THE USE OF ORIGINAL SOURCES IN THE CLASSROOM FOR LEARNING MATHEMATICS

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ABSTRACT

The teaching and learning of the history of mathematics contributes to the overall education of students, whether they are prospective mathematicians, engineers or teachers. The use of the history of mathematics as an implicit and explicit resource makes it possible to improve the teaching of mathematics and the comprehensive training of students. The subjects that deal with the history of mathematics convey a perception of mathematics as a useful, dynamic, human, interdisciplinary and heuristic science, while complementing the thematic study of the different parts of mathematics. It is important to think of mathematics as a discipline linked to society and culture, as shown by the many people who have advanced the discipline by solving the problems of the society at each time and place. History shows that mathematics can be considered as a scientific and cultural activity that helps to solve problems in every period. This particular contribution will be focused on the use of original sources in the classroom; that is, on practical activities based on the history of mathematics for learning mathematics. Such practical activities using original sources drawn from the history of mathematics can provide students with a broader comprehension of the foundations and nature of the discipline, as well as a deeper approach to the understanding of the mathematical techniques and concepts used every day in the classroom. The aim of this paper is therefore to reflect on the use of original historical sources for learning mathematics by means of practical activities, in order to provide new resources and ideas for teachers of mathematics.

1 The history of mathematics for scientific education in the classroom[1]

The teaching and learning of the history of mathematics contributes in two ways to the comprehensive training of students, whether they are prospective mathematicians, engineers or teachers of mathematics. On the one hand, it enhances the understanding and learning of some mathematical concepts and

[1] This research is included in the project "Mathematics, Engineering and Heritage: New Challenges and Practices (XVI-XIX)" (PID2020-113702RB-I00) of the Ministerio de Ciencia e Innovación. I am grateful to Fàtima Romero-Vallhonesta, Iolanda Guevara Casanova and Carles Puig-Pla. All us are members of the history of mathematics group at the ABEAM (Barcelona Association for the Teaching and Learning of Mathematics). Some of the practical activities presented have been worked together.

methods, and on the other provides a more authentic and accurate perspective of mathematics (Massa-Esteve, 2003).

The history of mathematics can be used as an explicit resource for introducing and achieving a greater understanding of certain mathematical concepts, methods and processes through the analysis in the classroom of selected original historical sources (Jahnke et al., 2000; Demattè, 2006 and Barbin, 2022). In addition, this analysis of historical sources enables students to acquire a vision of mathematics, not as a final and finished product but as a useful, dynamic, humane, interdisciplinary and heuristic science.

- A useful science. It is important to explain to students that mathematics has been an essential tool in the development of different civilizations. It has been used since antiquity for solving problems of counting, for understanding the movements of the stars and for establishing a calendar. There are many examples right down to the present day in which mathematics has proved to be vital in spheres as diverse as computer science, economics, biology, and in the building of models for explaining physical phenomena in the field of applied science, to mention just a few of the applications.

- A dynamic science. It is also necessary whenever appropriate to teach students about problems that remained open in a particular period, how they have evolved and the situation they are in now, as well as showing that research is still being carried out and that changes are constantly taking place.

- A human science. Teachers should reveal to students that behind the theorems and results there are remarkable people. It is not merely a question of recounting anecdotes but rather that students should learn something about the mathematical community; human beings whose work consisted in providing us with the theorems we use so frequently. Mathematics is a science that arises from human activity, and if students are able to see it in this way, they will probably perceive it as something more accessible and closer to themselves.

- An interdisciplinary science. Wherever possible, teachers of history of mathematics should show the historical connections of mathematics with other sciences (physics, biology, engineering, medicine, architecture, etc.) and other human activities (trade, politics, art, religion, etc.). It is also necessary to remember that a great number of important ideas in the development of science and mathematics itself have grown out of this interactive process.

-A heuristic science. We should analyze with students the historical problems that have been solved by different methods, and thereby show them that the effort involved in solving problems has always been an exciting and enriching activity at a personal level. These methods can be used in teaching to encourage students to take an interest in research and to become budding researchers themselves.

It is necessary to think of mathematics as a discipline rooted in society and culture, as shown by the many mathematicians who have made advances in the field by solving problems in society at each time and place. Indeed, the history of mathematics shows that the subject can be understood as a cultural activity, since societies develop as a result of the scientific activity undertaken by successive generations and that mathematics is being a fundamental part of this process. At the same time, the cultural and social influences involved in this historical development provide students with a view of mathematics as a subject closely linked to time and place, thereby contributing an additional value to the discipline itself (Radford, 2006).

Thus, when teaching the history of mathematics in the classroom, it is essential to analyze scientific discoveries in the context of both the time and place in which they occurred. To this end, the past, its contemporaries and its social and economic context must be taken into account. Some chronological accounts of the history of mathematics may make such discoveries appear as a mere linear correlation and give an impression of continuity that is not real. The objective of the historian of mathematics is not simply to compile lists of events and an enumeration of authors, but rather to shed light on influences and interactions, thereby helping to understand the origin of concepts and the different transformations of mathematics (Calinger, 1996).

The aim of this paper is to reflect on the use of original historical sources for learning mathematics through practical activities using new resources and ideas. In what follows, I would like to address this way of introducing the history of mathematics, which will lead to the analysis of some new resources and ideas, and also examine many of the practical activities, especially some drawn from the transformation of mathematics in the 17th century. The crucial aspects of this period will serve to attain improvements in the mathematical education of students by learning new mathematical ideas, procedures and proofs.

2 Using original historical sources for learning mathematics: practical activities.

I have been teaching the history of mathematics in many university courses for twenty years. The practices analysed herein have been implemented in the Interuniversity Master of Formation of Prospective Teachers of Mathematics (UPC). I teach the compulsory subject “Mathematics from a Historical Perspective”, in two groups, each consisting of approximately 30 students. This subject is included in the Module of Complements of Formation of the above-mentioned Master. In addition, in the University Degree in Superior Engineering (ETSEIB, UPC), I also teach the subject "The History of Applied Mathematics to Engineering" for the prospective engineers, in one group of approximately 25 students. In this case, the subject is not compulsory and the students show a keen interest in both the history of mathematics and the history of science. Finally, in the University Degree in Mathematics (FME), I teach the elective subject "History of Mathematics", in one group of approximately 15 students. I share the teaching of this course with Mónica Blanco (UPC), I teach the three first periods.

The basis of my courses on the history of mathematics consists of the use of practical activities for learning mathematics and for reflecting on its development. The practical activities using original historical sources provide students with a deeper approach to the understanding of the mathematical procedures and concepts used every day by the teachers in mathematics or engineering classes. History may serve as an explicit resource to introduce or understand better certain mathematical concepts through the analysis in the classroom of selected historical texts.

Therefore, it is necessary to explain how the practical activities are conducted and why, by the analysis of some examples, we regard this way of introducing the history of mathematics as very fruitful. In my courses on the history of mathematics I undertake a practical activity every week using original sources in accordance with a script of questions on the subject of the source, with the intention of clarifying any doubts or problems that may arise and together discuss the development of mathematical thought in each historical period. One aspect is the use of images or videos for introducing the source. It is not necessary to insist too much that the image in the source is not a mere complement to the explanation, but that, on many occasions, it has a leading role in the discourse or in the proof.

A further key point consists in the selection of these original sources. This is a complex matter, especially if we wish to ensure that it is reliable and really conveys the context or the idea that is to be communicated. Therefore, we want to remark that the choice of the sources for teaching the history of mathematics has to be considered carefully. We may pose some questions such as: Is the source related to the historical content of the curriculum? Does it contribute significantly to the improvement of the learning of mathematics? Is it essential to understand the origin of the mathematical concept being taught? Does it stimulate mathematical reflection? Has it represented the solution to any of the real problems of society? Does it arouse curiosity? Does it teach new methods? Does it enhance mathematical reasoning? (Jankvist, 2009; Mosvold, Jakobsen & Jankvist, 2014; Romero-Vallhonesta & Massa-Esteve, 2016).

The relevance of using historical sources in our courses is clear. From the results of our experience in the practical activities, we have observed that when students are faced with the historical mathematical text, they make it their own and create their own knowledge, which is the best way to learn mathematics. At the same time, thanks to these practical activities, students are able to learn, remember or review theorems, formulas or mathematical rules from another perspective. Finally, this way of introducing the history of mathematics transforms the class into a kind of laboratory in which ideas and concepts flow and are debated.

In the following, based on the sources, I briefly describe some practical activities with the aim of illustrating affirmative answers to some of the questions posed above. In my courses, the first three specific periods in the history of mathematics are regarded and will be addressed chronologically: "Mathematics in Antiquity", as example of this period, the three first practical activities; "From Arab science to Renaissance algebra", with the activities fourth and fifth as example and "The Birth of Modern Mathematics", with the last five activities, as example. Therefore, I have selected 10 practical activities from my courses.

1. Pythagoras' theorem in Euclid's *Elements* (300 BC)
2. The measurement of the circle in Archimedes' work (287 BC)
3. The distances to the Sun and the Moon using plane geometry in Aristarchus' work (280 BC)
4. The geometrical justifications of the solution of equations in Al-khwarizmi's work (813)

5. The measure of inaccessible distances in Tartaglia’s *Nova Scientia* (1537)
6. The negative roots of an equation in Girard’s work (1629)
7. The “specious” algebra in François Viète’s *In Artem Analyticen Isagoge* (1591)
8.The construction of quadratic equations in Descartes’ *Géométrie* (1637)
9. The Arithmetical Triangle in Pascal’s work (1654)
10.The quadrature of figures by using triangular tables in Mengoli’s work (1659 and 1672).

2.1 Analysis of these practical activities

1.“Pythagoras’ theorem in Euclid’s *Elements* (300 BC)”

The source is Euclid's *Elements* (300 BC), consisting of 13 books, which brings together the mathematical knowledge of different Greek schools and shows some geometric propositions that can be interpreted in terms of a second-degree equation or the Pythagorean theorem. This work, which is believed to be a collective endeavour, is second only to the Bible in the number of editions published (more than a thousand), being one of the most culturally influential works in the entire history of science.

There are many practical examples using propositions taken from Euclid's text. The Pythagorean theorem has been chosen because it is well known and it is demonstrated with equal geometric figures (triangles) that are compared with quadrilaterals, in an original and rigorous way (see Fig. 1).

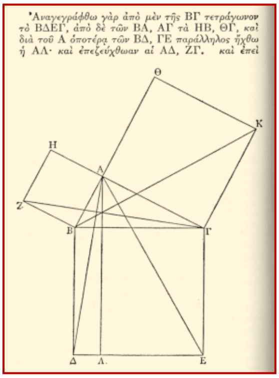


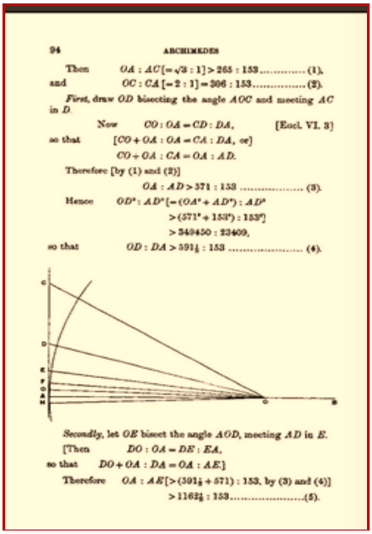
Figure 1. Proposition 47 of Book I in Euclid’s *Elements*

2. “The measurement of the circle in Archimedes’ work (287 BC)”

The source is *The Measure of the Circle* (approx. 287 BC) by Archimedes, where an approximation to the number π is calculated thus helping to understand its origin.

Proposition III in this book shows that the relationship between the length of the circumference and its diameter is between 3 10/71 and 3 1/7, which represents an approximation of the number π between 3.1408 and 3.1428.

Archimedes began by inscribing and circumscribing triangles in a circle, and by doubling the number of sides he arrived at polygons of 96 sides (see Fig. 2). To find this approximation he uses the bisector and the Pythagorean theorems and the relationship between the inscribed angles and the central angle, among other properties of the circumference.


94 ARCHIMEDES

Then $OA : AC [= \sqrt{3} : 1] > 265 : 153$ ....(1),

and $OC : CA [= 2 : 1] = 306 : 153$ ....(2).

First, draw OD bisecting the angle AOC and meeting AC in D.

Now $CO : OA = CD : DA$, [Eucl. VI. 3]

so that $[CO + OA : OA = CA : DA$, or$]$

$CO + OA : CA = OA : AD$.

Therefore [by (1) and (2)]

$OA : AD > 571 : 153$ ....(3).

Hence $OD^2 : AD^2 [= (OA^2 + AD^2) : AD^2$

$> (571^2 + 153^2) : 153^2]$

$> 349450 : 23409$,

so that $OD : DA > 591\frac{1}{8} : 153$ ....(4).

Secondly, let OE bisect the angle AOD, meeting AD in E.

[Then $DO : OA = DE : EA$,

so that $DO + OA : DA = OA : AE$.]

Therefore $OA : AE [> (591\frac{1}{8} + 571) : 153$, by (3) and (4)]

$> 1162\frac{1}{8} : 153$ ....(5).

Figure 2. The approximation of the number π (Archimedes, 1921)

3. “The distances to the Sun and the Moon using plane geometry in Aristarchus’ work (280 BC)”

The work “On the Sizes and Distances of the Sun and the Moon” by Aristarchus of Samos is an attempt to calculate the distances Sun-Earth and Earth-Moon, with an original, rigorous and correct method, using similar triangles, the bisector theorem and the Pythagorean theorem (see Massa-Esteve, 2005b for more). The geometric propositions that Aristarchus used are found mostly in Euclid's *Elements*. Eudoxus's theory of proportions from Book V of the *El-*

ements is used consistently and its properties of inverting, alternating, composing and multiplying are applied for both equal and unequal proportions. Aristarchus' work is also implicitly based on other relations, which we now see and identify as trigonometric, as if he knew them or considered them trivial.

An example can be seen in the proof of Proposition VII, where Aristarchus states: "The distance from the Earth to the Sun is greater than eighteen times, but less than twenty times the distance from (the Earth) to the Moon." Aristarchus builds a right triangle with vertices at the centres of the Earth (B), the Moon (C) and the Sun (A) with given angles or known by observation (see Fig. 3). As the Moon is shown to us split in two, the BCA angle is a right angle, the ABC angle is 87º (by observation) and the CAB is 3º. In fact, he shows that: 18CB < AB < 20CB, which we would say with trigonometry: 1/18> sin 3º = CB: AB > 1/20, where CB is the Moon-Earth distance, AB the Sun-Earth distance, and here the ratio of the distances is interpreted as the sine of the angle complementary to that between them. These ratios allowed Aristarchus to determine the upper and lower bounds of the value we are looking for.

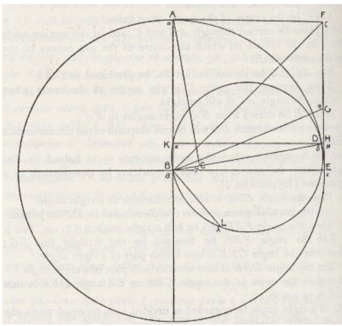


Figure 3. Image of Proposition VII (Aristarco de Samos, 2007: 50)

With this approach, it is necessary to highlight the four mathematical strategies required for the development of the proof of the first inequality (18CB < AB): the passage from the analysis of the problem of the triangle Sun-Earth-Moon to a similar triangle; the use of the relationship, as if it was trivial, between the tangents (current expression) and the angles (tg α: tg β> α: β, with the angles α, β of the first quadrant, and α > β); the establishment of a proportion between the segments that determines the bisector of an angle and the sides of the triangle (applying the proposition VI.3 of the *Elements*) and, the last one, the approximation of square root of 2, by 7: 5. At the end, Aristarchus transfers the result obtained in the similar triangle to the initial triangle ABC or the Sun-Earth-Moon triangle and concludes that AB > 18 CB.

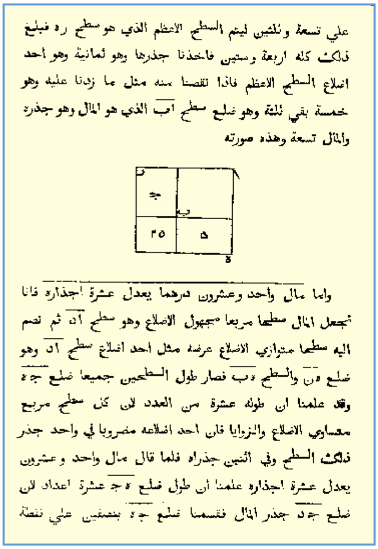
A comprehensive assessment of this work on astronomy must consider the close relationship that the beginnings of astronomy had with the origins of trigonometry, an aspect that contributes to a better understanding of the text and the evolution and utility of trigonometry.

4."The geometrical justifications of the solution of equations in Al-khwarizmi's work (813)"

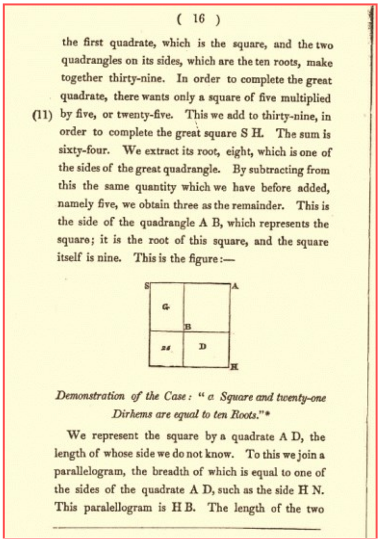
Arabs have played a fundamental role in the development of many branches of science. Arabs collected the abstraction of Greek knowledge and the pragmatism and calculation of Hindu knowledge, for growing and transforming this assimilated knowledge, creating new ideas based on the resources of their own civilization. Baghdad emerged as the great scientific centre that enabled the translation of the great Greek works such as *The Elements* of Euclid and the *Almagest* of Ptolemy, thanks to which it was also possible to draw up new astronomical tables. After Baghdad, other focal points of culture were: Cairo, Cordova, Samarkand, Isfahan, and others. The Arabs made important contributions to physics, observational astronomy, alchemy, medicine, geometry and especially in algebra (Romero *et al*., 2015).

Abu Ja'Far Mohamed Ben-Musa al-Khwārizmī, mathematician, astronomer and member of the House of the Wise of Baghdad, died in 850 (AD) and is regarded as the creator of the rules of algebra. His work *Kitāb al-Mukhtasar fī hisāb al-jabr wa'l-muqābala* (ca. 813) was translated into Latin by Robert of Chester with the title *Liber algebrae et almucabola* (Segovia, 1145), where the current name of algebra comes from. The work of al-Khwarizmi consisted of a theoretical part with the method for solving equations with positive coefficients (classified into six types, up to the second degree) and a practical part that contained problems concerning numbers, trade, dowries and inheritance.

The language was rhetorical, without the use of symbols and with some geometric justification of found solutions. The geometric justification used by the Arabs for the solutions of the equation of second-degree is based on the construction of a square of side "*x*", completing it with two rectangles of measures "*x*" and "b/2" and one square of side "b/2", in order to obtain a square of side “x + b/2”, as may be checked in the example of following figures (Figs. 4 and 5).


علي تسعة وثلثين ليتم السطح الاعظم الذي هو سطح ره فبلغ
ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد
اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو
خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره
والمال تسعة وهذه صورته

واما مال واحد وعشرون درهما يعدل عشرة اجذاره فانا
نجعل المال سطحا مربعا مجهول الاضلاع وهو سطح اد ثم نضم
اليه سطحا متوازي الاضلاع عرضه مثل احد اضلاع سطح اد وهو
ضلع هن والسطح هب فصار طول السطحين جميعا ضلع جه
وقد علمنا ان طوله عشرة من العدد لان كل سطح مربع
متساوي الاضلاع والزوايا فان احد اضلاعه مضروبا في واحد جذر
ذلك السطح وفي اثنين جذراه فلما قال مال واحد وعشرون
يعدل عشرة اجذاره علمنا ان طول ضلع هج عشرة اعداد لان
ضلع جد جذر المال فقسمنا ضلع جه بنصفين على نقطة


(16)

the first quadrate, which is the square, and the two quadrangles on its sides, which are the ten roots, make together thirty-nine. In order to complete the great quadrate, there wants only a square of five multiplied
(11) by five, or twenty-five. This we add to thirty-nine, in order to complete the great square S H. The sum is sixty-four. We extract its root, eight, which is one of the sides of the great quadrangle. By subtracting from this the same quantity which we have before added, namely five, we obtain three as the remainder. This is the side of the quadrangle A B, which represents the square; it is the root of this square, and the square itself is nine. This is the figure:—

*Demonstration of the Case: “a Square and twenty-one Dirhems are equal to ten Roots.”**

We represent the square by a quadrate A D, the length of whose side we do not know. To this we join a parallelogram, the breadth of which is equal to one of the sides of the quadrate A D, such as the side H N. This paralellogram is H B. The length of the two

Figure 4. Geometrical Justification in the translation by Rosen (1831)

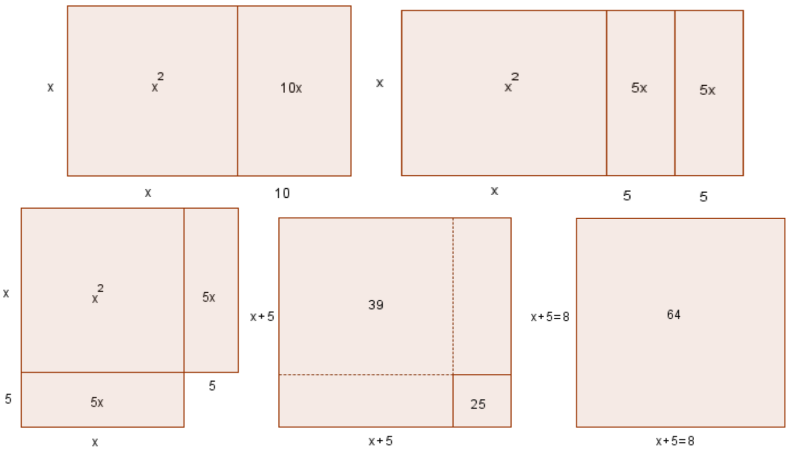


Figure 5. Solution of x^2 +10x=39 with visual reasoning

In this activity, students attempt to represent and solve equations of the second-degree using geometry in the Arab way, while reflecting on the relationships between two parts of mathematics that often complement each other in order to advance mathematical knowledge: algebra and geometry.

5.“The measure of inaccessible distances in Tartaglia’s *Nova Scientia* (1537)”

The following historical activity deals with the work *Nova Scientia* (1537) by Nicolò Fontana (Tartaglia) (1499/1500-1557). In order to implement the activity in the classroom, it is recommendable: to begin with a brief presentation of the epoch, the Italian Renaissance, and the character of Tartaglia himself; the aims of the author as well as the features of the work would then be analyzed; finally, students are encouraged to construct an instrument for measuring degrees and follow the reasoning of a significant proof in order to acquire new mathematical ideas and perspectives. This classroom activity would be implemented also in the last cycle of compulsory education (14–16-year-old) with the aim of introducing and motivating the study of trigonometry (see Massa-Esteve, 2014, for more).

Tartaglia constructs two gunner’s quadrants, one with a graduate arc to measure the inclination of the cannonball and the other an instrument for solving the problem of measuring the distances and height of an inaccessible object. In the third book, from the first proposition to the fourth proposition, he describes the material required for constructing the second gunner’s quadrant: the rule and the set square, and checks its angles in the following propositions (see Fig. 6).

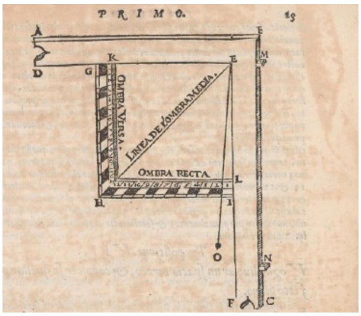
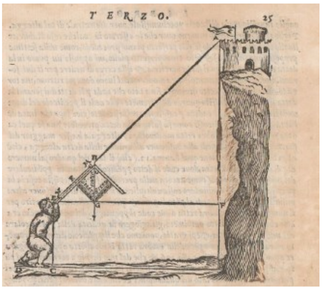


Figure 6. The gunner’s quadrant

This gunner’s quadrant is used by Tartaglia for measuring the height of inaccessible objects in the propositions of the third book. Tartaglia uses this gunner’s quadrant, while at the same time employing geometry in similar triangles in the proof for measuring the distances and height of an inaccessible object. In the implementation, students could be prompted to reproduce the reasoning of this proof with the geometry of triangles before introducing the trigonometry.

In Proposition VIII of third book, Tartaglia proves how to obtain the height of a visible but inaccessible object. The image of this proposition clarifies the geometric reasoning (see Fig. 7a):



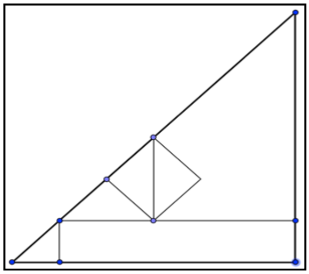


Figure 7a. Image of Proposition VIII. Tartaglia, 1537, 25r and **Figure 7b**. Reproduction of the mathematical problem

After explaining the construction of gunner's quadrant, accurately, together with the students the teacher could follow the reasoning of the proof using the similarity of triangles. For example, they could draw a figure with triangles that reproduces the geometric problem (see Fig. 7b). Together with the students, the teacher can reproduce the geometrical proof using similar triangles, Pythagoras' theorem and Thales' theorem. The teacher can also demonstrate the use of this figure to solve other problems in the classroom; for instance, the height of a house, using this procedure. In fact, these kinds of problems are solved today by trigonometry, and furthermore this historical activity also justifies the introduction of the teaching of trigonometry.

6."The negative roots of an equation in Girard's work (1629)"

Through our experience, we have realized that many students have difficulty understanding and handling negative numbers (see Romero-Vallhonesta *et al.*, 2021, for more). Accordingly, first we present an analysis of some relevant historical texts that deal with the product of negative signs like in Pacioli's work (1494) or with the negative numbers like in Cardano's work (1545). On the basis of these texts, classroom activities can be designed with contents related to numbering, which forms part of the numbering and calculation block of the curriculum. However, a deep understanding of negative numbers by students will come past the conceptual barrier of symbolic reasoning and once solving algebraic equations is introduced. For example, Girard wished to solve the quadratic equation: $5x^2 = 18x + 72$, using some rhetorical instructions that reminding the modern formula. In this activity they realize a clearer acceptance of the negative roots of a quadratic equation in this work by Albert Girard (1595-1632), *Invention Nouvelle en l'algèbre* (1629), situated after publication by Viète's work and before by Descartes' work.

"When x2 (2) equals to x (1) and number (0).
For example, if 5 x2 is equal to 18 x + 72.
The half of the number of the x's is +9.
Its square +81.
To which we add the product of 5 times +72, which is +360.
The sum + 441.
The root of the sum is +21,
which added, and subtracted from the first in this order will give 30 and -12.
Each of these divided by 5 will give 6 and also -12/5 values of x".

7."The "specious" algebra in François Viète: *In Artem Analyticen Isagoge* (1591)"

The implementation of this historical practical activity in a mathematics history course is appropriated for my courses and also for the bachelor's degree in mathematics or in the last cycle of compulsory education (14–16 years old) (see Massa-Esteve, 2005a and 2020, for more). This activity contains singular geometric constructions solving quadratic equations by François Viète (1540-1603), in the process of algebraization of mathematics, which was mainly the result of the introduction of algebraic procedures for solving geometrical problems; in turn, this process led to two fundamental transformations in mathematics: the creation of what is now known as analytic geometry, and the emergence of infinitesimal calculus (Mahoney, 1980; Mancosu, 1996). These disciplines became exceptionally powerful when connections between algebraic expressions and curves and between algebraic operations and geometric constructions were established.

In his *In Artem analyticen Isagoge* (1591), Viète used symbols to represent both known and unknown quantities, and was thus able to investigate equations in a completely general form. Viète introduced the specious logistic, a method of calculation with "species", kinds or classes of elements. The symbols of this analytic art (or algebra) could therefore be used to represent not just numbers but also values of any abstract magnitude, line, plane, solid or angle. In my courses, I analyze Viète's analysis as a method of solving all problems.

In fact, it is important to explain to students that Viète solved equations geometrically using the Euclidean idea of proportion: proportions can be converted into equations by setting the product of the medians equal to the product of the extremes (Viète, 1591: 2). This Viète's principle was taken directly from Euclid's *Elements* VII.19. (Euclid, 1956: 318-320). In Chapter 2 of *Isagoge,* Viète states: "And so, a proportion can be called the composition (*constitutio*) of an equation, an equation the resolution (*resolutio*) of a proportion".

In the classroom, Viète's claims concerning the quadratic equation ($x^2+bx=d^2$) and how he solved a geometrical problem with a singular construction are analyzed (see Fig. 8). In this construction, Viète set up the quadratic equation *A quadratum plus B in A, aequari D quadrato* by means of a proportion (A + B): D = D: A, using Viète's principle.

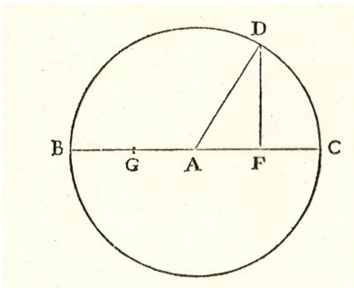


Figure 8. Viète's three proportional construction (Viète, 1646: 234)

Then, in the classroom, it is emphasized that Viète's geometrical construction procedures are based on the identification of terms of an equation, both known and unknown quantities, as terms of a proportion, or proportional lines through the height theorem.

8. "The construction of a quadratic equation in Descartes' *Géométrie* (1637)"

The other singular example that I analyze concerning the algebrization of mathematics is the geometrical construction in a quadratic equation found in *La Géométrie* (1637) by René Descartes (1596-1650). In the classroom, first, I explain the significance of Descartes' work and describe the contents of the three books in *La Géométrie*. I begin with the book I by describing the creation of an *algebra* of segments by Descartes and showing how Descartes adds, multiplies, divides and calculates the square root of segments with geometrical constructions (see Bos, 2001; Allaire &Bradley, 2021 and Massa-Esteve, 2020, for more). In the classroom, it is emphasized the use of Tales theorem for the product of segments, the introduction of the segment unity for the operations between segments and the height theorem for the extraction of the square root.

Next, I show how a quadratic equation (x^2 = ax +bb) may be solved geometrically by Descartes, reproducing the singular geometric construction (see Fig. 9):

"For example, if I have $z^2=az+bb$, I construct a right triangle NLM with one side LM, equal to b, the square root of the known quantity b^2, and the other side, LN, equal to ½ a; that is, to half the other known quantity which was multiplied by z, which I suppose to be the unknown line. Then prolonging MN, the hypotenuse of this triangle, to O, so that NO is equal to NL, the whole line OM is the required line z. This is expressed in the following way: $z= \frac{1}{2} a + (\frac{1}{4} aa + bb)^{1/2}$."

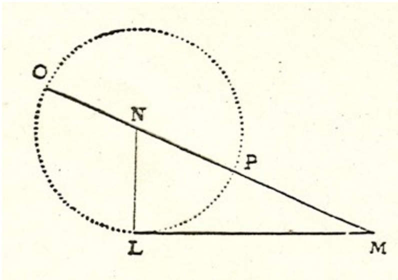


Figure 9. Descartes' geometrical construction (Descartes, 1637: 302)

In the classroom, after analyzing Descartes' geometrical construction, I could hold a discussion with the students. Note that the symbolic formula appears explicitly in Descartes' work. His geometrical construction corresponds to the construction of an unknown line in terms of some given lines without numerical coefficients. Therefore, the solution of the equation is given by the sum of a line and a square root, which has been obtained using the Pythagorean theorem. However, Descartes ignores the second root, which is negative, and did not mention that this geometrical construction could be justified by Euclid's *Elements* III. 36, in which the power of a point regarding a circumference is shown (Euclid, 1956: 75-77).

I ask questions for comparing the two geometrical constructions and reflect on the relationship between algebra and geometry. I reproduce Viète's and Descartes' geometrical construction and explain the procedure; questioning whether this geometrical construction could be used for any quadratic equation. Students should give suitable reasons to the following questions. What about negative solutions? How are the Pythagorean and the height theorem used? Explain their relationship with the solution of the equation. What is the main difference between Viète's geometrical construction and that by Euclid? What is the difference between Viète's and Descartes' geometrical constructions? Can we say that geometric reasoning reaches its full potential by relating algebra with geometry? One student of the course for prospective mathematicians (FME) answers this last question with these remarks:

"Thus, the tool that emerges from the fusion of algebra and geometry makes it possible to select the best properties of both sciences; from the first (algebra), the optimization of the treatment of mathematical concepts, obviating the need to represent the respective procedures and results of a demonstration, and at the same time providing more information intrinsic to the symbolism itself. From the second (geometry), the possibility of visualizing in a particular case the object studied algebraically, and at the same time having a large number of properties that could be used as an axis or complement to a proof. But this is not all; this combination not only allows for the construction of the mentioned method, but also catalyzes a much more effervescent development of both sciences, and consequently the creation (to be constructed later) of new fields of study within mathematics, as would be the case of analytical geometry or the convulsion that trigonometry triggered in the seventeenth century".

Students through this activity can learn that at the end of the process of algebrization, algebra and geometry became complementarians and that was from the coordination and conjunction of both branches that new fields of mathematics developed in the path of modern mathematics.

9. "The Arithmetic Triangle in Pascal's work (1654)"

The arithmetical triangle is the most famous set of numbers in mathematics arranged in a triangular table. It was useful in many fields and had been studied since ancient times and by many civilizations. Despite being used since the eleventh century, I may see that it is not until the seventeenth century when I find the first definitions of the arithmetic triangle, and where its properties are explained by Blaise Pascal (1623-1662). Indeed, the source of this activity, written in 1654 and published in 1665, is Pascal's work: *Traité du*

Triangle arithmétique, avec quelques autres petits traités sur la même matière. Usage du Triangle Arithmétique pour les ordres numériques, pour les combinaisons, pour trouver les puissances des binômes et des apotomes.... After defining the arithmetical triangle, Pascal wrote and subsequently published three further treatises in which he put forward and explained, in a very clear style, these three interpretations, their properties and uses (see fig. 10).

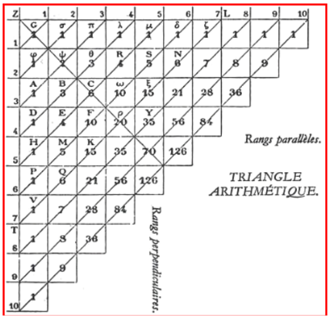


Figure 10. Pascal's Triangle (1654)

The rule for forming the arithmetical triangle is simple: every row begins and ends with 1, and the other numbers are obtained by the addition of two numbers closest to the row immediately above (see fig. 11).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Figure 11. The arithmetical triangle

The numbers that form the arithmetical triangle, arranged diagonally, are well known and date back at least as far the ancient Greeks, if not earlier. They are known as the figurate numbers (triangulars, tetrahedrals or pentagonals). The numbers in the rows of the triangle were subsequently recognized as the terms of a binomial development (now called binomial coefficients), and later on, as may already be seen in Pascal's arithmetical triangle, the numbers apply to solving combinatorial problems (see Edwards, 2002). In the classroom, the triangle is a source of ideas and enables us to calculate with combinatorial numbers (see fig. 12).

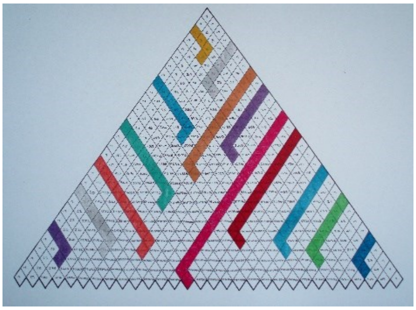


Figure 12. Pascal's triangle in the classroom (see Massa & Romero, 2009, for more)

$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}=\binom{n+3}{2}$$

As far as their applications can be appreciated, it is not only used to make combinatory or to find the coefficients of the Newton binomial, but can also be used to generalize, to calculate summations of powers or later summations of series, and even to calculate areas, as I explain in Mengoli's works and Leibniz's excerpts (see Massa-Esteve, 2017 and 2018, for more).

10. The quadrature of figures by using triangular tables in Mengoli's work (1659 and 1672)

The source is Mengoli's *Geometriae Speciosae Elementa* (Bologna, 1659), a 472-page text in pure mathematics with six *Elementa* whose title: "Elements of Specious Geometry" already indicates the singular use of symbolic language in this work and particularly in Geometry (see Massa-Esteve, 2006 for more). He unintentionally created a new field, a "specious geometry" modelled on Viète's "specious algebra" since he worked with "specious" language, that is to say, symbols used to represent not just numbers but also values of any abstract magnitudes.

Indeed, throughout the book he introduced triangular tables as useful algebraic tools for calculations. In the *Elementum primum,* the terms of the triangular tables are numbers and they are used to obtain the development of any binomial power (see fig. 13).

$$\begin{array}{ccccccccc} & & & & u & & & & \\ & & & a & & r & & & \\ & & a2 & & 2ar & & r2 & & \\ & a3 & & 3a2r & & 3ar2 & & r3 & \\ a4 & & 4a3r & & 6a2r2 & & 4ar3 & & r4 \end{array}$$

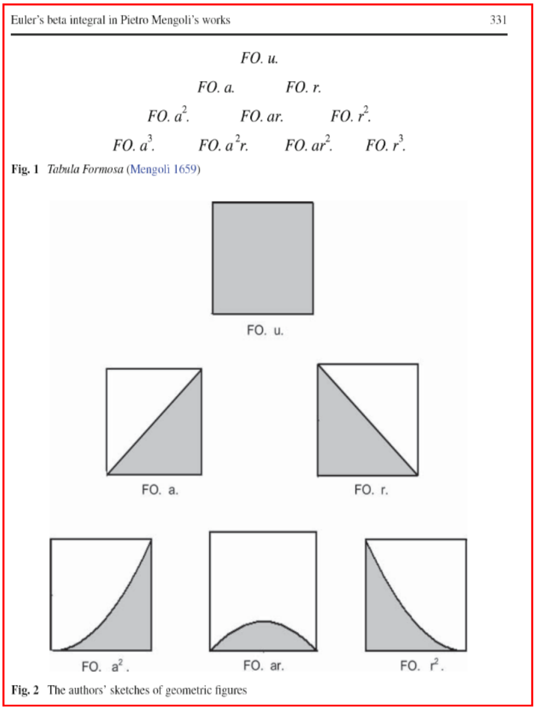
Figure 13. Table of binomial power

In the *Elementum secundum,* the terms are summations and are used to obtain their values (see fig. 14).

$$\begin{array}{ccccccccc} & & & & O.u & & & & \\ & & & O.a & & O.r & & & \\ & & O.a2 & & O.ar & & O.r2 & & \\ & O.a3 & & O.a2r & & O.ar2 & & O.r3 & \\ O.a4 & & O.a3r & & O.a2r2 & & O.ar3 & & O.r4 \end{array}$$

Figure 14. Table of summations

Finally, in the *Elementum sextum* of *Geometria* and in the *Circolo,* the terms are geometric figures or forms and triangular tables are used to obtain the quadratures of these geometric figures (see fig. 15).



Euler's beta integral in Pietro Mengoli's works 331

FO. u.

FO. a. *FO. r.*

FO. a^2. *FO. ar.* *FO.* r^2.

FO. a^3. *FO.* a^2r. *FO.* ar^2. *FO.* r^3.

Fig. 1 *Tabula Formosa* (Mengoli 1659)

FO. u.

FO. a. FO. r.

FO. a^2. FO. ar. FO. r^2.

Fig. 2 The authors' sketches of geometric figures

Figure 15. Table of geometric figures (Massa-Esteve & Delshams, 2009: 331)

I analyze with the students (prospective mathematicians) that Mengoli's originality did not stem from the presentation of these tables but rather from his treatment of them. On the one hand, he used the combinatorial triangle and symbolic language to create other tables with algebraic expressions, clearly stating their laws of formation; on the other hand, he employed the relations between these expressions and the binomial coefficients to prove results like for instance the sum of the pth-powers of the first *t-1* integers. Mengoli found a rule in which the value of the sum of the *p*th powers is obtained. However, in addition to stating the rule, Mengoli also proved it and used it to perform these values expressing all calculations in symbolic language.

Mengoli's idea was that letters could represent not only given numbers or unknown quantities, but variables as well: that is, determinable [but] indeterminate quantities. The summations are indeterminate numbers, but they are determinate when we know the value of *t*. By assigning different values to *t*, Mengoli explicitly introduced the concept of "variable", a notion that was quite new at the time. He applied his idea of variable to calculate the "quasi ratios" of these summations. The ratio between summations is also indeterminate, but is determinable by increasing the value of t. From this idea of quasi ratio, he constructed the theory of "quasi proportions" taking the Euclidean theory of proportions as a model, which enabled him to calculate the value of the limits of these summations. This theory constitutes an essential episode in the use of the infinite and would prove to be a very successful tool in the study of Mengoli's quadratures and logarithms.

Nevertheless, Mengoli's principal aim was the computation of the quadrature of the circle. Instead of just computing it, Mengoli created a new and fruitful algebraic method which involved the computation of countless quadratures. He explicitly identified these geometric figures with the values of their areas, which were also displayed in another triangular table (now called the harmonic triangle) (see fig. 16 and 17). It is noteworthy that in the *Geometria,* there are only three drawings of the geometrical figures whereas in the *Circolo,* he did not include any drawing.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1/2 & & 1/2 & & & \\ & & 1/3 & & 1/6 & & 1/3 & & \\ & 1/4 & & 1/12 & & 1/12 & & 1/4 & \\ 1/5 & & 1/20 & & 1/30 & & 1/20 & & 1/5 \end{array}$$

Figure 16. Values of the quadratures. Harmonic Triangle

$$\int_0^1 1\,dx = 1$$

$$\int_0^1 x\,dx = 1/2 \qquad \int_0^1 (1-x)\,dx = 1/2$$

$$\int_0^1 x^2 dx = 1/3 \quad \int_0^1 x(1-x)\,dx = 1/6 \quad \int_0^1 (1-x)^2 dx = 1/6$$

$$\int_0^1 x^3 dx = 1/4 \quad \int_0^1 x^2(1-x)\,dx = 1/12 \quad \int_0^1 x(1-x)^2 dx = 1/12 \quad \int_0^1 (1-x)^3 dx = 1/4$$

Figure 17. Identification of quadratures and his values in modern notation

In other Mengoli's work, *Circolo* (1672), basing in the harmonic triangle, by interpolation, he computed quadratures between 0 and 1 of mixed-line geometric figures determined by $y = x^{n/2}(1\text{-}x)^{(m-n)/2}$, for natural numbers m and n. Note that in the special case m=2 and n=1, the geometric figure is the semicircle of diameter 1.

However, I argue in my courses that the most innovative aspect of Mengoli's algebraic procedure was his use of letters to work directly with the algebraic expression of the geometric figure. On the one hand, he expressed a figure by an algebraic expression, in which the ordinate of the curve that determines the figure is related to the abscissa by means of a proportion, thus establishing the Euclidean theory of proportions as a link between algebra and geometry. On the other hand, he showed how algebraic expressions could be used to construct geometrically the ordinate at any given point. This allowed him to study geometric figures via their algebraic expressions and calculated its areas.

3 Some reflections

These kinds of practical activities are very rich in terms of competency-based learning, since they allow students to apply their knowledge in different situations and from different points of view, rather than to reproduce exactly what they have learned.

The practical activities based on the analysis of historical texts using original sources contribute to improving the students' overall education, providing them with additional knowledge of the social and scientific context of the periods involved. Students acquire a vision of mathematics, not as a final product, but as a science that has been developed on the basis of seeking answers to questions that mankind has been asking throughout history about the world around us.

These practical activities oblige students to tackle some significant historical demonstrations with different procedures, while at the same time encouraging debate and reflection, thereby transforming the classroom into a laboratory of ideas. Showing the difficulties that have been encountered throughout history in answering certain questions can help to motivate students who sometimes believe that mathematics consists of a series of formulas and rules that understanding is preserved for privileged minds.

Geometry has a great visual and aesthetic value and offers a beautiful way of understanding the world. The elegance of its constructions and proofs makes it an area of mathematics that is highly appropriate for developing the student reasoning process and providing proofs, as well as for incorporating geometrical constructions as a part of the heuristic in solving problems. Geometric proofs have a great potential for linking geometrical and numerical reasoning in some of the activities proposed, and geometrical and algebraic reasoning in others. In this way, students are able to establish connections among numbers, figures and formulas; that is to say, calculations, geometric constructions and algebraic expressions.

In addition, with these practical activities, students can work with problem-solving, reasoning and proof processes, thus addressing connections, communication and representation. By analysing historical texts, students are introduced to different ways of working from different perspectives (transversal competences), which enables them to tackle mathematical problems by developing their mathematical thinking.

Finally, I conclude that this "way of introducing" the history of mathematics will enable prospective engineers, mathematicians and teachers of mathematics to more readily recognize the most significant changes taking place in the mathematical discipline, and above all to reflect more deeply on the formation of their scientific thought.

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