THE HISTORY OF MATHEMATICS IN ITALY THROUGH THE AGES:

Sources, correspondences and editions

Maria Teresa BORGATO
Department of Mathematics and Computer Science, University of Ferrara, Italy
bor@unife.it

ABSTRACT

The study of primary sources is the starting point for the research in the history of mathematics, and increasing those resources is one of the most significant and enduring contributions that scholars can make to the discipline. In past decades, Italian historians of mathematics have contributed to it with critical editions of works, and the publication of numerous inedited documents, manuscripts and correspondences. The enhancement of primary sources for the history of mathematics in Italy was a determining element for the foundation of our scientific journal *Bollettino di Storia delle Scienze Matematiche*. In the teaching of mathematics, quoting meaningful passages from these sources can be a useful strategy to introduce arguments and concepts, or to raise awareness of historical development. In my talk, I will present a certain number of examples and suggestions, which go from the late Middle Ages and the Renaissance, to the first half of the twentieth century. Starting from original and recent research, it is possible to propose less usual themes connected to different mathematics topics and directed to different age groups, including: abacus mathematics and Archimedean tradition, the resolution of algebraic equations of third and fourth degrees, the pre-Newtonian systems of the world, the foundation of infinitesimal calculus, the science of waters and the origin of hydrostatic and hydrodynamic laws, new approaches in elementary geometry (fusionism, paper folding).

1 Introduction

I should like to thank the organizers of this Summer School for their invitation to hold this lecture. This is a particularly important occasion for those like me, who have made the history of mathematics their main field of research and have always sustained the importance of the history of mathematics in the teaching of mathematics. I feel that my presence here today is also due to my position as President of the Italian Society of the History of Mathematics and consequently it is my duty to provide as wide a picture as possible of the research works that have so far been carried out.

As the community of historians of mathematics has been quite active since the last quarter of the past century (1980s) it would not be possible here today to give an adequate account of its output, so I shall limit myself to the contents of research on primary sources. The enhancement of primary sources is one of the main objectives of the *Bollettino di Storia delle Scienze Matematiche*, which includes many of these contributions. The use of historical sources in the introduction of topics in the history of mathematics in the classroom is, moreover, a persistent theme in the international literature in the teaching of mathematics, which runs through other interventions in this conference. As the title of section 6 indicates ("History of Mathematics in Italy"), I will focus on manuscripts and works of Italian mathematicians, investigated by Italian historians of mathematics, which may provide students with accessible material and which are therefore of use in the context of teaching. As a result, I will not be covering research works, which do not concern the history of mathematics in Italy, or mathematical works of advanced content, or historical essays of synthesis or interpretation.

2 Long tradition of historical mathematical studies in Italy

First of all, let me remind you that research in the history of mathematics in Italy is based on a secular cultural heritage, which preceded the unity of the nation itself. The historiography of mathematics in Italy has a long tradition dating back to the Renaissance (Bernardino Baldi, 201 biographies in his *Vite de' matematici*). Moreover, Italian scholars contributed to the transmission and history of science with translations, commentaries and editions of ancient classics, and chronologies were included in encyclopedic treatises in the seventeenth century (Giuseppe Biancani, Giovanni Battista Riccioli, and Claude François Milliet Deschales). In a modern sense, however, the historiography of mathematics begins in the eighteenth century, when the history of mathematics was considered an area of the history of human thinking. In Italy it was initially developed as a part of Italian literature and inserted into general works (Giovanni Andres, Girolamo Tiraboschi).

Critical analysis of mathematical theories and their historical foundations can be found in works of mathematicians like Joseph-Louis Lagrange, Gregorio Fontana, Pietro Cossali, and Giambattista Guglielmini. An important work, which combined general overview, technical and archival investigation came out later: the famous *Histoire des sciences mathématiques en Italie* by

Guglielmo Libri (4 volumes 1838-41, see Del Centina & Fiocca, 2010). For this first period you can see (Borgato, 1992) and the articles collected in the volume of the conference proceedings (Barbieri & Cattelani, 1989).

The historiography of mathematics developed in the second half of the nineteenth century, after the political unification of Italy, with Baldassarre Boncompagni's foundation of the *Bullettino di bibliografia e di storia delle scienze matematiche e fisiche* (20 volumes), which supplied both an international diffusion and primary sources. Antonio Favaro was the editor of Galileo's collected works, and Pietro Riccardi published a bibliographic work still of great importance, the *Biblioteca matematica italiana*. In the first half of the twentieth century, two mathematicians became the main historians of mathematics: Gino Loria and Ettore Bortolotti.

The Second World War constituted a sort of breakdown also for this type of studies so Italian historians of mathematics had to re-launch their activity in the light of a new relation with the international environment. In the last quarter of the century, the community of Italian historians of mathematics developed so that Italy became known as one of the main centres for the historiography of mathematics. In particular, their research was supported by the publication, from 1981 on, of the *Bollettino di storia delle scienze matematiche* (40 volumes so far) and, in 2000, the foundation of the national society: Società Italiana di Storia delle Matematiche (SISM web site: http://www.sism.unito.it/), which, among the objectives of its statute, has the training of teachers and the promotion of mathematical culture in the country. Besides numerous contributions in journals and volumes (see Barbieri & Pepe, 1992), in recent times Italian historians of mathematics have collaborated with historians and philosophers of science to important editorial projects:

- Archivio della Corrispondenza degli Scienziati Italiani: https://www.olschki.it/catalogo/collana/acsi/p2
- Edizione Nazionale Mathematica Italiana: http://mathematica.sns.it/
- Edizione Nazionale dell'Opera Matematica di Francesco Maurolico: www.maurolico.it/Maurolico/index.htm
- Edizione Nazionale Ruggero Giuseppe Boscovich: http://www.brera.inaf.it/~boscovich/pagine/

3 Importance of primary sources for the history and teaching of mathematics

The study of primary sources is the starting point for the research in the history of mathematics, and increasing those resources is one of the most significant and enduring contributions that scholars can make to the discipline. In past decades, Italian historians of mathematics have contributed to it with critical editions of works, and the publication of numerous inedited documents, manuscripts and correspondences. Correspondences, in particular, allow us to investigate the circulation of scientific thought and the origin of mathematical ideas. Primary sources are also a useful tool for introducing topics of history of mathematics into mathematics teaching, with some precautions, however.

The benefits of a historical approach in the teaching and learning of mathematics are widely recognized, as:

- The lack of interest in mathematics can be overcome through the emotional involvement of the story
- In conjunction with epistemological obstacles, it promotes learning and self-esteem
- By placing the mathematical discipline within the more general history of culture promotes holistic learning
- Through the awareness of intrinsic difficulties in the discipline, it gives reasons for formal definitions and choices that may appear obscure and excessively artificial or abstract
- With the awareness that the development of mathematics is not the merit of a single country and a single civilization, it favors integration
- It places mathematics alongside the other sciences, in which theories, hypotheses, languages, and representations, are linked to the time and place (to the society) that produced them.

The risks in the historical approach in the teaching and learning of mathematics are sometimes underestimated by teachers, they include:

- Failure to adapt to the level of pupils in proposing the themes
- Trivialization in an attempt to simplify a complex topic
- Introduction of misconceptions
- Superficiality due to little knowledge of the history of mathematics as a discipline.

In conclusion, not all mathematics topics lend themselves to a historical approach, which is not limited to narrative, and biographical or bibliographic references, and requires attention on the part of teachers in choosing the topics that must be commensurate with the students' age and level of understanding, and also the sources to refer to, which must be scientifically valid.

The reference to original works emphasizes advantages and difficulties, since it requires the interpretation of a language that is not immediately translatable and must be contextualized. On the other hand, it emotionally involves the student even more, in that it allows the most direct contact with the author and the social and cultural environment of the time. Finally, it enriches the mathematical topic with other elements of historical investigation and lends itself to an interdisciplinary development.

4 An overview of the workshop project

To provide a broad picture of topics to be developed in the classroom, connected to recent research on primary sources, conducted by Italian historians of mathematics, I proposed a workshop divided into two sessions (of two hours each) connected to my lecture.

However, as I had to make a selection for this, I thought of a temporal distribution of the themes aimed at covering the fourteenth to seventeenth centuries in workshop n.1, and in workshop n. 2, the eighteenth to the twentieth centuries. The first session is entitled:

Starting from the history of mathematics in Early Modern Italy: From primary sources to mathematical concepts

It includes the following topics:

- *Abacus mathematics and Archimedean tradition*
- *The resolution of algebraic equations of third and fourth degrees*
- *The law of free fall*
- *The pre-Newtonian systems of the World.*

The organization of this workshop is shared with Alessandra Fiocca (University of Ferrara) and Veronica Gavagna (University of Florence) and makes use of the collaboration of Elena Lazzari (University of Ferrara).

The second session is entitled:

Starting from the history of mathematics in Late Modern Italy: From primary sources to mathematical concepts

And deals with:

- *The dissemination of infinitesimal calculus in Italy*
- *The science of waters as the main field of applied mathematics*
- *Re-launching Italian education and research after political unification*
- *The geometry of paper folding and the resolution of problems of third degree*

The organization of workshop 2 is shared with Maria Giulia Lugaresi (University of Ferrara) and Paola Magrone (University of Rome) with the collaboration of Elena Lazzari (University of Ferrara) and Elena Scalambro (University of Turin).

4.1 Abacus mathematics and Archimedean tradition

Abacus mathematics developed in Tuscany and then spread over a period of time that traditionally goes from 1202 (*Liber abbaci* by Leonardo Fibonacci) to 1494 (*Summa* by Luca Pacioli.)

The meticulous research of manuscripts and documents carried out in Tuscan archives for forty years by Raffaella Franci, Elisabetta Ulivi and Laura Toti-Rigatelli has allowed both the publication of various texts used in these schools and the extensive reconstruction of this network of schools, devoted to the formation of professional figures in the fields of commerce, crafts and finance. These schools contributed to the growth of middle-class professions, which were to become the foundation of the new Renaissance society.

Figure 1. Gentile da Fabriano, *Arithmetic*, fresco. Foligno, Palazzo Trinci 1411-1412

The *Liber Abbaci* by Leonardo Pisano, which marks the start of the dissemination of abacus mathematics in the West, is a very rich and complex text, the first critical edition of which was recently published by Enrico Giusti in collaboration with Paolo D'Alessandro (2020). From the *Liber Abbaci* by Leonardo Pisano various rules may be extracted in order to carry out operations with numbers, as well as problems of mercantile mathematics (purchases and sales, barters, corporations, and coins), or linked to the Fibonacci series. From the *Pratica Geometrie* of the same author, we find other problems of practical geometry. Many teaching sequences and materials were designed on these themes, the first by *Il Giardino di Archimede*, mostly experimented in Italy. Franco Ghione and Laura Catastini have recently set up a website devoted to the *Liber Abba*ci with a translation in Italian and numerous teaching proposals: https://www.progettofibonacci.it/

In parallel, a "cultured" mathematics was developing in medieval universities and courts of enlightened princes, which originated from the rediscovery and retrieval, and at times from the reconstruction of works from classical antiquity. Later on, the invention of printing favored the process of enlargement of the scientific community. Among the leading protagonists of this retrieval is the mathematician Francesco Maurolico (works of Archimedes, Theodosius, Menelaus, Euclid and Apollonius). Maurolico's legacy included printed works and manuscripts: after several years of partial editions and essays, an editorial enterprise under the direction of Pier Daniele Napolitani with the collaboration of Veronica Gavagna, was undertaken to produce the National Edition of the Mathematical works of Francesco Maurolico: *Edizione Nazionale dell'Opera Matematica di Francesco Maurolico*.

In spite of the philological competence that the reading of the documents requires, some topics suitable for use in secondary schools can be drawn from themes of medieval mathematics, as well as from passages of the ancient classical works rediscovered. Veronica Gavagna develops this topic, showing, with direct examples from the texts, the influence of the Archimedean tradition in the works of Piero della Francesca and Luca Pacioli.

4.2 The resolution of algebraic equations of third and fourth degrees

Bringing together different traditions, of commercial mathematics and the mathematical culture of universities and academies, the sixteenth century saw the conquest of the Italian algebraists Del Ferro, Tartaglia, Ferrari, Cardano; i.e. the general resolution of third and fourth algebraic equations. The contrasting events of this discovery and the mathematical challenges that have accompanied it lend themselves to an engaging narrative for students, and have been the subject of presentations, including theatrical ones. See in particular the performance "La formula segreta” (The secret formula) staged by Daniele Squassina and Maurizio Lovisetti, which, in addition to the pleasant show, presents a careful reconstruction of some details of the life of Tartaglia and figures connected to him. The theme also lends itself to mathematical insights, even if not trivial. Tartaglia's famous poem containing the resolving rule can be an example of a text to be interpreted and the methods of resolution can be re-proposed.

A mathematical formula in poetry

The resolution formula of the cubic equations of the form $x^3 + bx + c = 0$ is expressed in the first nine verses of the poem that Tartaglia communicated to Cardano on the night of March 25, 1538, shown below with the paraphrase alongside the modern algebraic symbolism:

Quando chel cubo con le cose appresso	When $x^3 + bx$
Se agguaglia à qualche numero discreto	$= c$
Trovan dui altri differenti in esso.	find u and v such that $u - v = c$
Che'l lor produtto sempre sia eguale	and $u \times v =$
Al terzo cubo delle cose neto,	$(b/3)^3$
El residuo poi suo generale	Then follows
Delli lor lati cubi ben sottratti	$\sqrt[3]{u} - \sqrt[3]{v}$
Varrà la tua cosa principale.	$= x.$

In his 1545 treatise *Ars Magna*, Cardano showed that the cubic equations also containing the second degree term (of the general form $ax^3 + bx^2 + cx + d = 0$) could be transformed with suitable artifices into equations of the reduced form without this term : $x^3 + bx + c = 0$.

We can then go on with the *imaginary numbers*, introducing the work by Rafael Bombelli who in *Algebra* (1572, 1579) examines the solutions of the various cases of the third degree equations, including the so-called *irreducible case*, which in Cardano’s formula presents the square root of a negative number. The imaginary roots, called “sylvan quantities” and complex numbers (referred as to “plus of minus” and “minus of minus” for + *i* and -*i*) are then examined, establishing their calculation rules (addition and multiplication). Later Descartes was to coin the term *imaginary.*

Bombelli’s *Algebra* represents a more mature result of sixteenth century algebra, remaining for over a century the most authoritative text on advanced algebra. Unlike several of his contemporary mathematical writers, in his printed editions and manuscripts Bombelli used a sophisticated form of mathematical notation, introducing, in particular, the exponents to indicate the powers of the unknown.[25]

To use this work as a primary source in classroom, we can cite some passages from the first book of Bombelli’s *Algebra* in which syncopated algebraic language is used (so p. stands for “plus”, m. for “minus”, R. q. means “square root” etc. for example see p. 95 of the first edition). We can read the rules to operate with imaginary numbers (addition, subtraction, multiplication) for example on p. 169, 189, 192, of Book I. The irreducible case can be studied using resolving algebraic formulae and the graph of the corresponding polynomial.

Alessandra Fiocca, who has published some inedited works by Ludovico Ferrari and a new edition of the third book of Bombelli’s *Algebra* (Fiocca & Leone, 2017), deals with this part of workshop n. 1.

4.3 Galileo’s law of free fall

The seventeenth century is Galileo's century, and we can certainly take cues from his works and the mathematicians of his school, such as Torricelli, Viviani, Cavalieri and Borelli, introducing for example, Torricelli’s hyperbolic solid as a famous example of an unlimited solid with finite volume and an application of Cavalieri’s principle. We prefer to focus on the law of free fall. The law of uniformly accelerated motion, the space proportional to the square of time (*the time-squared law*) as well as the *law of odd numbers*, which follows from the former, are to be found in a letter from Galileo to Paolo Sarpi dated 16th October 1604:

[25] Bombelli’s *Algebra*, like many other reproductions of fundamental and rare printed works of Italian mathematicians can be downloaded from the site of the project carried out by the Scuola Normale Superiore of Pisa: *Mathematica Italiana*: http://mathematica.sns.it/opere/9/, where you can also find a description of the main features of the work. The project was carried out by a scientific board that included Luigi Pepe, Mariano Giaquinta, and Paolo Freguglia.

"gli spazi passati dal moto naturale esser in proportione doppia de' i tempi et per conseguenza gli spazi passati in tempi uguali esser come i numeri impari ab unitate".

Galileo later developed a mathematical theory of accelerated motion, which he derived from a fundamental principle, firstly identified in the proportionality of speed to space, and only later in the proportionality of speed to time (Giusti, 1990).

We are used to representing the Galilean law of fall as a single result, summarized in a formula that describes a vertical motion, in which the traveled space grows proportionally to the square of time, and the constant of proportionality is the same for all bodies, corresponding to half the acceleration of gravity. Each of these assumptions, however, has registered different opinions and positions and has been the subject of debate and experimentation (Borgato, 2014).

In fact, the questions raised by Galileo's law in relation to scholastic philosophy are many and of diverse nature which are to be considered separately:

- non uniformity (*difformity*) of the fall and the distinction between levity and gravity
- the law of uniformly accelerated motion ('uniformiter diformiter') of heavy bodies, that is the Times Square Law, and the equivalent Odd-Number Law
- the constant of proportionality, that is the ratio between the distance traversed and the square of time–intervals
- velocity independent of weight and the simultaneous fall (in a vacuum or in the air)
- the trajectory of a freely falling body in 'absolute' space

An in-depth study of these issues will require the contribution of the philosophy teacher. However, a useful exercise in classroom, starting from Galileo's direct quotation, could also be to compare the two forms of Galileo's law, that is the *Times Square Law* and *Odd-Number Law* and try to demonstrate their equivalence, even without using the infinitesimal calculus:

$$s = \frac{1}{2} g t^2 \qquad s(t+1) - s(t) = \frac{1}{2} g (2t+1)$$

$\Rightarrow$ Trivial

$\Leftarrow$

$$s(0) = 0,\ s(1) = \frac{1}{2} g\,,$$

$$s(2) = \frac{1}{2} g(3+1) = \frac{1}{2} 4\, g\,,$$

$$s(3) = \frac{1}{2} g(5+4) = \frac{1}{2} 9\, g\,, \dots$$

$$s(t) = \frac{1}{2} g(2t-1) + s(t-1) = \frac{1}{2} g(1+3+5+\cdots+2t-1) = \frac{1}{2} g t^2$$

4.4 The pre-Newtonian systems of the World

I would like to point out another source, which refers to the school in opposition to the Galilean school, that Jesuit school which also has merits in the physics-mathematical research of the seventeenth century, especially in those disciplines that did not interfere with the dictates of Catholic orthodoxy. This theme brings us to celestial mechanics and cosmological systems before the definitive affirmation of the Copernican system according to Kepler's hypothesis. Giovanni Battista Riccioli was a famous astronomer of his day, who is currently living a sort of renewed celebrity thanks to the many volumes and articles devoted to him.[26] One of his merits was the first direct experimental proof of Galileo's law of free fall, carried out by Riccioli with the help of many members of his brotherhood in the 1650s, as they performed various launches of heavy spheres from churches and buildings in Bologna and in particular from the Asinelli Tower.

But now let's turn to the proposal of a teaching activity. In his major work, the *Almagestum Novum*, two great folio volumes, Riccioli compares the various astronomical systems of the world proposed from antiquity to his day and discusses all the evidence in favor or against the motions of the Earth.

[26] The first of these books, was: (Borgato, 2002).

Riccioli's aim is represented in the extremely famous and often reproduced frontispiece of the *Almagestum Novum* (1651).

Figure 2. In it, two characters confront each other: Astraea representing theoretical astronomy, and Argus Panoptes, the many-eyed giant in Greek mythology, representing observational astronomy. In the center a balance in which the heliocentric Copernican system and the Tychonic system are opposed. The balance hangs in favor of the latter, with the variants introduced by Riccioli: the Sun, Jupiter and Saturn revolve around the Earth while Mercury, Venus and Mars orbit around the Sun. The Ptolemaic system lies on the ground neglected.

The representations of the various cosmological systems in comparison can be the starting point for introducing the students to the various hypotheses that followed one another: the Ptolemaic system, the Copernican system, the Tychonic system, the Ricciolian system and others.

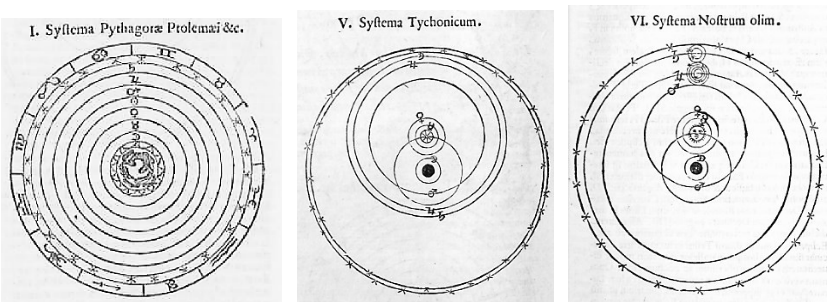

Figure 3. The Ptolemaic Tychonic and Ricciolian systems.

In the pre-Newtonian systems, the planetary orbits that explained the apparent motions of the celestial bodies with respect to the Earth, were obtained by composing different circular motions according to a complex system of epicycles, deferents, eccentrics and equants, which can be explained to the students through figures and then reconstructed in the simplest cases using the GeoGebra software (roulettes, epicycloids). Elena Lazzari presents a laboratory on this type of curves.

T the center of the Earth
AOPF the eccentric
LK the epicycle
C the equant
HRV the ecliptic
HV the line of apsides

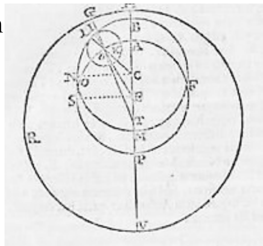

Figure 4. The system of eccentric epicycles in Riccioli's Almagestum Novum

The epicycloid is the trajectory of a point on a circumference that rolls outside of another circumference. The rolling circumference could represent the epicycle, while the deferent would be the circular trajectory of the center of the epicycle. If the ratio *k* between the radii of the major and minor circles is rational the curve is closed (the trajectory is *periodic*). Otherwise, it is open (*aperiodic*).

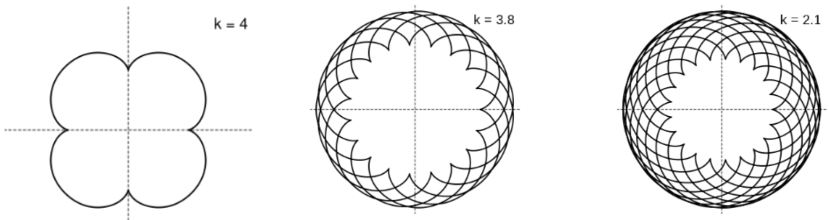

Figure 5. Epicycloids corresponding to different values of k

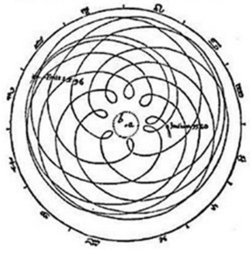

Figure 6. The aperiodic trajectory of Mars

In this figure, the complicated and aperiodic trajectory of Mars, drawn by Kepler using Tycho Brahe's data, for the model in which the Earth is still and at the center of the universe.

4.5 The dissemination of infinitesimal calculus in Italy

This proposal of classroom activity is related to the first half of the eighteenth century, and inspired by the research carried out by Luigi Pepe and Silvia Clara Roero, on the dissemination of infinitesimal calculus in Italy, as well as a treatise by Lagrange, which remained unknown until the 1980s and edited by myself. As is well known, in the October of 1684, Leibniz published, in the *Acta eruditorum*, his *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus*. This is traditionally considered as the official birth of infinitesimal calculus. In this short memoir with a long title, Leibniz introduced directly the rules of differentiation.

Almost twenty years before the publication of Leibniz's *Nova Methodus*, Newton had already formulated the fundamental elements of his calculus, based on the systematic use of series expansions. In works written in the 1670s but published only later, he presented the problem in terms of finding the relation between "fluxions" (that is, the velocity of variations) of given "fluent" quantities (i.e the variables). The first publication of Newton's results did not take place until 1687, under the title *Philosophiae naturalis Principia mathematica*, so shortly after the appearance of Leibniz's *Nova Methodus*. The bases of the calculus are contained in the first book, where some lemmas introduce the method in the form of "first and last ratios of evanescent quantities", and in the second book with the algorithms of differentiation. In the first edition of the *Principia*, Newton recognized in these the foundation of his own method as well as that of Leibniz, but in the third edition, this reference disappeared. In the meantime, the well-known controversy on the priority of discovery had developed, which involved and divided the mathematicians for decades.[27]

The dissemination of the infinitesimal calculus in Italy is characterized in a first period by an adherence to the Leibnizian method, favored by Leibniz's trip to Italy. This journey, started by Leibniz to reconstruct the origins of the House of Hanover of the Dukes of Brunswick-Lüneburg, for whom he served as librarian (and related to the Este princely family), was documented in his meetings with Italian mathematicians by André Robinet (2007).

The initial spread of infinitesimal calculus in Italy owes much to the many works written by Guido Grandi (1671-1742), one of Leibniz's correspondents, as well as to Gabriele Manfredi, the author of the first work on infinitesimal calculus ever published in Italy (*De constructione aequationum differentialium primi gradus*, Bologna, 1707). Other figures who also greatly contributed to this dissemination were Jacopo Riccati (1676-1754), for his studies on differential equations, and Maria Gaetana Agnesi (1718-1799), author of the first treatise devoted to a wider public of students and scholars: *Instituzioni Analitiche*, published in 1748.[28]

However, when it was necessary to establish "rigorous" theoretical bases, mathematics scholars turned to the Newtonian conceptualization of the "ultimate ratios of vanishing quantities". By the expression *ultima ratio* (*ultimate ratio* in the English translation) in the final scholium of Section I, Book I of *Principia*, Newton attempted to give a meaning to the ratio 0/0 which two variable quantities assume when they become equal to zero. A strong criticism of the foundations of both the Leibnizian infinitesimal calculus and the Newtonian theory of fluxions arose in Great Britain following the publication of a pamphlet by George Berkeley: *The Analyst: or, a Discourse addressed to an Infidel Mathematician...,* London 1734.

In Italy, Joseph-Louis Lagrange composed a treatise for the Artillery Schools of Turin in which he outlines his personal conception of the founda-

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[27] More details in (Giusti 2007), and useful teaching materials in the related exhibition: web.math.unifi.it/archimede/archimede_NEW_inglese/mostra_calcolo/pannelli/3.html

[28] The reconstruction and documentation of this phase of the Leibnizian calculus in Italy, through correspondences and unedited documents, in (Pepe 1981) and (Mazzone-Roero 1997).

tions of infinitesimal analysis deriving from the Newtonian theory of fluxions and series: *Principi di Analisi Sublime* (1754~).[29]

An attempt to found the calculus on purely algebraic basis, with no assumption of motion or consideration of compound quantities of infinitely small parts, was developed by John Landen, who also influenced Lagrange's later research on this matter: J. Landen, *A Discourse Concerning the Residual Analysis: A New Branch of the Algebraic Art*, London, Nourse, 1758.

Turning to primary sources of the Italian scientific panorama, to compare the two different approaches, students can start from the treatise by Agnesi (*Instituzioni Analitiche*, 1748), read some passages on the study of curves, and others with the rules of calculus, and draw some curves with GeoGebra (in particular the "Versiera" attributed to her). Then examine some passages from the *Principi di Analisi Sublime* (1754~), by Lagrange in which differential and integral calculus is introduced starting from the calculus of differences and finite sums, according to a definition that goes back to Newton.

The rules of differential calculus are preceded by the algebraic calculus of finite differences:

$$\Delta(x+y) = \Delta x + \Delta y$$

$$\Delta(x \cdot y) = x\Delta y + y\Delta x + \Delta x \cdot \Delta y$$

$$\Delta\left(\frac{x}{y}\right) = \frac{y\Delta x - x\Delta y}{y(y+\Delta y)}$$

$$\Delta x^m = \Delta x[(x+\Delta x)^{m-1} + x(x+\Delta x)^{m-2} + x^2(x+\Delta x)^{m-3} + \cdots + x^{m-1}]$$

$$\Delta \sqrt[m]{x} = \sqrt[m]{x} - \sqrt[m]{x+\Delta x} =$$

$$= \frac{\Delta x}{\left(\sqrt[m]{x+\Delta x}\right)^{m-1} + \sqrt[m]{x}\left(\sqrt[m]{x+\Delta x}\right)^{m-2} + \left(\sqrt[m]{x}\right)^2\left(\sqrt[m]{x+\Delta x}\right)^{m-3} + \cdots + \left(\sqrt[m]{x}\right)^{m-1}}$$

The differential calculus is then introduced in accordance with Newtonian conceptualization as: "the calculus of the ultimate ratios of the differences, i.e. the ultimate terms, to which the general ratios of the differences continually approach, while these continually decrease".

[29] This treatise, which remained in private hands until the 1950s, was published for the first time only some years ago (Borgato 1987).

The resort to infinitesimals, just as in the Leibnitzian tradition, is instead accurately avoided, and is introduced only at the end of the demonstration as a useful tool to simplify the calculus.

The integral calculus, instead of being introduced as the inverse of differentiation (as in Agnesi) is derived from the rules of summation:

$$\int x\Delta x = \frac{x^2 - x\Delta x}{2}$$

$$\int x^2\Delta x = \frac{x^3}{3} - \frac{x^2\Delta x}{2} + \frac{x(\Delta x)^2}{6}$$

Then the integral is presented in geometrical terms clearly inspired by Newton (cfr. *Principia*, Book I, Section I, Lemma II).

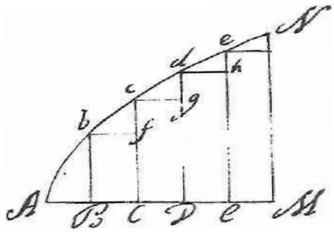

Figure 7. The integral presented in geometrical terms

4.6 The science of waters as the main field of applied mathematics

In Italy there is a solid tradition of hydraulic studies in all eras, and mathematicians had an extremely important role as theoretical and practical hydraulic experts in territorial planning. Leonardo da Vinci designed some of the most important hydraulic works of the 16th century in Lombardy and France, Bombelli was an expert hydraulic engineer who worked in Bologna and Rome. Galileo, Castelli, Torricelli, Cassini, Guglielmini, Grandi, Manfredi, Poleni, Zendrini all wrote about hydraulics and were used as technicians in the different states into which Italy was divided.

The territory of our peninsula underwent many hydraulic works over the centuries, which greatly modified its structure. Exploitation of the waterways, important not only for irrigation, but also as channels of communication and commerce, caused controversy among the states and involved works, at times damaging, on the course of rivers, which protected the interests of some to the detriment of others.

Some examples of century-old hydraulic problems and interventions include: the Lagoon of Venice and the deviation of rivers that flowed into it, floods of Po River, Adige River and the entire Po Valley hydraulic system, the convergence of the River Reno into the River Po, vast extensions of marshlands (Valli Grandi Veronesi, Paludi Pontine, …), conflict for irrigation (for the use of Tartaro River, between Mantua and Verona…) and so on.

Figure 8. Benedetto Castelli. *Della misura delle acque correnti*, In Roma, nella Stamperia Camerale, 1628, frontispiece

Some historians of mathematics such as Luigi Pepe, Alessandra Fiocca and myself have devoted part of their research works to the science of hydraulics in Italy. Maria Giulia Lugaresi has also made in-depth studies on the theoretical aspects of the science of water, or rather the laws which have been formulated for the study of fluid mechanics in ideal conditions, on the one hand, and, on the other, the empirical laws and instruments of measurement which have been devised for practical hydraulic works (Lugaresi 2015). In this field, too, many inedited works taken from the abundance of documentation in the Archives of Venice, Rome, Milan, Paris and Vienna have been published.

Figure 9. A manuscript map with details of the Po River and the Panaro River, 1808 (State Archive of Ferrara).

In her workshop, Maria Giulia Lugaresi has designed a teaching sequence starting from a historical perspective of practical problems and direct citations to introduce students to basic hydrodynamic laws.

4.7 Re-launching Italian education and research after political unification

The unification of Italy was preceded by the simplification of the political framework operated in the Napoleonic period and by the impulse given to the sciences and to scientific teachings (Patergnani & Pepe, 2011). After political unity (1861), there was a strong resumption of mathematical research and education. On the one hand, a connection with the most advanced sectors of European mathematical research, on the other, a colossal commitment to the creation of a national education system.

As for the teaching of mathematics at secondary school level, there was, as in the rest of Europe, a return to synthetic geometry, without the admixture of algebra. The original Euclidean text was initially revived but later on new texts were produced and a new didactic proposal emerged to blend plane and solid geometry (*fusionism*) and introduce new results in elementary geometry.

The 1867 ministerial programs written by Betti and Brioschi made Euclid's *Elements* compulsory in the teaching of geometry in gymnasiums and lyceums. It was also Betti and Brioschi who edited the famous publication of the *Elements* the following year, which was based on the seventeenth century translation by Vincenzo Viviani with notes by Luigi Cremona, and an appendix containing Archimedean results on cylinders, cones and spheres. Following the attempt to re-propose Euclid's text directly in secondary schools, there appeared new presentations of elementary geometry, always however within a purist approach, edited by some of the leading mathematicians of the end of the nineteenth century. These texts, besides their teaching aims, were guided by a concern to maintain the utmost rigor, since studies on the foundations of geometry had revealed the incompleteness of Euclid's system of axioms (Giacardi, 2006). The phenomenon of *fusionism* arrived in Italy later than other European countries, and dates back to the publication of the treatise by Riccardo De Paolis in 1884: *Elementi di Geometria* (Loescher, Turin). More than a scholastic text, it is a treatise on fusionism and the foundations of geometry. The text by De Paolis was addressed to first grade secondary schools, but without detracting from its rigor and completeness, it was too difficult for the students and too innovative for some teachers. The *Elementi di Geometria* by Giulio Lazzeri and Anselmo Bassani, published firstly in 1891, and then in 1898, was more successful. They wrote a book which was better suited to students as it was the result of direct experimentation, and was used as a textbook in various secondary schools (Borgato 2016). The structure of the book is the same as that of De Paolis, with a few variations. The postulates are divided into twelve groups, with an additional group regarding points, lines and surfaces. It is divided into five books. What is original was the introduction of the theory of radical axes and planes, and the theory of the homothetic figures, free from the theories of equivalence and proportions. This text formed the basis of some experimentations in various schools and also had an influence on other texts destined for use in secondary schools. A heated debate arose within Mathesis, the Italian association of mathematics and physics teachers founded in 1895, on the didactic value or otherwise of fusionism.

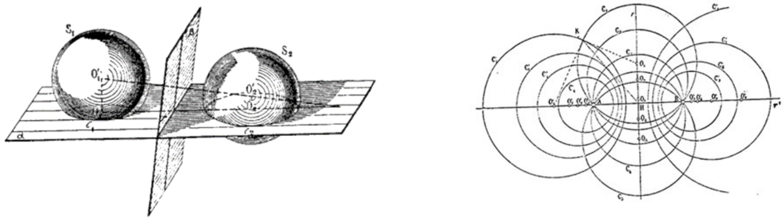

Figure 10. Lazzeri & Bassani, Elementi di Geometria, Livorno, 1891, pp. 188, 192

I do not want to go further on this theme, studied by Livia Giacardi and Elisabetta Ulivi as well as by myself. More recently, Marta Menghini has also made some contributions. In the laboratory proposed by Elena Lazzari, the starting point is one of the most modern parts of the treatise by Lazzeri and Bassani, who wanted to include, from a fusionist point of view, new results in elementary geometry resulting from the research of Poncelet, Moebius, Gergonne, Klein… of that century. Elena focuses on the theory of circles and spheres, that is, of the infinite systems of these entities, in which the fusionist method allows us to deduce plane theorems from the three-dimensional theorems in a more evident and direct way.

4.8 The geometry of paper folding and the resolution of problems of third degree

The last laboratory proposal concerns elementary geometry of the twentieth century. This research originates from an exhibition I organized on Women and Mathematics in Italy, which highlighted some aspects of Margherita Beloch's teaching activity in Ferrara. At the same time in Rome, other research was being carried out by Paola Magrone on this professor of descriptive and projective geometry, who also linked her name to photogrammetry and roentgen photogrammetry, the forerunner of CT (computed tomography). Beloch also made an interesting contribution to the geometry of paper folding.

This geometry was introduced by the Indian mathematician Tandalam Sundara Row in 1893 and was favourably received in the West particularly following its appraisal by Felix Klein. In this geometry, the use of the ruler and the compass is replaced by the folding of paper, and Sundara Row makes use of five fundamental folds to obtain all the results of Euclidean geometry and some results of the geometry of algebraic and transcendent curves.

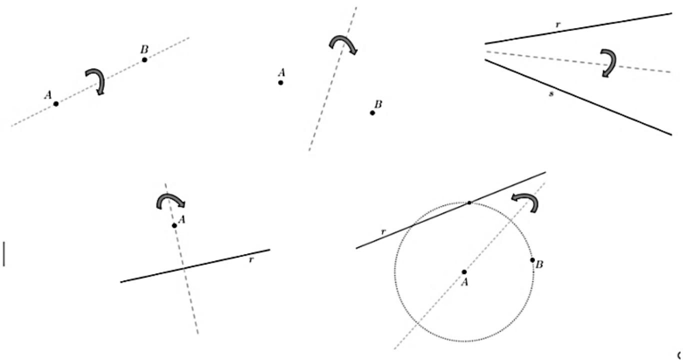

Figure 11. Sundara Row's basic folds

Beloch introduced a new folding that made it possible to solve classic problems impossible with ruler and compass, such as the trisection of an angle or the duplication of a cube. It corresponds to the possibility to find, by folding, the common tangent to two parabolas (Magrone &Talamanca 2018; Borgato & Salmi 2018).

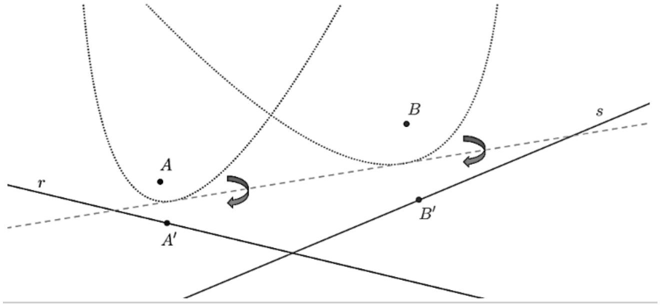

Figure 12. Margherita Beloch's new fold

In the laboratory proposed by Paola Magrone we find some applications of what is now called *origami geometry* in the construction of curves, and also the general Beloch method to solve all the geometric problems of third degree.

5 *Conclusions*

Even in the limited space of this conference, I have tried to give a fairly broad picture of the resources offered by the history of mathematics in Italy to build educational paths starting from the original sources. I based myself on topics accessible to secondary school students, and mainly on materials of my own direct investigation. I have also tried to include different historical periods and different mathematical disciplines, favoring interdisciplinarity. I hope that my contribution will serve to foster collaboration between teachers, educators and historians of mathematics in relation to the teaching and learning of this discipline, designing new proposals which are inspired by the most current research in an attempt to offer original ideas. This collaboration also guarantees the necessary competence to build quality teaching materials and to avoid superficiality and misconceptions.

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