trend-Art Exemp. haff i act:
1. Multiplication er først/
2. Division er oc dernest:
3. Proportion følger saa best.

PARTITIO EXEMPLORUM

Learn «de tri» for reasons given

Three arts example have in mind:

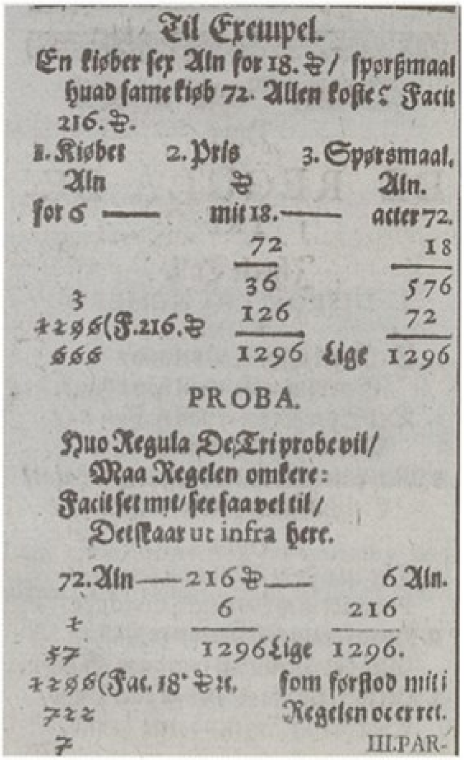
1. Multiplication is first
2. Division is also thereafter:
3. Proportion then follows best.

PARTITIO EXEMPLORUM

Herhos lær De tri grund saa sact

Trend-Art Exempel haff i act:

1. Multiplication er først
2. Division er oc dernest:
3. Proportion følger saa best


Til Exempel.
En kiøber sex Aln for 18. ß/ spørßmaal huad same kiøb 72. Allen koste? Facit 216. ß.
1. Kiøber Aln for 6 —— 2. Pris ß mit 18. —— 3. Spørsmaal. Aln. acter 72.
72 36 126 1296 Lige 1296
PROBA.
Huo Regula De Tri probe vil/
Maa Regelen omkere:
Facit set mit/ see saa vel til/
Det staar ut infra here.
72. Aln —— 216 ß —— 6 Aln.
6 216
1296 Lige 1296.
1296 (Fac. 18 ß rc. som forstod mit i Regelen occerret.
III.PAR-

As Example

One buys six Alen for 18 ß. Question: what same buy 72 Alens cost? Facit 216 ß.

1. Buyer	2. Price	3. Question
alen	ß	alen
for 6 -------	with 18 -----	gives 72

Equal

Check
The one who wants to learn Regula De Tri
must the rule remember:
Facit insert and make sure
It stands outside from in here.

72 Alen -------- 216 ß ----------- 6 Alen
6 216
1296 Equal 1296

It fits with the rule and is correct.

Students may become curious about many issues, such as:

- Why are parts of the text written in verse? Was that normal for mathematics textbooks?
- What kind of school was this book meant for? Who were the students?
- What does Regula de tri mean and why haven't we heard of it before if it used to be in textbooks?
- What is the strange symbol?
- What is "Aln" (or "Alen")?
- What goes on in the calculation? Where can we find the multiplication and the division that is mentioned in the rule? And where does the number 216 come from?
- Why was the answer (fact) given immediately next to the question – and was that normal?
- How do the algorithms for multiplication and division work? (See more about this division algorithm in chapter 2.)

- Why is 126 placed in the way it is below 36? (My personal idea: maybe the printer moved 126 a bit to make space for the division to the left.)

The excerpts are from Geir Botten: Min lidle norske regnebog: noen dypdykk i ei lærebok i matematikk fra 1645, Universitetsforlaget 2009, p. 65. https://urn.nb.no/URN:NBN:no-nb_digibok_2019120207035

10.3.3 Comments from a mathematics education perspective on *regula de tri* tasks

Nowadays, students would perhaps regard *regula de tri* tasks as equations that they would solve by setting up an equation involving fractions and then solve for x. Learning about other solution methods, such as *regula de tri*, might provoke some discussion on why this works and what happens when solving the equations.

From quite another perspective, it is interesting to look at *regula de tri* tasks from the point of view of authenticity. Many mathematics tasks lack authenticity, and in regula de tri tasks there is often an unmentioned assumption that the rate (price per unit, speed per hour etc) is constant. To help discuss this, some humorous examples of non-authentic regula de tri tasks can be used:

- A plumber takes 40 minutes to repair the plumbing under the kitchen sink. How long time would 10 plumbers spend on the same job?
- A woman spends 9 months to carry a child. How much time would three women spend?
- Four weightlifters take five minutes to carry a one-ton-car 200 metres. How long would one weightlifter need to carry the car 50 metres?

10.4 Other examples

Here, there will be other examples of pieces of textbooks of interest – each example taking no more than half a page. Possible examples include: old algorithms used in Norway's first mathematics textbook, old Norwegian measuring units, New Maths, …

Chapter 11 Biographical information

11.1 Why work on biographical information?

History of mathematics is, of course, so much more than just mathematicians' biographies. But still students should get to know one or more mathematicians during their school years to help "humanize" the subject. Including information about the people who developed or used mathematics throughout history, can have the effect on some students of turning mathematics from a stale, technical issue to a human endeavour. Moreover, getting to know the people, we also find out how these people have at different points faced and solved problems. If we are careful in our selection of examples, we may also be able to show that mathematicians are a diverse crowd, counteracting some unhelpful stereotypes about who can succeed in mathematics.

11.2 Historical introduction: Mathematics in the 19th century

A historical introduction will go here.

11.3 Main example 1: Florence Nightingale

Here, there will be an example on Florence Nightingale, with introduction, a teaching unit and some comments from a math. ed. perspective – on statistics, mathematics' role in war etc.

11.4 Main example 2: Niels Henrik Abel (and the quintic equation)

11.4.1 Introduction

One of the greatest mathematicians of all time was the Norwegian Niels Henrik Abel (1802-1829). I will give a sketch of his life, stressing some episodes that can give an insight into the person Abel.

11.4.2 Teaching unit

Algebra	Grade 10	<1h	1800s	Development	Attitudes, view

The teaching "idea" in this teaching unit is simply to tell the story of Niels Henrik Abel.

He was born in Finnøy on August 5th, 1802, grew up on Gjerstad and in 1815 he became a student at Oslo Cathedral School. Niels Henrik's father was an important person, both a priest and parliamentarian. However, both he and his wife developed a drinking habit. The family's economy deteriorated, and it got even worse when the father died in 1820. Then Niels Henrik was left to take care of his little brother.

In November, 1817, one of Abel's fellow students, Henrik Stoltenberg, died after having been beaten up by one of the teachers, Hans Peter Bader. Bader lost his job, and Abel got a new mathematics teacher, Bernt Michael Holmboe. Holmboe was a mathematics teacher who understood mathematics and could see Abel's talent. Today, a prize for good mathematics teachers is named after Holmboe.

In 1820, Abel believed that he had found a general solution for the quintic equation. He showed it to his teacher and to the professors at the university, and thereafter to professor Degen in Copenhagen. Not even Degen could find a mistake, but he asked Abel to give the proof in a more detailed form and to try some concrete numbers. Abel eventually realized that he had made a mistake. Some years later, in 1824, he was able to publish a proof that there is no such general solution.

At that time, he had already been a student at the university in Christiania for a while, and he got a travel scholarship to go to Copenhagen, Göttingen and Paris to learn from the major mathematicians at the time. However, he had a tendency to travel where his friends were rather than traveling to the great mathematicians – he described it like this: "I happen to be like this – I can absolutely not, or at least with great difficulty, be alone. I tend to become melancholy and then I am not in the best mood to get anything done".

In the summer of 1823, he had met a girl, Christine Kemp. They got engaged at Christmas time in 1823, and thereafter the only thing that was needed was for him to get a permanent position before they could get married. After the trip abroad mentioned above, he got a temporary job, but a permanent position did not come his way. He spent the summer of 1828 on Froland, where Christine had gotten a position as a nanny. Here Niels Henrik Abel died of tuberculosis on April 6th, 1829. Two days later, the university in Berlin offered him a permanent position, but it was too late.

11.4.3 Comments from a mathematics education perspective on identity, on proving impossibilities etc.

The story of Abel is of course especially relevant for Norwegian students, since he is considered our foremost mathematician, in a close tie with Sophus Lie. In the story of his life, one can see that a child from modest conditions can become a great mathematician, but also that even mathematicians can make major mistakes. And that even mathematicians can fall in love… Research shows that many students are positioned as less able and that many students believe that making mistakes is a sign of low abilities – history of mathematics may help.

The biography of Abel may also be a start of a discussion on the non-solution of the quintic equation, and on how mathematics may at times be about what is not possible.

11.5 Other examples

Here, there will be other examples of biographical information that could be useful in school, no more than half a page per example. Relevant examples: al-Khwarizmi, Ramanujan, Ahmes (the scribe who created the Ahmes papyrus), Archimedes, Pythagoras [where the idea would be to point out how little we know], Lewis Carroll, Andrew Wiles and the proof of Fermat's theorem, Turing and code theory, Galois, Benjamin Banneker… [I need some biographies of users and learners of mathematics as well…]

3 Postscript

As expected, the group discussions went in many different directions, and no two groups discussed exactly the same issues. Groups discussed:

- Issues in history of mathematics
- How history of mathematics can enrich mathematics education – and how suited the examples are (and whether other examples would be better)
- Issues in mathematics education that can be discussed in light of the examples
- What teachers need in order to teach with history of mathematics
- Ideas on how to develop such a book

I will give examples from each of these directions. However, let me repeat that part of the goal of the workshop was to elicit insightful discussions of importance for the HPM group, and to be welcoming to newcomers. As this workshop was on the first day of the conference, I was happy to see that people got to know each other by sharing their own experiences, knowledge, and views on including HM in teaching mathematics. I heard several references to this workshop later in the conference, which is a sign that the groups had interesting discussions.

Obviously, some of the discussion concerned the history of mathematics in itself – people were talking about other examples of *regula de tri* and provided pointers to other articles to read, details to check and so on.

People also discussed the goals of including history of mathematics in mathematics education, including their own experiences. The Abel biography includes little mathematics, and some participants would prefer to include mathematicians where the usefulness of mathematics – to sciences or practical work – was more obvious. One example could be Isaac Newton.

Issues in mathematics education were discussed in connection with the question about giving the answer directly after the question. Mathematics education researchers advocate for less focus on the answer and more on the process. Giving the answer in advance could be a way of ensuring that students did not believe that the point of mathematics is to find the answer. However, most likely the students in the 1600s did not have a book each – it is more likely that the questions were read to the students and that the answer in the book was to the benefit of the teacher.

Another such issue concerns my use of modern, "humorous" examples of "non-authentic regula de tri tasks". Discussions at the conference – and feedback from a reviewer – have made me uncertain about whether the use of these are wise, as they are clearly created to be funny, and can be easily dismissed. It would perhaps be better to find authentic regula de tri-tasks with problematic modelling assumptions (for instance, that 1 kg and 1000 kg of a product would have the same cost per kg, with no room for bargaining, or that it is possible to travel at a constant speed).

Yet another question is what a teacher needs to know in order to start teaching (meaningfully) with history of mathematics. To avoid serious mistakes, a teacher needs to know history quite well, but few teachers will include history of mathematics if they have to study history of mathematics for years first. Participants discussed whether the examples gave enough information to be used safely.

There was also a discussion in the workshop and after the workshop on how such a book should be developed. The idea of a website instead of a book was proposed. A website can be easily accessed, which is both good and bad. More teachers will see it, but more teachers will also see just one part of it without bothering to look at the whole. A website makes it possible to include "work in progress", but this may lead teachers to believe that nothing on the website is "finished". Still, I consider creating a website with preliminary versions of chapters, partly to get more input from teachers and classrooms, and partly to get things "out there" while waiting for the book to be finished. One major issue at the moment, is that each example has been tested little or not at all, and a website could help with that.

My conclusion is that the workshop helped to raise many questions, some of which were very helpful to me, but some were probably also interesting for the participants.

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