DIRECT COMPARISON BETWEEN OBJECTS

Discrepancies between the ancient and the modern world

Ernesto ROTTOLI, Gianstefano RIVA
Gruppo di Ricerca sull'insegnamento della matematica per la scuola primaria, Università Milano Bicocca, Milan, Italy
erneroTT@tin.it

ABSTRACT

Philosophical practice on "direct comparison between objects" indicates an unusual approach to the familiarization of primary school children with fractions.

1 Introduction

The workshop is divided into three parts, each of which is based on activities of direct comparison between objects: 1) Direct comparison as addressed by Davydov. 2) Direct comparison as performed by the Pythagoreans. 3) Activities of direct comparison as the source of the concept of logos/ratio. The aim is to propose moments of philosophical practice related to these activities. As an introduction to the workshop, after some brief information about our research group, we have presented: 1) our idea of philosophical practice and 2) the basic activity of direct comparison between objects.

1.7 Philosophical practice

Accompanying classroom practice with philosophical practice characterizes our Group, which has met for more than twenty years at Milano Bicocca University. Our effort aims to keep philosophical practice at the heart of teaching practice.

We have introduced this topic, proposing to discuss the prominent meanings we attribute to the term "philosophical": a) bringing out questions, b) pursuing new answers, c) transmitting new meanings.

1.8 Direct comparison between objects

We met direct comparison activities while addressing the question of the long persistence of unsatisfactory results in teaching and learning fractions. This question was both directly affecting our didactic practice and persistently emerging from the scientific literature. At the same time, we found in the scientific literature Davydov's proposal about direct comparison activities: he indicated these activities as the source of the concept of fractions. He proposed a series of situations that led primary school children from direct comparison to familiarization with the concept of fractions. (Davydov et al., 1991) We have explored Davydov's situations, starting to apply them in some primary school classes and accompanying this classroom practice with philosophical practice.

2 First part of the workshop: Direct comparison between objects as addressed by Davydov.

2.1 Direct comparison between two mugs of different shapes

We started this part of the workshop by proposing the activity of direct comparison between two mugs of different shapes. The answer was of complete "dépaysement". This answer is not surprising because it is like the answer given by our students in laboratories of "Scienza della Formazione Primaria" at the University of Milano Bicocca. Having to compare real objects instead of mathematical objects, brings out meanings of the adjective "direct", to which the mathematics teacher is often not accustomed. We will see that this "dépaysement" is closely linked to the history of direct comparison in the Western world.

2.2 From direct comparison between objects to comparison of their measurements.

To understand how Davydov proceeds to answer the problem of direct comparison, we have proposed to the workshop some texts by Sierpinska.

a) "Davydov spent a lot of time thinking about the meaning and the sense of fundamental mathematical concepts taught in grades 1-3, such as number, multiplication, and fractions, coming up with activities on which these concepts have their source." (Sierpinska, 2019)

b) *"Children were first introduced to numbers in the context of direct comparison (by juxtaposition, superposition) of objects relative to qualities such as length, width, height, weight, etc."* (Bobos & Sierpinska, 2017)

c) "Children were taught to record the results of their comparisons using symbols such as +, <, >." (Bobos & Sierpinska, 2017).

d) *"Next, the situations were changed to make the direct comparison of objects difficult or impossible; this forced the children to measure each object using sticks, pieces of string, and other devices and compare the measurements, not objects."* (Bobos & Sierpinska, 2017)

So Davydov solves the problem by moving from the direct comparison of objects to the comparison of their measurements.

This solution proposed by Davydov is identical to that generally proposed in the modern Western world. It is the same solution we have adopted in our classroom practice: grade 3 children (8-year-old) carry out direct comparison activities by comparing the measurements of the objects. We showed it by photocopies of the record of activities carried out in the classroom.

Bringing the direct comparison between objects back to the comparison of their measurements, makes the aforementioned "dépaysement" disappears.

2.3 Two key features of Davydov's proposal

Before moving on to the second part of the workshop, we have presented some texts that highlight two very significant features of Davydov's proposal: 1) his idea of abstraction/generalization and 2) his use of symbolic language in primary school.

As for abstraction/generalization, we have proposed the following texts:

"... The generalization reduces the diversity in the specific examples. Davydov argues that we ought to conceive of learning differently. The specific examples should be seen as carrying the generalizations within them; the generalization process ought to be one of enrichment rather than impoverishment." (Kilpatrick, 1990)

"Davydov's views on abstraction, his ascent to the concrete, refers to the development of an idea via a dialectical "to and fro" between the concrete and the abstract." (Monaghan, 2016)

In our didactic practice of fractions in primary school, we proposed abstraction/generalization as an ascent to "the objective content" of fractions.

As for use of symbolic language in primary school, we followed Davydov's suggestion, but not as systematically as he does. Our aim is to develop "friendliness with symbols" in children. At this point, we presented to the workshop our idea of "friendliness with symbols", which results from the synthesis of two utterances: a) "*Friendliness with numbers precedes number sense*" by Howden, and b) "Making students friends with symbols" by Arcavi.[35] In the second part of the workshop, we will see the role of friendliness with symbols in coming to a writing that favors the ascent to the objective content of fractions.

3 Second part of the workshop: Direct comparison between objects as addressed by Pythagoreans.

As was customary throughout the ancient world, the Pythagoreans performed the comparison between objects directly, without bringing it back to the comparison of their measurements. That comparison, which has the goal to seek the highest common unit between the two compared quantities, was named "Anthyphairesis" by the Greeks.

3.1 Activities on anthyphairesis

First, we presented to the workshop the following text by Fowler: "*Anthyphairesis = anti-hypo-hairesis, 'reciprocal sub-traction'. It is described in Euclid Elements X,2): "... when the less of two unequal magnitudes is continually subtracted in turn from the greater, ...". The phrase 'is continually subtracted in turn from', describes the anthyphairesis*".

Then a PowerPoint has been projected, in which the mutual subtraction of two quantities is shown step by step.

Subsequently, the participants performed the direct comparison of two natural numbers, being careful to write the development in two columns. This development will constitute a central text in the third part of the workshop.

3.2 Philosophical practice on anthyphairesis

We drew the attention of the participants to the fact that while the objective practice of anthyphairesis indicates this activity as elementary and apparently scarcely significant, the philosophical practice brings out its profound meaning. To this, we showed to the workshop two Leont'ev's statements [Sierpinska, 2002].

[35] The path towards friendliness with symbols develops through activities on change: a) to register the change; b) to discuss and describe the change; c) to recognize the change; d) to get the quantity that produces the change. [Bonetto et al., 2008]. We have shown the workshop some of these activities.

The first statement is: "*Meaning belongs first of all to the world of objective-historical phenomena*". As for the “objective” aspect of phenomena, Davydov points to the direct comparison as the source from which the search for the "objective content" of the concept of fractions must begin.

As for the “historical” aspect of phenomena, Leont’ev's second statement clarifies the direction in which Davydov has developed his interpretation: “*Individual mind is a result of an assimilation of the experience of the previous generations of people*". But the discovery of incommensurability, moving the direct comparison of objects to the comparison of their measurement, causes the forgetfulness of the Pythagorean comparison [Fowler, 1979]. Consequently, the assimilation of the experience of direct comparison by the generations of people before the Pythagoreans, is cut off. That forgotten knowledge stays “*deposited in the culture*” [Asenova et al., 2020]. It demands an interpretation of history that goes beyond Leont'ev's assimilation.

In search of this interpretation, we resort to Toth.

3.3 Toth: “There is something else”

We met with Imre Toth in 1999. His suggestion to listen to *hidden meanings that could still be kept in Pythagorean mathematics* and the reading of his book “Lo Schiavo di Menone” have introduced us to the anthyphairesis, and have initiated us into philosophical practice.

Just at that time, we were starting to practice in some classes Davydov’s situations about the familiarization of primary school children with fractions. The intertwining of Davydov-inspired classroom practice and Toth-inspired philosophical practice has led us, over the years, to make changes both to the practice and to the interpretation of Davydov's situations.

3.4 A split in the development of Greek mathematics.

To show the changes, we proposed to the workshop some considerations on incommensurability, taking up the previous text by Fowler. *The definition of incommensurability is found in Euclid Elements X, 2: “If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable.”*

The discovery of incommensurability by the Pythagoreans introduced a split in the development of Greek mathematics. This split makes it possible to distinguish between “mathematic before the discovery of incommensurability” and “mathematics after the discovery of incommensurability”.

The activity carried out in the workshop on anthyphairesis helps to highlight how, before the discovery of incommensurability, Greek mathematics was characterized by two founding and independent acts: measurement and comparison. A measurement is a number that requires the concept of unit of measurement. A comparison is a “pair of numbers in relation” - a logos and requires the concept of the highest common unit. The discovery of incommensurability has led to abandoning the anthyphairesis, to neglecting the direct comparison, and to downgrading the search for the highest common unit.

After the discovery of incommensurability, measurement becomes the founding concept, while comparison, as it is reduced to two measurements, becomes a derived concept; the comparison of two objects is moved to the comparison of their measurements; the “highest common unit” is subordinated to the choice of a “unit of measurement”. The crisis of incommensurability channels in this direction the development of Greek mathematics. This direction is preserved in the development of Western “academic” mathematics.

We kept these considerations in mind as we addressed the question of “the long presence of unsatisfactory results in teaching and learning fractions”. This resulted in our didactic proposal regarding the familiarization of primary school children with fractions.

3.5 Hypothesis

Philosophical practice led us to link the “long persistence” with the split created by the discovery of incommensurability and with the forgetfulness that ensued. Our hypothesis, presented at the workshop, is the following: In the forgotten procedure of anthyphairesis there are indications that allow treating the question of “long persistence” in a different way from the usual ones. But be careful: In our classroom activities, we have never made any direct recourse to the procedure of Pythagorean comparison. Rather, through Davydov's situations, we have rediscovered the activities of direct comparison between objects that characterize the Pythagorean comparison. Thanks to that, we returned to the centrality of the search for the highest common unit, making use of two tools: 1) A "new" language. During the actions of each activity carried out in the classrooms, the language is built on three keywords, two referring to the quantities to be compared, the third referring to the common unit. 2) Symbolic writing. In familiarizing primary school children with fractions, we have introduced symbolic writing to record the comparison activities.[36]

Our classroom practice develops in a movement between the objective phenomenon and the symbolic writing, mediated by the new language. During the workshop, we traced the structure of our classroom activity by means of photocopies of the children's notebooks. There are three steps. 1)The first consists of direct comparison activities. These are developed in the language of the three quantities: the two compared quantities and the common unit. Symbolic writing is M;R = 9;3. Children read this form in this way: “The comparison between the two mugs "M" and "R" is the pair of numbers 9;3. That is, mug M contains 9 times the common unit, and mug R contains 3 times the common unit.” 2) The next step represents the transition to measurement. It is obtained by introducing an order between the compared quantities, with the choice of the reference quantity W. Symbolic writing is C/W = 16/4 = 4. Children read: “The measurement gives me a pair of numbers that determine how many egg cartoons I can pack.” This is the measurement by comparison. 3) Finally, as activity evolves toward division, symbolic writing evolves toward Euclidean division: Z/W = 17/5 = 3+2/5. “The fraction 17/5 equals 3 integers plus 2 common units.”

Symbolic writing is not formal. It evolves with classroom practice by crossing the “sub-constructs” ratio, measurement, and division of the “construct” fraction.[37] So, children become accustomed to thinking of “ratio, measurement, and division” as related to each other in the concept of fractions.

4 Third part of the workshop: Logos / Ratio

The third part, the shortest, has focused on the concept of logos/ratio. The main tools of this part were some texts by Toth and Fowler.

[36] For this purpose, we took advantage of children's friendliness with symbols, which had already been developed previously.

[37] Kieren [1980] introduced five sub-construct of the construct rational number: ratio, measure, division, part-whole, and operator.

4.1 Al tempo dei Pitagorici il logos ha una natura puramente aritmetica.[38]

This text by Toth emphasizes the purely arithmetic nature of logos, before the discovery of incommensurability.

What happened after the discovery of incommensurability is described by Toth as follows: "*La penetrazione dell'àlagon nell'universo puramente aritmetico del logos ... produce un salto nella concezione del logos… Il linguaggio aritmetico scompare e il discorso risulta trascinato da un flusso verbale di sostanza differente*"[39].

In our teaching practice with primary school children, thanks to the direct comparison between objects, we tried to restore the purely arithmetic nature to the concept of logos/ratio: this concept is associated with a pair of numbers.

4.2 A ratio is a relationship between two numbers, not just 'two numbers'

This text by Sierpinska introduced the workshop to the theme of the relationship between the two numbers of logos. The following two texts indicated how, after the discovery of incommensurability, the search for this relation has resorted to a language that has lost its arithmetic character:

(In the Elements) no alternative definition of ratio apart from V, Definition 3 is proposed: "A ratio (logos) is a sort of relation in respect of size between two magnitudes of the same kind". (Fowler, 1979)

"*L'espressione ποια σχέσις (una certa relazione) rinvia a una relazione il cui carattere concreto è forse fluido e resta ancora nel vago.*"[40] (Toth, 1988)

The philosophical practice led us to attempt to extend the purely arithmetic nature to the relationship between the two numbers of logos.

4.3 Purely arithmetic nature of the relationship

To eliminate the vagueness of the meaning of the relationship between the two numbers of the logos, we have represented the anthyphairesis by "a

[38] "*At the time of the Pythagoreans, logos has a purely arithmetic nature*".

[39] "The penetration of the àlagon into the purely arithmetic universe of logos ... produces a leap in the conception of logos ... The arithmetic language disappears, and the speech is dragged by a verbal flow of different substance".

[40] "The expression ποια σχέσις (a sort of relationship) refers to a relationship whose concrete character is perhaps fluid and still remains vague."

modern symbolic writing". We first showed to the workshop the following text by Fowler "*We now consider the suggestion that the ratio of two numbers or magnitudes was defined by their anthyphairesis*". We then proposed the following activity: 1) Take up again the sheet on which, in the second part of the workshop, you had written the development of the anthyphairesis in two columns. 2) Add two outer columns. 3) Write in these columns the letter S, if the subtraction has been performed in that column, or the letter C if the number has remained constant. In this way, they got a writing that reproduces the Pythagorean subtractive procedure step by step. The double column that survives in this writing is the key tool for our conclusions. We showed the workshop the following text by Fowler which contains another writing of the anthyphairesis: "*We call this procedure 'anthyphairesis', and refer to the sequence* $n_0, n_1, \ldots$ *as 'the anthyphairesis of A and B\ sometimes writing Anth(A, B) =* $[n_0, n_1, n_2, \ldots]$." We are faced with two different writings: 1) The two-column writing reproduces step by step the subtractive nature of the Pythagorean comparison. 2) The one-line writing translates the Euclidean algorithm and its multiplicative character. The first writing needs some considerations. It reproduces the feature of the Pythagorean comparison of moving between two columns. It is the "story" of this "movement" and generates the unusual property of the unique additive partition of the pair of numbers. As an example, we consider the pair (11; 3). This pair has the unique additive partition (1 + 1 + 3 + 3 + 3; 1 + 2). When the number 11 belongs to the pair (11; 7), it has another partition: (1 + 3 + 7; 1 + 1 + 1+ 4). The unique partition reproduces step by step the subtractive procedure of the anthyphairesis. Academic mathematics and its extraordinary effectiveness are founded on the unique factoring of natural numbers. They are generated by the choice of the Euclidean algorithm and by the abandonment of the Pythagorean comparison. But does the attention to Pythagorean comparison suggest rethinking foundations? The Pythagorean comparison suggests new concepts of measurement and quantity: the number 11 considered in the abovementioned example takes on meaning by the number it is paired with. Could this observation open a dimension in which to investigate dual phenomena? (Rottoli & Riva, 2021)

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