A HIDDEN THREAD: IDEAS AND PROPOSALS ON CHILDRENS' MATHEMATICS EDUCATION THROUGHOUT HISTORY

Ana MILLÁN GASCA
Roma Tre University, via del Castro Pretorio 20, Rome, Italy
anamaria.millangasca@uniroma3.it

ABSTRACT

Dipoi imprendano l'abaco e quanto sia utile geometria; le quali due sono scienze atte e piacevoli agli ingegni fanciulleschi, e in ogni uso ed età non poco utili saperle. Poi ritornino a' poeti, agli oratori, a' filosofi ...
Leon Battista Alberti, *I libri della famiglia* (1433-1441)[2]

Children are virtually mute in the sources of history.
Willem Frijhoff (2012)

1 Beyond the history of teaching arithmetic: mathematics for children 1800-1950

The first steps in mathematics of the child in primary school and preschool are today an issue of considerable concern of parents, teachers, and education researchers all over the world. Educational overall methods and specific recipes are on the market in many countries, with national and cultural specificities.

This concern emerged distinctly in Europe around 1800 and for two hundred years it pushed the conception of many educational ideas regarding mathematics for young children as well the marketing of an array of proposals

[2] Then they should take up the abacus [written arithmetic] and what is useful in geometry; which two are suitable and pleasant sciences for childish minds, and in every use and age it's not a little useful to know them. Then they return to poets, orators, philosophers ... [Eugenio Garin quotes this passage in his *Education in Europe 1400-1600*, Garin 1976, p. 143]

– textbooks addressed to adults (schoolteachers or parents) or to pupils, picture books, toys and educational aids. Contributions were put forward in several countries and languages, which expressed civil commitment, showed trust the child's early "elective affinity" with number and form, and displayed the authors' creativity in devising methods and designing materials. Several among the innovators – starting from the late 18th century – deployed their proposals in close alliance with the publishers and manufacturers of books and other gadgets for children, who aimed at potential relevant sales in the growing childhood-oriented market. At the turn of the 20th century there was a golden age of reflection on mathematics and children, with the contribution of several women.

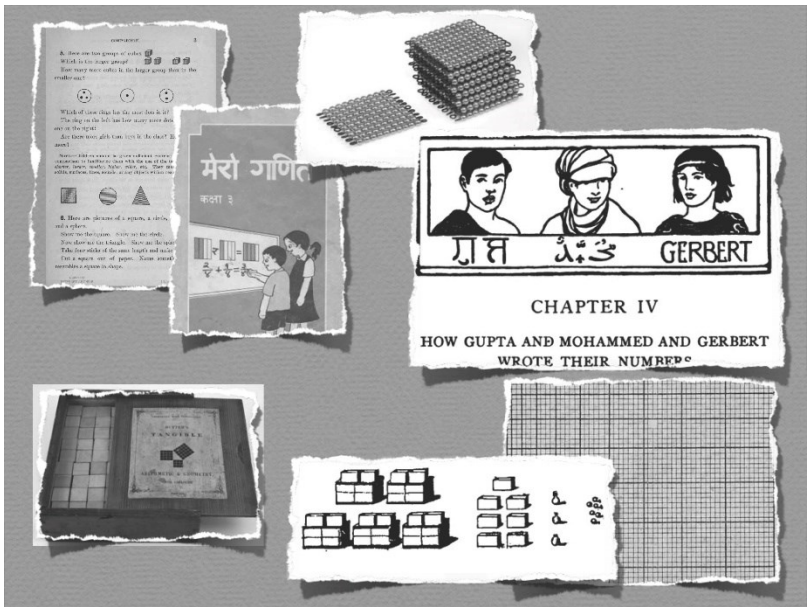


Figure 1. Objects and books display the increasing attention to children as mathematical learners, with a playful dimension of enjoyment. Some examples from the 19th-20th centuries: educational aids and toys, children's books with illustrations, books for educators, primary school textbooks, conveying new views on elementary mathematics for early education and approaches.

The issue of children's first steps in mathematics throughout history belongs to a certain extent to the history of teaching arithmetic,[3] as the teaching of oral numeration, measures, and written symbols for quantities (numerals) has always been (until nowadays) the core of the early contacts of children with mathematical knowledge. The tradition with which the innovators in the Late Modern age were confronted focused on numeracy.

On the other hand, innovators did not restrict themselves to written arithmetic, but paid attention to oral counting and calculation and (most importantly) considered also geometry as suitable for a smooth and enjoyable – at the same time, more far-reaching – contact with mathematics. Pure reckoning was insufficient to meet the ambition, in certain sectors of European intellectuality, to offer a generalist education to all, particularly in primary school, a school level that emerged as a school of the people.

The growing sensitivity and understanding of the world of children, especially in early and middle childhood (up to age 10), led several men and women to seek new educational solutions. To *improve* (methods of teaching), to *innovate* (contents and methods), to *adapt* (to children's minds and sentiments) have been keywords all along this evolution. Innovators were confronted with a *tradition* inherited from the past that conflicted with *the new*: with children themselves, "new to the world" as Hannah Arendt writes,[4] as well as with the very conditions of an ever-changing world.

Since the late 18th century, the first contact of children with number and form happened in a complex educational universe including both schooling in institutions of various kinds (including infant schools) and domestic learning with tutors or parents; from a social point of view, it included both males from affluent families before attending secondary school (where they would further study Euclid), and the children of low classes in charity or popular schools. An audience continuously growing to include the poor, the girls, and even disabled children, which was to merge in unified state systems of compulsory and universal primary education.

Let's consider briefly some aspects of the *status quo* that innovators focusing on young children had to face. Next, I introduce some examples of ideas and proposals in Great Britain around 1800, before considering some general

[3] See Bjarnadóttir 2014 for the state of the art on the history of teaching arithmetic.

[4] Arendt 1961.

trends emerging from available studies. We shall see that ideas and proposals often aimed at developing a genuine "mathematics for children".

2 Children learn to reckon: contexts and contents

Starting from the pioneering contribution by Augustus De Morgan (1806-1871), the concern with and attention to the teaching of arithmetic to the young has been a powerful impulse to explore the historical roots of current practices and contents of arithmetic in elementary education (that involving minors). Around the turn of the 20th century historical and bibliographical research made it possible to identify the persistence, in the study programs of both secondary and primary schools, of the contents transmitted by what was identified as an European successful printed book genre in its own right during the XVI-XVII centuries, the commercial elementary arithmetics.[5]

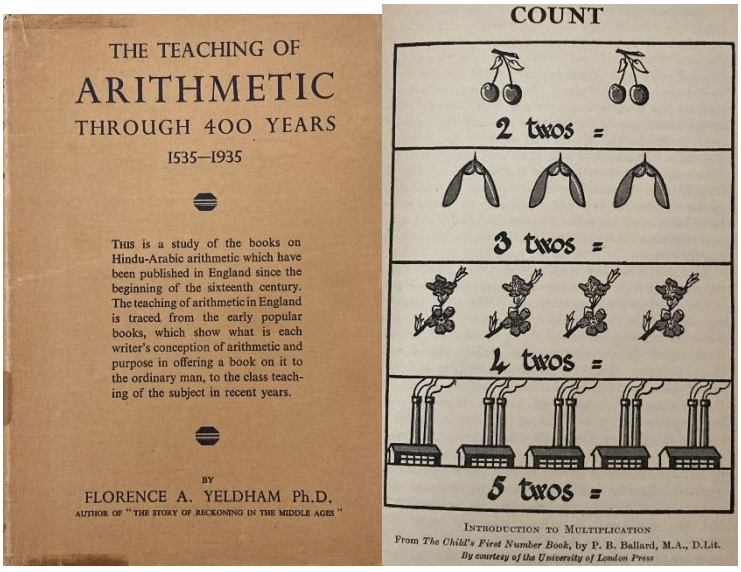


Figure 2. (left) Cover Florence A. Yeldham's (1877-1945) essay on English arithmetic books, a companion to her book on reckoning in the Middle Ages (London/Bombay/Sidney: George G. Harrap &Co, 1936). She made wide use in it of De Morgan's *Arithmetical books* (1847). (right) The reproduction of an illustrated page on multiplication using numerals, number words and pictures from *The child's first*

[5] See Schubring 2014. Many contributions were published in Germany and the USA, countries where the extension of literacy was ahead, as underlined by Cipolla (1969).

number book (2nd ed. 1933) by Philip B. Ballard (1865-1950), included in the last chapter of Yeldham's essay, devoted to "Arithmetic in recent years": "Two other new movements, namely, the custom of sending children to school at a much earlier age than before and the better equipment for school buildings, are in great measure responsible for the important advance of modern years, the approach to the subject through practical work." (Yeldham 1936, p. 131)

Printed books were written and published with a view to adult buyer-readers, it was in fact "mathematics for sale"; let's read the synthetic reconstruction by Frank Swetz (1992):[6]

While commercial arithmetics emerged as a mathematical genre in the late fifteenth century and throughout the sixteenth century, their influence prevailed in the teaching of arithmetic up until the beginning of the twentieth century. Their content, format and instructional style, with a great reliance on the use of problems, set the standard for arithmetic teaching for centuries. The impact of the commercial texts had raised the popular understanding of arithmetic to a new level; [...] it became the basis for accessible, lucrative careers. Quite simply, arithmetic became noticeably useful. Unfortunately, in a sense, this usefulness focused it as a kind of vocational training. [..] Throughout the seventeenth and into the eighteenth century, arithmetic books grew larger in content and became more comprehensive but still closely resembled their fifteenth and sixteenth century ancestors.

Thus, a long-standing tradition was at hand for any family or teacher (both tutoring at home or in schools) facing the first steps with numbers of a child in the 18th-19th centuries. This European tradition was conserved and transmitted by means of arithmetic books; its vitality can be linked to some trends in European cultural, social and economic evolution: the diffusion of literacy, the development of national written cultures in vernacular languages; commercial and manufacturing development in conditions of relative freedom, including the book market pushed by the increasing number of consumers.

As for contents, basic arithmetic started from the reading and writing of natural numbers using the decimal numeration system (with digits 1, 2,.... , 8, 9 and the use of 0), measures, and then the operations and its use in the solution of problems consisting of short quantitative scenarios of practical daylife

[6] These contributions, encouraged by the aim to improve teaching, are among the first in the history of mathematics education.

and work life, also with "broken" numbers, culminating with proportion for coin change and measure conversion, as well as distribution of profits, payments and so on (see Bjarnadóttir 2014, p. 434).

Content and methods can be been explored mainly through arithmetic books, and through other sources (manuscript notebooks, autobiographical writing), as well as through the criticisms and alternative proposals of the innovators. As for methods, they appear to be based more on practice and remembering than on understanding. For example, John Denniss (2009), considering both arithmetic books and pupils' notebooks in the case of Great Britain, writes:

Both Recorde and Newton had long ago pleaded for more imaginative teaching of arithmetic, based on understanding rather than rote learning. At the end of the 19th century such ideals had still barely begun to be put in practice.

The lack of a "more imaginative teaching" could potentially have a greater negative impact as the age of the learners/pupils decreased. Now, to what degree and in which modalities were young children granted numeracy? Institutionalization of schooling needs to be put in the picture, considering both the early stages of secondary education and the schools for the people (popular schools and vocational schools). It is one of those aspects of daily life of the past which risks going unnoticed by historical studies, hidden from attention by issues of a wider nature, within such fields as childhood history, the history of literacy or the history of elementary education. And yet, even scattered examples reveal a world of practices, of feelings, of meanings and expectations linked to the first steps of children in an adult world of number and measure.[7]

Medieval reckoning schools were a turning point, as their (vocational) training was addressed to children. In the pioneer essay by Carlo Cipolla, *Lit-*

[7] There is an ancient history also regarding children and mathematics. For example, Werner Jaeger has pointed out that Plato, in his last work, *Laws*, writes about the learning of number and measure in childhood by play and with pleasure (in his 1944 essay *Paideia*). Henri Marrou depicted lively scenes of the introduction to numeration of children in his *History of education in antiquity* (1948), and has brought up that Augustin, in his *Confessions* (I, 13), remembers "the odious song: one and one, two; two and two, four". Thus, Marrou stressed the struggle of young learners.

eracy and development in the West (1967), numeracy and specifically children's education as one aspect of literacy spreading is not a central issue. Yet, he quotes an episode dating back to the 14th century regarding the Italian reckoning teaching to the youngest: a document from the Italian town Lucca dated 1382 proclaimed that prosperity of citizens depended upon trade and it was thus indispensable that they learnt to read and reckon, and after four years city authorities put in practice their advice hiring a master so that "children would be taught arithmetic, so as to became wiser and shrewd in business."[8] Yet the fact that the schools devoted to the teaching of the new Hindu-Arabic reckoning technique have as attendants children (not adults) has a relevance in itself in the history of children and mathematics in Europe. Raffaella Franci has paid attention to this circumstance:

The topics of the curriculum were finely divided into teaching units called [in Italian] muta (shift). Thus, for example, reading and writing numbers formed one pack, learning the multiplication tables formed another. The multiplication between integers was divided into several shifts according to the number of digits of the multiplier, and similarly the division relative to the digits of the divisor. The pupils, upon entering the school, were placed in a muta according to the level of their knowledge, they passed to the next wetsuit only when they demonstrated full mastery of the techniques taught in the previous one. The presence in the school lasted the whole day, the teaching was based on the repetition of numerous exercises, both written and oral. In addition to the numerous exercises done at school, homework was also provided. The teaching, aimed at learning the techniques rather than understanding the methods, was very repetitive, many practical rules were taught, above all to deal with the complicated system of coins, weights and measures. The techniques relating to commercial operations (companies, barters, merits, etc.) were taught by proposing problems of gradually increasing difficulty, in which an attempt was made to envisage all the cases that could have arisen in the effective exercise of the trade. [Franci 2000, p. 130-131]

[8] He quotes from a 1905 essay by Paolo Barsanti on public teaching at Lucca in the 14th-18th centuries.

Figure 3. A page from Filippo Calandri's (1468-1517/18) *Arithmetic*, manuscript conserved at the Florence library Biblioteca Riccardiana. Source: Ulivi 2002

Recent historical research has ascertained that the first European arithmetic printed books in turn stemmed from the content and teaching practices of medieval reckoning schools attended by children.[9] A working hypothesis is that method or "learning path" developed by Italian teachers (*maestri d'abaco*) would last for centuries, being extended as recipients both to self-taught

[9] See the overall presentation in Bjarnadóttir 2014. Historical research in past decades has thoroughly analyzed the origin of reckoning schools for trading professions in Italy in the late Middle ages: their relevance for the history of mathematics, for the history of accounting techniques, for European economic history, and even for the history of vocational education had been considered. Contents and teaching methods were developed by reckoning masters) in an extraordinarily fruitful cultural season, thanks to the confluence of social factors (the flourishing of urban centres); to the activation of economic life (manufacturing, commerce, finance); to the recovery of classical mathematical knowledge. Reckoning included learning the oral numeration of course, but the main focus was the written numeration of Indian origin that – after its diffusion in the Islam area – had reached Europe.

adults and (indirectly, through teachers) to children. Let's read the synthetic reconstruction by Swetz, who underlines the contribution of Pestalozzi:

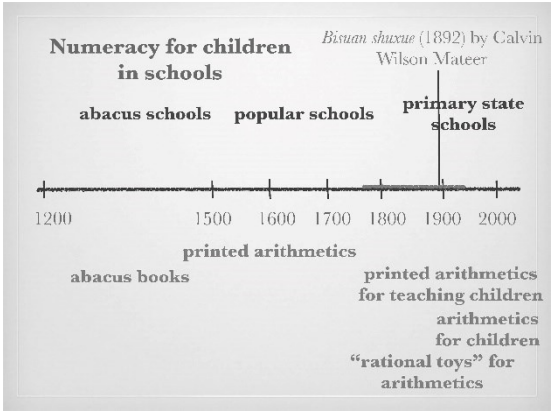
The applied arithmetic taught in secondary schools and private academies of the nineteenth century was, for all practical purposes, commercial arithmetic. When the educational reformer Johann Pestalozzi (1746-1827) established a curriculum for the education of young children, he drew on the subject content of the higher schools. Thus, the primary arithmetic studies he designed were heavily influenced by commercial arithmetic and monetary transactions. This overriding influence remained until reforms early in the twentieth century refocused education on the child as an individual with æsthetic and intellectual need beyond mere vocational training. Although arithmetic today is taught as a subject from which a child may draw many meanings and uses, its teaching had been irrevocably shaped by the commercial needs of the early Renaissance. [Swetz 1992, p. 377]

Now, Pestalozzi's curriculum for the educational establishments founded and managed by him was not only a direct adaptation of the subject content of gymnasia or collèges and the same. Early arithmetical training for children of higher classes could be part of the initial curriculum in secondary schools; besides, in modern Europe, arithmetic was combined with reading and writing in popular schools, created and maintained by charitable privates or towns or religious groups. And Pestalozzi's initiatives were originally intended for orphans and children of the working class (Horlacher 2011).

For example, the Catholic area, in the Piarist schools of José de Calasanz (1557-1648) the children were taught free of charge reading, writing, arithmetic and Catholic doctrine. Moreover, the idea of "readiness" of very young children, as put forward in the Protestant area by Comenius (Jan Amos Komensky, 1592-1670), included mathematics: he envisaged in his *Didactica magna* a mother school (the first six years of a child's life) where the seeds of arithmetic will be planted if the child understand what is meant by "much" and "little," can count up to ten, can see that three are more than two, and that one added to three makes four" [...] He will possess the elements of geometry if he know what we mean by "large" and "small," "long" and "short," "broad"

and “narrow,” “thick” and “thin”; what we signify by a line, a cross, or a circle, and how we measure objects (in different length measures).[10]

Swetz pointed out that possibly both the persistence of the educational practical goals (commercial/arts and crafts) needs and the fossilizing of an emphasis on procedures, receipts, and formulae hindered the exploration of a first approach to number – to mathematics as a whole – from a “æsthetic and intellectual” encounter, emphasizing conceptual clues (such as decomposition, equality, comparison, ratio), problems as challenges, beauty enclosed by regularities in mathematical properties and patterns, and the pleasure that could be obtained from number and form. Exploring precisely this new dimension in the encounter of young children with mathematics was a fascinating task in front of several scholars who shared their commitment to education and the nearness to the child's world. Their work was developed in the decades in which the goal of a state universal, free of charges primary school spread in many countries in the world. While the Modern age brought both the *sentiment de l'enfant* (Ariès 1960) and popular schools, mass education and the schooling of young children during the 19th century prompted coping with the need to adapt the arithmetic curriculum and training methods to the specificities of children's minds.



[10] Comenius 1896 (1657), p. 426. Besides, the elements of statics will have been learned if the children see objects weighed in scales or acquire the power of telling the approximate weight of objects by weighing them in their hands.

Figure 4. A tentative time line regarding educational contexts and the literary tradition handing down content. The timeline does not suggest continuity in aims, but different contexts which should be taken into account. An example is the textbook *Bisuan shuxue* (1892) on pen-calculation arithmetic by the U. S. A. missionary Calvin Wilson Mateer, the "dawn of modern Chinese mathematics education" (Ma 2010, pp. xiv-xv; see Fisher 1911, Dauben 2003).

3 "Making minor mathematicians": a glimpse at some early contributions

The following dialogue between a 5 year old child and his father appears in the first pages of the lengthy treatise *Practical education* (1798) by Maria Edgeworth and her father Richard Lovell Edgeworth. This essay is an early example of the aim of developing a genuine science of education in Europe. It offers us a shot of a young child approaching numbers at home in late 18th century Great Britain:[11]

We will give you an instance: arithmetic is one of the first things that we attempt to teach children. In the following dialogue, which passed between a boy of five years old and his father, we may observe that till the child followed his father's train of ideas he could not be taught.

Father. S —, how many can you take from one?

S—. None.

Father. None! Think; what can you take nothing from one?

S—. None, except that one.

Father. Except! Then you can take one from one?

S—. Yes, that one.

Father. How many then can you take from one?

S—. One.

Father. Very true; but now, can you take two from one?

S—. Yes, if they were figures I could, with a rubber-out. (This child had frequently sums written for him with a black lead pencil, and he used to rub out his figures when they were wrong with Indian rubber, which he had heard called rubber-out)

Father. Yes, you could; but now we will not talk of figures, we will talk of things. There may be one horse or two horses, or one man or two men.

S—. Yes, or one coat or two coats.

[11] Edgeworth and Edgeworth 1798, pp. 58-59. On Maria Edgeworth, see Fantaccini, L.,& Leproni, R. (Eds.) (2019).

Father. Yes, or one thing or two things, no matter what they are. Now, could you take two things from one thing?

S—. Yes, if there were three things I could take away two things, and leave one.

This narrative is presented by the Edgeworths as one of their first examples of the need to enter in the cognitive and sentimental sphere of the child, and it regards precisely arithmetic. The scene happens in a full oral framework, using (counting) number words and expressions such as "take from", "how many", together with objects and the hand gesture of rubbing out. Yet a reference is made to written arithmetic: figures and sums. The Edgeworths, as any other educated European, were heirs to a consolidated tradition of training in reckoning. The time was ripe, however, for taking account of a child's encounter with numbers in its specific characteristics. For example, through the awareness on the part of the teachers of the fact that it was a question of acquiring a technique carried out in writing, but nevertheless learning passed through a relationship with the pupil where the spoken words and the body (the hand, the eye, motion, and rhythm) played a crucial role.

In the Edgeworths' essay, moreover, the chapter on arithmetic is followed by one on geometry, and both contained references to the "rational games" mentioned in the chapter *Toys*, that opens the 2 volume-work with a critique of expensive and fashionable play objects for children:

The first toys for infants should be merely such things as may be grasped without danger, and which might, by the difference of their sizes, invite comparison: round ivory or wooden sticks should be put into their little hands; by degrees they will learn to lift them to their mouths, and they will distinguish their sizes: square and circular bits of wood, balls, cubes, and triangles, with holes of different sizes made in them, to admit the sticks, should be their playthings. No greater apparatus is necessary for the amusement of the first months of an infant's life. [...] To gratify the eye with glittering objects, if this be necessary, may be done with more safety by toys of tin and polished iron, a common steel button is a more desirable plaything to a young child then many expensive toys; a few such buttons tied together, so as to prevent any danger of their being swallowed, would continue for some time a source of amusement.

A contemporary of the Edgeworths, Ellenor Fenn (1744-1813), an author of readings for children who published her works – under pseudonyms – with John Marshall and Newbery, promoted what she called "teaching in sport",

and for this purpose she designed a «Set of toys», produced in the 1790s by Marshall, consisting of three wooden boxes or trays included in a large box, each of them divided into compartments containing a variety of materials in cardboard or wood, dedicated respectively to grammar, spelling and figures. In the Figure Box, compartments presented a purse of counters, a sack with beans for merchandise, and a set of cards for multiplication (pairs of cards with cuts of animals which when matched together produce the entries multiplication tables) published by J. Aldis in London.[12]

Fenn's boxes are an early example of educational aids which sharply marks the distinction between self-taught adults and children, together with the books to be directly used by children, such as *Marmaduke multiply's merry method of making minor mathematicians*, published by John Harris, around 1816, using of pleasant story-images as *aide-mémoire* for numerical facts (Fig. 1).

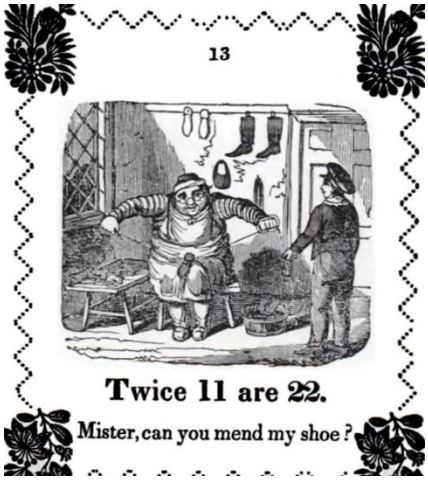


Figure 5. A page of the booklet *Marmaduke multiply's merry method of making minor mathematicians* (see Denniss 2009).

The possibilities of making money on children's goods in Britain at the turn of the 1800s gave wings to the creativity of authors in alliance with publishers. Consider the box «Tangible arithmetic & geometry» designed and sold by Henry Butter, author of primers for spelling and reading (Fig. 6).

[12] Fossa 2021; on Fenn see Stocker 2007.

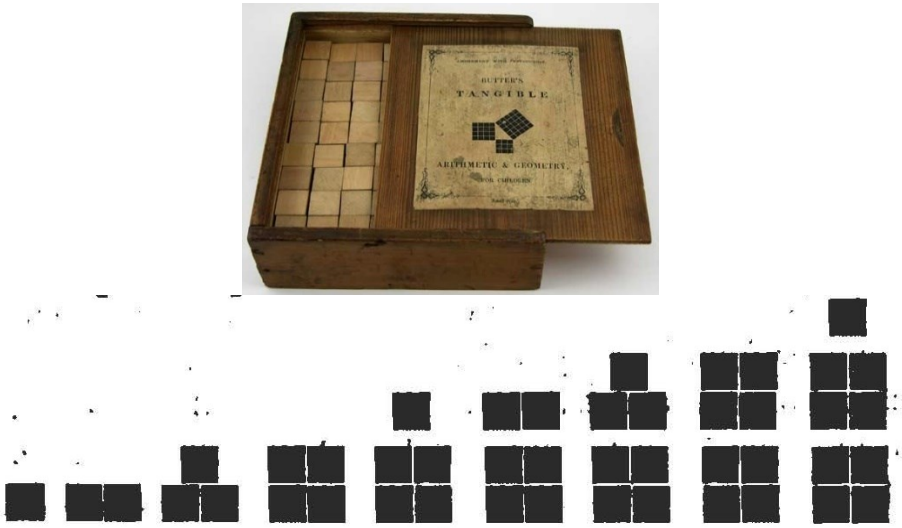


Figure 6. The box of cubes Tangible arithmetic and geometry by Henry Butter.

Explanations were presented in a booklet (35 pp, with some end pages of advertising of other works by Butter) *Tangible arithmetic & geometry: The most effectual method of teaching addition, multiplication, subtraction, and division, and the analysis and composition of numbers; also the formation of squares, parallelograms, triangles, rhomboids, cubes, parallelopipeds, prisms, pyramids, &c. It will likewise convey, with ease and certainty, a far more accurate knowledge of fractions than has been attainable by any means hitherto devised. Illustrated by figures, and by one hundred and forty-four cubes in a box. Forming a Permanent fund of amusement and instruction for all ages*, London (printed by J. S. Hodson). The booklet was illustrated with geometric drawings (squares representing cubes). The introduction opens with the following recommendation:

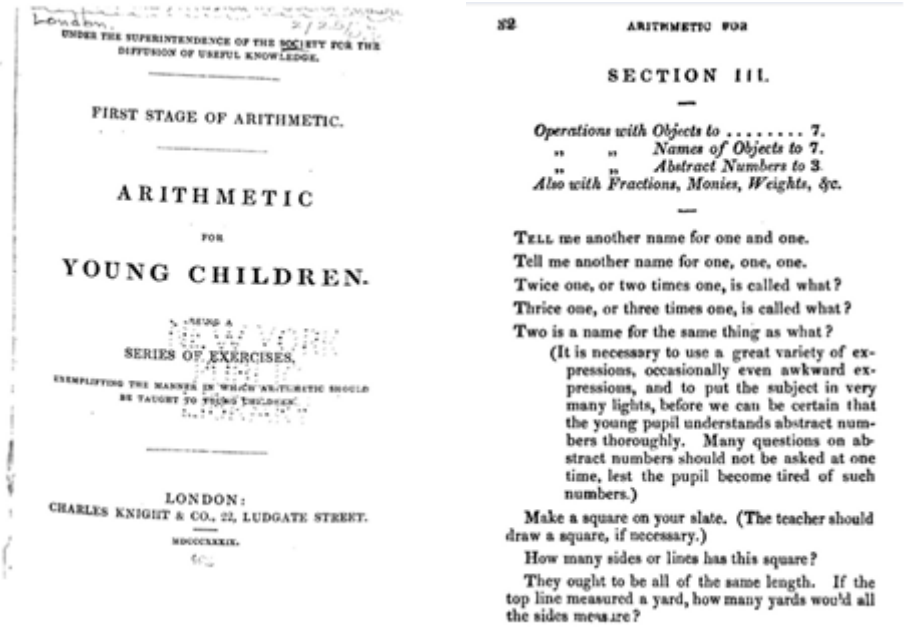
"The Author strongly recommends that young children should not be required to learn the numerical characters till they are very familiar with numbers, for which those characters are mere arbitrary signs. Instead of which he begs to suggest that for Addition, Subtraction &c., they should be accustomed to place the cubes in regular groups, for the various numbers up to nine, so that they may at a glance see what number of cubes are in a group, without having to count them."

Imagination, narration, and play were introduced in mathematics thanks to drawings, stories, counters and dice, sticks and cubes to compare sizes, for composition and decomposition to be linked to numerical operations. This was part of the general development of toys and children's books, which has received increasing attention in recent years. Educational aids for learning mathematics would become a kind of rational toys, and a flourishing commercial production until our days.

The books published by Horace Grant (1801-1859) represent both the focus on young children and the fact that the extension of early mathematics instruction to working-class children also fostered innovations (Calabrese 2015). In fact, he belonged to the radical milieu which in Great Britain sought to obtain a law establishing public primary education, and which founded (in 1826) the Society for the Diffusion of Useful Knowledge. While De Morgan wrote treatises on mathematics for the Society, Grant was entrusted with a «Library for the young children». In a biographical note published in 1861, Edwin Chadwick wrote: "[he] will be found worthy to be classified with Comenius, Pestalozzi, the Abbé Gaultier, and the Abbé Girard, men of great humanity and eminent ability, who devoted themselves zealously to the special study of the minds of children, and to the best means of cultivating them" [Chadwick 1861, p. v].

The child's first steps in oral counting were for Grant a moment to be seized with care, through a shelling of deliveries and small problems, between observation and imagination, between sounds, movements and small problems. The experience with children of various social backgrounds led him to an overall cultural project composed of three works: *Arithmetic for young children* (1835), *Exercises for the improvement of senses* (1835), which contains many exercises in geometry with objects such as wooden cubes or through measurements, and an unpublished work, *Course of exercises in the first elements of form, adapted for the use of mothers and early teachers (Geometry for children)*.[13]

[13] He also published *Drawing for young children* (1838), *Second stage of arithmetic for schools and families* (1841) and *The elements of practical geometry for schools and workmen* (1852) (Calabrese 2018)



UNDER THE SUPERINTENDENCE OF THE SOCIETY FOR THE DIFFUSION OF USEFUL KNOWLEDGE.

FIRST STAGE OF ARITHMETIC.

ARITHMETIC

FOR

YOUNG CHILDREN.

SERIES OF EXERCISES,

LONDON:
CHARLES KNIGHT & CO., 22, LUDGATE STREET.

82 ARITHMETIC FOR

SECTION III.

Operations with Objects to ..... 7.
" " *Names of Objects to* 7.
" " *Abstract Numbers to* 3.
Also with Fractions, Monies, Weights, &c.

TELL me another name for one and one.

Tell me another name for one, one, one.

Twice one, or two times one, is called what?

Thrice one, or three times one, is called what?

Two is a name for the same thing as what?

(It is necessary to use a great variety of expressions, occasionally even awkward expressions, and to put the subject in very many lights, before we can be certain that the young pupil understands abstract numbers thoroughly. Many questions on abstract numbers should not be asked at one time, lest the pupil become tired of such numbers.)

Make a square on your slate. (The teacher should draw a square, if necessary.)

How many sides or lines has this square?

They ought to be all of the same length. If the top line measured a yard, how many yards would all the sides measure?

Figure 7. Cover page and sample page of *Arithmetic for young children* (1835), published under the superintendence of the Society for the Diffusion of Useful Knowledge by the publisher Charles Knight. The dialogue form was present in arithmetic books, but in this case the purpose is to exemplify how to introduce number to the young child through graduated questions using number words, involving a variety of examples such as the body, real life objects, geometrical figures, coins and measures. Figures are introduced in the final pages. Educational comments are included in between questions. Thus, before the production of books for children on number, educational books were published with a focus not only on contents, but also on what we would call teaching methodology.

4 The traces of the past in our present: researching ideas and proposals regarding mathematics and children.

A research program has been developed on the subject of this paper in the Department of Education of Roma Tre University, leading to several Master theses in history of mathematics education regarding single authors. The research was tightly connected with the development of courses and workshops in mathematics and mathematics education for the preschool/primary school education degree (ca. 170 teaching hours in the first and third years, following the Italian 2011 state regulations). Historical research work was coupled with the design of mathematical activities for children inspired by ideas and proposals, which were tested in practice during the traineeship in preschools and primary schools.

The issue of the relationship between methods and conceptions presented in university courses and "traditional" practices in school is deeply felt among students, some of them actually already working in schools with temporary jobs. Trying to avoid dogmatic blame on school habits, historical perspective appeared potential useful. Some examples analyzed in single master thesis were presented to students in experiential workshops, where the historical case was coupled with the discussion of school activities inspired by the historical material.[14] Subsequently, a panorama of the evolution in history of contacts between children and mathematics, was introduced in the syllabus of the Roma Tre courses and practical lessons.[15] On the one hand, tracing the historical origin of present practices (such as the limit of 20 for natural numbers in 1st degree (6 years olds), the emphasis put on written algorithms and

[14] Those experiential workshops constantly arose strong engagement, self questioning and active participation of many students with comments and questions. One should take into account, besides and independently of the historical contents, the fact that listening to a course mate as well as storytelling, photographs and videos regarding actual school and children were important factors of success. Experiential workshops have been developed further in the context of the Erasmus + project ANFoMAM (www.unavarra.es/anfomam)

[15] Millán Gasca 2016, chapters 2 and 3, presents an overview of some trends among innovators, considering their interaction with practices and contents in the tradition of introduction to elementary arithmetic. The course and workshops include also an overview of the evolution of mathematics in history with some detail on the history of classic geometry, of numerations systems and of measure.

on conversion of measures in the decimal metric system, the lack of geometrical contents) was helpful to give meaning to a student's puzzlement or to their personal records. On the other hand, considering and comparing the work of innovators could help in analyzing contemporary proposals (including marketed ones); and in enhancing students' skills in designing activities.

The research program on ideas and proposals on mathematics for children in 18th-20th centuries started from this remark on Pestalozzi and Friedrich Fröbel (1782-1852) in a reflection by the Italian mathematician Federigo Enriques (1871-1946) in a chapter devoted to teaching in his 1938 essay *Mathematics in history and culture* induced to investigate early childhood and popular elementary schools:

The educational value of mathematics is revealed [...] also in the first grades of childhood and working class education; because mathematical intelligence is very precocious. Two pedagogues above all worked to bring mathematical knowledge into the education of the child, as an element of his intellectual development: Pestalozzi and Fröbel. The first teaches "as Geltrude teaches her children", showing them early on the awareness of the relationships of number and measure, which they must learn quickly and clearly. The second, already in his first gifts, in the games and exercises in his gardens, offers children a vision of geometric figures and their symmetries, and interests them in increasingly difficult observations, with a methodical progression that responds to a precise educational design. For infant schools as for popular schools, what was observed above is above all true, that the educational direction is not separated from the utilitarian, which creates the acceptance of the things taught with interest.

Moreover, the essays included in the 1988 edition of Condorcet's arithmetic book by Gert Schubring and Charles Coutel, and Renaud D'Enfert's discussion of Francoeur's linear drawing for mutual schools (D'Enfert 2014), showed that Pestalozzi's contribution should be considered in a wider context, instead of considering him essentially an isolated figure and focusing on his followers.[16] Moreover, was there a Pestalozzi's revival (Denniss 2009) in the late

[16] Such as José Mariano Vallejo (1770-1846) *Aritmética de niños para uso en las escuelas del reino* (1804) in Spain, Warren Colbourn (1793-1833) *An arithmetic on the plan of Pestalozzi* (1821) in the United States (Kirkpatrick), Hippolyte Léon Denizard Rivail (1804-1869) *Cours d'arithmétique pratique et théorique d'après la méthode de Pestalozzi* (1824) and Hip-

19th century? The attention to two books, Jean Macé's (1815-1894) children's book *Grand papa's arithmetic* (Colella 2012) and Charles Laisant's *Initiation mathématique* (Lamandé 2011, Schiopetti 2015), was suggested by a short essay published in 1907 by the Spanish mathematician Zoel García de Galdeano (1846-1924).[17] A PhD thesis at the University of Zaragoza on mathematics with children with Trisomy 21 included a historical analysis of Édouard Séguin's (1812-1880) geometrical educational aids and principles, with applications to the design of activities. The analysis has a starting point the goal of researching an author that had inspired some among Maria Montessori's (1870-1952) educational aids for mathematics with young children, designed around 1900.[18]

Subsequently, a group of works have been developed in the last five years on selected topics regarding several periods and countries, considering single books or single authors.[19] A landscape has begun to be depicted, starting from

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polyte Vernier's *Petite arithmétique raisonnée* (1832) in France.

[17] See Millán Gasca 2015. I am working on a paper containing an analysis of the correspondence between Macé and his publisher Hetzel on the children's book *Arithmétique du grand papa*.

[18] Gil Clemente 2016; Cogolludo-Augustin & Gil Clemente 2019. Some aspects of the role of geometry in the awakening of conciousness in intellectually disabled children (between plane and solid geometry are considered in Gil Clemente, Millán Gasca 2021).

[19] John M. Colaw and John K. Ellwood's textbook *School arithmetic* (1900) (Di Clemente 2018) in the USA; on Jules Dalsème's (1845-1904) conception of "natural geometry" (Zannoni 2018); on the Adriana Enriques' 1928 schoolbooks (Peccarino 2018); on mathematics in the primary school state unique book in fascist Italy (Cammillucci 2020); on Montessori's treatment of ratio and proportions in *Psychoarithmetic* (1934) (in Parenti 2017); on Margalida Comas' (1892-1972) contribution to primary school mathematics (Mazza 2018); on Horace Grant's (1801-1859) books for young children for the Society for the Diffusion of Useful Knowledge (Calabrese 2018) and on Mary Boole's Lectures on the logic of arithmetic (Tomasello 2019); on the origins of George Cuisenaire's proposal of colored rods for reckoning (Di Tella 2021); on the toybox «Initiateur Mathématique» designed by Jacques Camescasse (Panichelli 2020); on mathematics elementary instruction in the pedagogical reflection of Emilia Santamaria Formiggini (1877-1971) in the early 20th century (Stramaccioni 2021); mathematics in Steiner's thought and pedagogy (Di Marco 2022): on mathematics in Richard L. and Maria Edgeworth's *Practical education* (1798) (Fossa 2022). See also in this volume the workshops Gil Clemente, Migliucci & Panichelli, Magrone.

contributions in the late 18th century and arriving to the 1920s and 1930s. In the 1930 the first contributions by Alina Szeminska (1907-1986), working at Geneva with Jean Piaget, were published, leading to the essay *La genèse du nombre dans l'enfant* (1941). The working hypothesis is that of a discontinuity – the age of new math – that made many among these contributions fall into oblivion.

A cultural thread can be distinctly identified linking ideas and proposals, as different authors (mathematicians and educators, women and men) shared a common tradition of mathematics teaching that they try to cope with, taking into account the needs of the child and the changing times. In some cases, the aim was to prepare and easing the struggle with written numbers and operations; in other cases, an overall proposal of radical, ambitious transformation was put forward.[20] This thread was linked to the evolution of views and social practices regarding childhood as well as to numeracy as a social goal, including the self education of adults. Entrepreneurial initiatives were prompted by these ideas and proposals, and at the same time their supporters were also encouraged by political visions of a modern, equal society.

Some general, interwoven trends are the following:

- the attention to the specificities of young children, thus of initiation or preparation to number (or to mathematics proper) rather than the teaching or "study" of mathematics;
- the changing emphasis from literacy (written notations and procedures) to orality in the educational setting;
- the extension of aims from raw numeracy to a general introduction to mathematics, with a pivotal role of geometry,[21] also supporting the understanding of number and measure;
- the design of mathematical gadgets intended as "rational games"

[20] The consideration of the impact and effective diffusion of these ideas and the commercial success of the proposals, as well as the impact on school practices – all certainly aspects of great interest – remain outside the scope of this contribution, but many contributions can be easily placed in the context of intellectual- political networks supporting the diffusion of ideas, or can be linked to the fortune of specific publishers/manufacturers. One should notice that, as innovators often proposed counter-measures to soften the rigidity of established practices, they offer indirect information of the persistence and varieties of such practices.

[21] Millán Gasca 2015.

openness to the multiple dimensions of playfulness and beauty (also thanks to the connection with the entertainment and pleasure market)

And yet it's a hidden thread, because of several circumstances. On one side, the issue of children, education and mathematics in history is placed at the crossroads of several historiographical perspectives. Historians of education and of the pedagogical thought in the contemporary age have neglected arithmetic (compared to grammar, history and other subjects) in the evolution of education and schooling (but see for example Roggero 1994 for popular numeracy, Terrón Bañuelos, & Alonso Velázquez 1999 for arithmetic as a school subject); and geometry may be hidden behind the teaching of drawing (which has recently received considerable attention). Besides, the role of mathematics (not only as a basic school subject, but from a more general point of view) in educational thought has been sometimes overlooked: for example, even for the iconic Pestalozzi and for the father of the Kindergarten, Friedrich Fröbel (but see Bullynck 2008, Spranger 1939, Friedman 2021), or in Rudolf Steiner, as well as in pioneers such as Édouard Séguin for special needs education or Wilhelm Lay for experimental pedagogy).[22]

Historians of mathematics and of mathematics education may have disregarded this issue as little relevant for the reconstruction of the mathematical universe throughout the ages, concentrating rather on secondary and higher education. Increasing attention to the sphere of childhood can be considered as part of a more general shift towards taking into account the history of non-elite mathematics. Yet, as teaching arithmetic (Bjarnadóttir 2014) and in particular numeracy in history regarded both adults and children, the young addressees risk to be concealed, as in the case of the arithmetic books, that did not reach the hands of children but reached them rather through adults.

On the other side, sources that could be helpful in unveiling the issue of children, education and mathematics – such as toys and educational aids, notebooks, school textbooks or picture books (Denniss 2009, 2012, Moyon 2016) – are often difficult to find, as they have not been properly archived and preserved in libraries and museums because they do not belong to "high" culture. Besides, single's contributions appear mainly to arise from a cultural and

[22] On mathematics in Steiner's pedagogy, see De Marco 2021; on Séguin, see Gil Clemente, & Millán Gasca 2021.

political background even if mainly as isolated initiatives: a single figure appears to have had a very influential role, Pestalozzi, who inspired innovators until the early 20th century. In that period, the French mathematician Charles Laisant explicitly acknowledged authors inspiring his work, which started from mathematics but lead him to promote a much more ambitious program regarding science for children.[23] Of course, further research could throw light on cultural connections.

The evolution of conceptions in the 18th-20th century was accompanied by myths of success (from Pestalozzi to Montessori, Rudolf Steiner or Georges Cuisenaire), by believers and by fashions driven by advertising and storytelling: sometimes these myths are still at work[24] in present days nourished by the above mentioned deep concerns of the general public regarding children's mathematics education, and implied the risk of a biased historical analysis. Conversely, the challenge coping with an early introduction to maths for the social goal of an inclusive, equal society, as well as for our digital future, is encouraging a recognition of the evolution in different countries or cultural areas – national and linguistic specificities need to be taken in to account – which in turn may represent the basis for comparative analysis so as to find the traces of the past in the present.

REFERENCES

Arendt, H. (2006/1961). "The crisis in education". In *Between past and future* (pp. 170-193). London: Penguin.

Ariès, Ph. (1960). *L'enfant et la vie familiale sous l'Ancien Régime*. Paris: Plon.

Auvinet, J. (2013). Charles-Ange Laisant. Paris: Herman.

Becchi, E. & Julia, D. (Eds.) (1998). *Histoire de l'enfance*. Paris: Seuil.

Bjarnadóttir, K. (2014). History of Teaching Arithmetic. In A. Karp &, G. Schubring (Eds.), *Handbook of the history of mathematics education* (pp. 431-457). New York: Springer.

Brueckler, F. M., Stilinović, V. (2013). Teaching arithmetic in the Habsburg Empire at the end of the 18th century–A textbook example. *Historia Mathematica*, *40*(3), 309-323.

[23] Lamandé 2011, Schiopetti 2015; on Laisant see Auvinet 2013.

[24] See Dhont et al 2015 for Rudolf Steiner.

Bullynck, M. (2008). The transmission of numeracy: integrating reckoning in Protestant North-German elementary education (1770-1810). *Pedagogica Historica 44*(5), 563-585.

Calabrese, F. (2019). *Horace Grant (1801-1859) e l'istruzione matematica nella prima infanzia, nel contesto delle iniziative della Society for the Diffusion of Useful Knowledge (1826 - 1848)* [Horace Grant (1801-1859) and early mathematics education, in the context of the initiatives of the Society for the Diffusion of Useful Knowledge (1826-1848)] (Master of Education Dissertation, Roma Tre University).

Chadwick, E. (1861). A biographical note of the author. In Grant, H., *Second stage of arithmetic. Arithmetic for schools* (New edition). Bell and Daldy, pp. v-xxxv.

Cipolla, C. (1969) *Literacy and development in the West*. London: Penguin.

Comenius, J. A. (1896) *The Great Didactics of John Amos Comenius*, translation by M. W. Keatinge. London: Adanmand Charles Black.

Condorcet, J.-A.-N. de Caritat (1988/1798). *Moyens d'apprendre à compter sûrement et avec facilité*. Paris: Art, Culture, Lecture Éditions.

Dauben, J. (2003). Internazionalizing mathematics east and west. In K. H. Parshall, &A. C. Rice, *Mathematics unbound: The evolution of an international mathematical research community, 1800-1945* (pp. 253-280). Providence: American Mathematical Society.

Decayeux-Cuvillier, M. (2019). *Des mathématiques pour les filles? L'exemple de l'enseignement primaire dans la Somme de 1881 à 1923*. Villeneuve d'Ascq: Presses Universitaires du Septentrion.

D'Enfert, R. (2003). Inventer une géométrie pour l'école primaire au XIXè siècle. *Tréma*, 22, pp. 41-49.

D'Enfert, R. (2014). Jomard, Francœur et les autres… Des polytechniciens engagés dans le développement de l'instruction élémentaire (1815-1850). *Bulletin de la Sabix*, 54, 81-94.

Denniss, J. (2009). Learning arithmetic: textbooks and their users in England 1500-1900. In E. Robson, & J. Stedall (eds.), *The Oxford Handbook of the History of Mathematics* (pp. 448-467). Oxford: Oxford University Press.

Denniss, J. (2012). *Figuring it out. Children's arithmetical manuscripts 1680-1880*. Oxford: Huxley Scientific Press.

Dhondt, P., Van de Vijver, N., & Verstraete, P. (2015). The possibility of an unbiased history of Steiner/Waldorf education? *Contemporary European History*, *24*, 639-649.

Di Clemente, C (2018). *Analisi storico-didattica di "School Arithmetic" (1900) di*

John M. Colaw e John K. Ellwood nel contesto dell'evoluzione delle idee sull'istruzione matematica infantile in età contemporanea [Historical-didactic analysis of "School Arithmetic" (1900) by John M. Colaw and John K. Ellwood in the context of the evolution of ideas on children's mathematics education in the contemporary age] (Master of Education Dissertation, Roma Tre University).

Di Marco, M. (2022) *La matematica nella pedagogia steineriana, dagli scritti di Rudolf Steiner (1861-1925) alla lettura attuale di Ernst Schuberth* [Mathematics in Steiner pedagogy, from the writings of Rudolf Steiner (1861-1925) to the current interpretation by Ernst Schuberth] (Master of Education Dissertation, Roma Tre University).

Di Tella, E. (2020). *Indagine sulle "reglettes" di Georges Cuisenaire (1891-1976)* [Investigation on Georges Cuisenaire's (1891-1976) rods] (Master of Education Dissertation, Roma Tre University).

Edgeworth, M., & Edgeworth, R. L. (1798). *Practical Education*. London: J. Johnson.

Fantaccini, L.,& Leproni, R. (Eds.) (2019). S*till blundering into sense. Maria Edgeworth, her context, her legacy*. Firenze: Firenze University Press.

Fisher, D. W. (1911). *Calvin Wilson Mateer, Forty-five years a missionary in Shantung China: A biography*. Philadelphia: Westminster Press.

Fossa, V. (2022). *Giocattoli, matematica e scienza: l'istruzione matematica infantile nell'opera* Practical Education *(1798) di Richard Lovell Edgeworth (1744-1817) e Maria Edgeworth (1767-1849)* [Toys, mathematics, and science: mathematics education in childhood in *Practical Education* (1798) di Richard Lovell Edgeworth (1744-1817) and Maria Edgeworth (1767-1849)] (Master of Education Dissertation, Roma Tre University).

Franci, R. (2000). L'insegnamento dell'aritmetica nel Medioevo. In P. Freguglia, L. Pellegrini, & R. Paciocco (Eds.) *Scienze Matematiche e insegnamento in epoca medioevale. Atti del Convegno internazionale di studi Chieti 2-4 maggio 1996*. Napoli-Roma: Edizioni Scientifiche Italiane, pp. 111-132.

Friedman, M., & Muñoz Alvis, J., (2021). Haüy, Weiß, Fröbel: the influence of nineteenth-century crystallography on the mathematics of Friedrich Fröbel's Kindergarten. (in two parts), *Paedagogica Historica* on line

Frijhoff, W. (2012). Historian's discovery of childhood. *Paedagogica Historica*, *48*(1), 11-29.

Garin, E. (1976). *L'educazione in Europa 1400-1600. Problemi e programmi*. Roma-Bari: Laterza, 3rd edition (1st ed. 1957, 2nd ed. 1966).

Gil Clemente, E., & Millán Gasca A. (2016). Integrating history of mathematics with

foundational contents in the education of prospective elementary teachers. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.) *Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between the History and Pedagogy of Mathematics* (pp. 427-440). IREM de Montpellier.

Gil Clemente, E., & Millán Gasca A. (2021). Geometry as "forceps of intelligence": lines, figures and the plane in Édouard Séguin's pedagogical thought. *Bollettino di storia delle scienze matematiche*, *41*(2), 315-339.

Horlacher, R. (2011). Schooling as a means of popular education: Pestalozzi's method as a popular education experiment. *Pedagogica Historica*, 47 (1-2), 65-75.

Jackson, L. L. (1906). *The educational significance of sixteenth century arithmetic from the point of view of the present time*. New York (printed for the author)

Karp, A. (2014). The history of mathematics education: Developing a research methodology. In A. Karp &, G. Schubring (Eds.), *Handbook of the history of mathematics education* (pp. 9-24). New York: Springer.

Lamandé, Pierre (2011). Une personnalité du monde de l'Éducation nouvelle: Charles Ange Laisant (1841-1920) et son combat politique pour une éducation rationnelle fondée sur la science. *Pedagogica Historica*, 47 (3), 283-301.

Ma, L. (2010). *Knowing and teaching elementary mathematics: teachers' understanding of fundamental mathematics in China and the United States*. New York: Routledge (original edition 1999)

Magrone, P., & Millán Gasca, A. (2023). Children shall know what lies at the heart of genuine Mathematical science: The *Lectures on the logic of arithmetic* (1903) by Mary Everest Boole (1832-1916), to appear on Bollettino di storia delle scienze matematiche.

Massenzi, S. (2016). *Il contributo di Mary Everest Boole (1832-1916) all'istruzione matematica infantile* [Mary Everest Boole's (1832-1916) contribution to mathematical education in childhood] (Master of Education Dissertation, Roma Tre University).

Millán Gasca, A. (2015). Mathematics and children's minds: The role of geometry in the European tradition from Pestalozzi to Laisant. *Archives Internationales d'Histoire des Sciences, 65*(2)-175, 261-277

Millán Gasca, A. (2017). Zoel García de Galdeano y las matemáticas para niños hacia 1900, *Gaceta de la Real Sociedad Matemática Española*, 20 (3), 611-634.

Millán Gasca, A. (2021). Educational writing by women in the early 20th century: mathematics for children in Mary Everest Boole, Maria Montessori, Grace Chisholm Young and Margalida Comas Camps. In R. Leproni (Ed.) *Women on*

women. De-gendering perspectives (pp. 141-153). Franco Angeli.

Moyon, M. (2016). Initiation au calcul et éducation nouvelle: la "méthode Havránek" au catalogue du Père Castor. *HISTEMAT. Revista de História da Educação Matemática*, *2* (3).

Panichelli, V. (2021). *L'innovazione didattica a scuola come questione culturale: il caso di Jacques Camescasse e della scatola-gioco* Initiateur Mathématique (1910) [Educational innovation at school as a cultural issue: the case of Jacques Camescasse and the toy box *Initiateur Mathématique* (1910)] (Master of Education Dissertation, Roma Tre University).

Pasquali, C. (2020). *Augustus De Morgan (1806-1871) nella storia della istruzione matematica* (Master in Mathematics Dissertation, Roma Tre University).

Peltzman, B. R. (1998). *Pioneers of early childhood education. A biobibliographical guide*, Westport, Greenwood Press, 1998.

Roggero, M. (1994). Conti sulle dita, calcoli a penna. L'aritmetica elementare a fine Settecento. *Studi Storici*, 35(4), 1039–1060.

Sanford, V. (1930). *A short history of mathematics*. New York: Houghton Mifflin Company.

Schiopetti, V. (2015). *Charles-Age Laisant nella storia dell'istruzione matematica infantile* [Charles-Age Laisant in the history of mathematical education in childhood] (Master of Education Dissertation, Roma Tre University).

Schubring, G. (2014). On historiography of teaching and learning mathematics. In A. Karp &, G. Schubring (Eds.), *Handbook of the history of mathematics education* (pp. 3-8). New York: Springer.

Spranger, E. (1939). *Aus Friedrich Fröbels edankenwelt.* Berlin: de Gruyter, Berlin.

Stedall, J. (2012). *The history of mathematics. A very short introduction*. Oxford University Press.

Stoker, D. (2007). Ellenor Fenn as "Mrs. Teachwell" and "Mrs. Lovechild": A Pioneer Late Eighteenth-Century Children's Writer, Educator, and Philanthropist. *The Princeton University Library Chronicle*, 68 (3), pp. 817- 850.

Stramaccioni S. (2021). *L'istruzione matematica infantile nell'opera di Emilia Santamaria Formìggini (1877-1971)* [Children's mathematical education in Emilia Santamaria Formìggini's (1877-1971) work] (Master of Education Dissertation, Roma Tre University).

Swetz, F. (1992). Fifteenth and sixteenth century arithemtic texts. What can we learn from them?, *Science & Education*, *1*, pp. 365-378.

Tomassoni, L. (2019) *Little Mathematicians: approccio narrativo e approccio con-*

cettuale nelle Lectures on the Logic of Arithmetic *(1903) di Mary Everest Boole* [Little Mathematicians: narrative and conceptual approach in Mary Everest Boole's *Lectures on the Logic of Arithmetic* (1903)] (Master of Education Dissertation, Roma Tre University).

Ulivi, E. (2002). "Scuole e maestri d'abaco in Italia tra Medioevo e Rinascimento", in Giusti (ed.), *Un ponte sul Mediterraneo. Leonardo Pisano, la scienza araba e la rinascita della matematica in Occidente*, Firenze, Edizioni Polistampa, pp. 121-159.

Ulivi, E. (2014). Masters, questions, and challenges in the abacus schools. *Archive for the History of Exact Sciences*, *69* (6), pp. 651-670.

Zannoni, I. (2017). *La geometria naturale di Jules Dalsème: sforzo divulgativo e contributo all'istruzione matematica primaria nella Francia della Terza Repubblica* (Master of Education Dissertation, Roma Tre University).

Yeldham, F. A. (1936).*The teaching of arithmetic through 400 years 1535-1935*. London/Bombay/Sidney: George G. Harrap &Co.