ROLES OF THE HISTORY OF MATHEMATICS IN THE MATHEMATICAL KNOWLEDGE FOR TEACHING

Abraham ARCAVI
Weizmann Institute of Science, Rehovot, Israel
abraham.arcavi@weizmann.ac.il

ABSTRACT

"Mathematical Knowledge for Teaching" (MKT) is the knowledge required to practice and accomplish the work of teaching mathematics. In this contribution, I describe its essential components and I propose that history of mathematics can be a bountiful source to enhancing of teachers' MKT through examples and the morals thereof.

1 *General Introduction*

I am very grateful to Marta Menghini and to all the members of the organizing committee for honoring me with the invitation to deliver this opening lecture at the 9^{th} European University on the History and Epistemology in Mathematics Education. I am humbled by this honor.

I begin with a proper disclosure: I am not a historian of mathematics and it has been many years since I worked on issues of history of mathematics in mathematics education. However, history of mathematics has been always close to my mind and to my heart and I salute the European Summer University as well as the HPM for the ongoing relevant and excellent work. A large part of my work in the last two decades is in the area of mathematics teacher knowledge, teaching practice and teacher professional development, and it is from that perspective that I would like to talk today. Thus, I am addressing the roles of history on the knowledge required for teaching mathematics, in the spirit of the work I did in my PhD many years ago (Arcavi, 1985). For that purpose, I am relying on the construct "Mathematical Knowledge for Teaching" (MKT) which is receiving much attention in the last two decades, as a framework for both research and the practice of mathematics education.

2 *Mathematical Knowledge for Teaching (MKT)*

Mathematical Knowledge for Teaching (MKT) is defined as the specialized mathematical knowledge required to practice and accomplish the work of teaching mathematics (Ball et al, 2008). Its two main components are Subject Matter Knowledge and Pedagogical Content Knowledge.

Subject Matter Knowledge consists of

- *Common Content Knowledge*: Competence with the mathematical topics (concepts, procedures and their underlying ideas); meta-mathematical ideas and nature of mathematical activity and problem solving.
- *Specialized Content Knowledge*: Ways of presenting mathematical ideas, answers to “why” questions; resourcefulness to find appropriate examples and counter-examples; acquaintance with nature and limitations of different representations and knowing how to link among them; knowledge of applications.
- *Horizon Content Knowledge*: “Horizontal depth” that covers extensions, which may go beyond the topic required by the curriculum and acquaintance with contents students will meet in their future academic studies (in order to unpack and stress the required relevant “predecessor” knowledge).

Pedagogical Content Knowledge consists of

- *Knowledge of Content and Students*: Anticipation of (and sensitivity towards) student idiosyncratic ways of knowing, thinking and doing; competence with attentively listening and interpreting student questions and their often unpredicted answers; discernment of difficulty levels and a repertoire of pedagogical resources to deal with them.
- *Knowledge of Content and Teaching*: Design and sequencing of instruction; judicious choice for appropriate tasks and problems, opportune assignment of different modes of student activity (group-worthy activities, digital labs, individual enquiry) according to the nature of the contents and their affordability.
- *Knowledge of Content and Curriculum*: Familiarity with diverse curricular approaches’ and proficiency in comparing and contrasting them in order to make thoughtful selections of materials for their classes and having the versatility for implementing them flexibly.

Figure 1 is a graphical representation of these components.

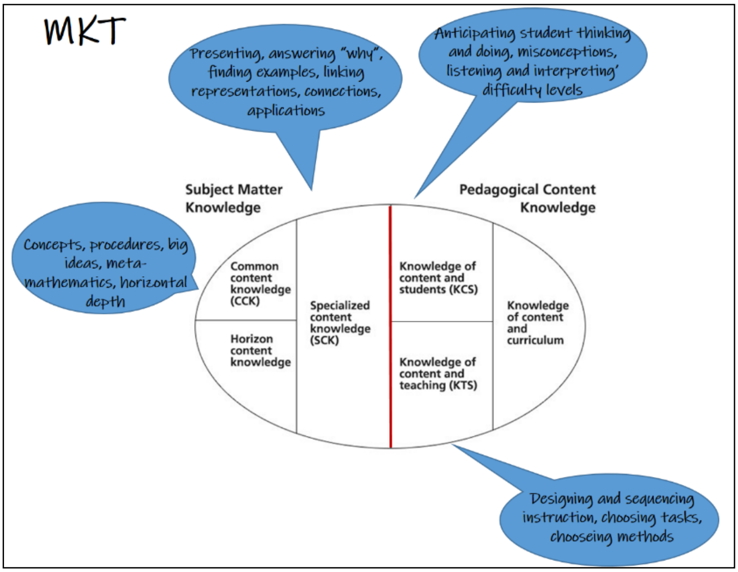


Figure 1. Graphical representation of the components of Mathematical Knowledge for Teaching.

3 Roles of History of Mathematics

History of mathematics can play several roles in supporting and enhancing the mathematical knowledge for teaching. In the following, I describe and exemplify some of these roles:

- Source of problems
- Learning to listen
- Revisiting what is taken for granted
- Original texts as interlocutors

3.1 Roles of problems

"*Where can I find some good problems to use in my classroom?" is a question I am often asked by mathematics teachers. My answer is simple: "the history of mathematics*." (Swetz, 2000, p. 59)

The history of mathematics contains a wealth of material that can be used to inform and instruct in today's classroom. Among these materials are historical problems and problem solving situations" (Swetz, 2000, p. 65)

The following are just a few vivid examples of the above quotes.

The Rhind Papyrus is one of the oldest extant mathematical documents, it is dated around 1500-1600 B.C. and it became widely known less than 200 years ago. It contains arithmetic and geometric problems, some of which are intriguing even today, for example the arithmetic of unit fractions (fractions whose numerator is 1). Consider the following problem, written in modern decimal notation, and borrowed from the Papyrus: "decompose 2/9 into the sum of two unit fractions". An obvious answer is 1/9 + 1/9. However, if we add the restriction that the two unit fractions be different, the problem becomes more interesting. One may resort to trial and error, and then go on to search for more systematic methods, for example looking for a decomposition of 2/9 into different unit fractions, such that in our initial obvious solution of 1/9 + 1/9 we can replace one of the addends by the sum of different unit fractions. We may remind of the following general property:

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)} \longrightarrow \frac{1}{9} = \frac{1}{10} + \frac{1}{90}$$

in order to find a first decomposition as follows: 2/9 = 1/9 + 1/10 + 1/90.

Another result can be helpful, by remembering that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$$

which serves as the basis for

$$\frac{1}{n} = \frac{1}{2n} + \frac{1}{3n} + \frac{1}{6n} \longrightarrow \frac{1}{9} = \frac{1}{18} + \frac{1}{27} + \frac{1}{54}$$

Thus 2/9 = 1/9 + 1/18 + 1/27 + 1/54.

History provides us with yet another way called the Fibonacci – Sylvester method, in honor of these famous mathematicians Leonardo of Pisa, a.k.a Fibonacci (c.1170 – c.1250) and James Joseph Sylvester (1814 – 1897). According to this method, in each step we add the largest unit fraction, which is smaller than what remains. In our case, the largest unit fraction smaller than 2/9 is 1/5. Then, in this case, what remains is a unit fraction – problem solved since 2/9 = 1/5 + 1/45, which is already a third decomposition.

The Egyptian table for the decomposition of 2/n (n ≤101), shows yet another possible decomposition (2/6 + 2/18), for which no method is provided (see Figure 2).

n	a	b	c	d
5	3	15		
7	4	28		
9	6	18		
11	6	66		
13	8	52	104	
15	10	30		
17	12	51	68	
19	12	76	114	
21	14	42		
23	12	276		
25	15	75		
27	18	54		
29	24	58	174	232
31	20	124	155	
33	22	66		
35	30	42		
37	24	111	296	
39	26	78		
41	24	246	328	
43	42	86	129	301
45	30	90		

n	a	b	c	d
47	30	141	470	
49	28	196		
51	34	102		
53	30	318	795	
55	30	330		
57	38	114		
59	36	236	531	
61	40	244	488	610
63	42	126		
65	39	195		
67	40	335	536	
69	46	138		
71	40	568	710	
73	60	219	292	365
75	50	150		

n	a	b	c	d
77	44	308		
79	60	237	316	790
81	54	162		
83	60	332	415	498
85	51	255		
87	58	174		
89	60	356	534	890
91	70	130		
93	62	186		
95	60	380	570	
97	56	679	776	
99	66	198		
101	101	202	303	606

Figure 2. The modern notation of the Egyptian table for decomposition of 2/n into unit fractions

These problems are borrowed from history. We note and stress the "borrowing from" in order to illustrate how history can be a bountiful source of nice problems, and nice solutions, using our knowledge and symbolism. Furthermore, we can also exemplify a question of historic-mathematical interest: how did the Egyptians arrived at their result (not obtained by any of the methods proposed above) and why did they prefer it to others.

These examples illustrate how history may support and enhance the mathematical knowledge for teaching by providing a repertoire of interesting problems (Specialized Content Knowledge). Moreover, this example enriches the knowledge of a seemingly simple topic like fractions, and enacts aspects of mathematical activity such as evoking connections to previous knowledge, bringing it to bear in order to develop systematic methods of expanding fractions into unit fractions and to ponder about their generality (Common Content Knowledge).

History shows how ancient this mathematical topic is and yet in spite of being elementary, it includes unsolved problems as of today, such as the Erdös-Strauss conjecture proposed in 1948 and still unproven: for every $n \in N$, $n>1$, $4/n$ can be expressed as the sum of three unit fractions (e.g. Graham, 2013).

Another very interesting problem from historical sources and its use in a classroom is the following: "In 1355 the Italian professor of law Bartolus of

Saxoferrato (1313-1357) wrote a treatise on the division of alluvial deposit. The problem he discussed is the following… Some landowners … Gaius, Lucius and Ticius, have neighbouring properties besides the bank of a river. The river deposits silt so that the new land is formed at the riverside. How is the new fertile soil to be divided up?" (Van Maanen, 1992, p.37).

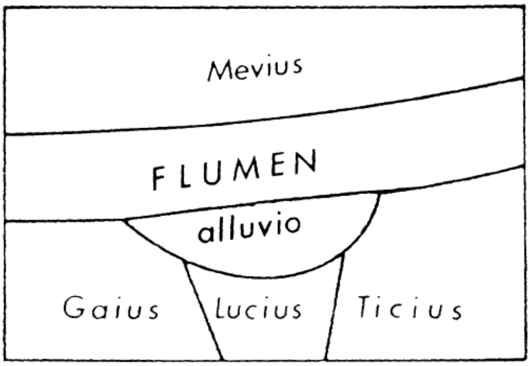


Figure 3: Map of the division of the alluvial deposit problem

The following are some of the goals of using this problem in the classroom: "to demonstrate the importance of mathematics in society; …to integrate disciplines; to let pupils to discover a number of constructions with ruler and compasses… to let pupils solve some juridical problems using the constructions that they had discovered earlier" (van Maanen, 1992, p. 42).

Laurence Sherzer, a mathematics teacher in a Florida school reported on an eight-grade mathematics lesson he had taught about the betweenness property of rational numbers. The class worked on methods of finding a rational number between two given rational numbers. They focused on the average and on how to calculate it (adding the two given numbers and dividing the sum by 2). The students not only practiced the procedure of adding fractions and dividing by 2, but they also had a procedural way to be convinced that since it is always possible to find an average between two given rational numbers, there is always one number between two given ones. Then "a student who had not been paying much attention but had been scribbling furiously suddenly interrupted. "Sir, you don't have to go to all that trouble to find a fraction between two fractions, all you have to do is add the tops and the bottoms." (Sherzer, 1973, p. 229). Sherzer admitted that he was going to reject outright the idea, possibly having in mind the typical erroneous procedure many students have for adding two fractions (i.e. numerator plus numerator over denominator plus denominator). On second thoughts, he decided to go along with the student suggestion. He suggested to go back and to apply the student's procedure to the examples they had already worked out by finding averages, and the new method indeed yielded a number in between. The class became excited and they tried many examples, until the teacher suggested to try to find a general proof to show that the procedure proposed indeed always yields a number in between two givens. The algebraic proof is rather simple (for a visually very convincing geometrical proof, see Arcavi, 2003). The teacher acknowledged that he did not know such property and the class was over, he "thought of that one moment when I was about to tell Mac Kay [the student's name], 'No, that's not the way it's done'" (Sherzer, 1973, p. 230).

This property, which the teacher named as "MacKay's Theorem", was firstly documented in the book *Le Triparty en la Science de Nombres* by the French mathematician Nicolas Chuquet (1445?-1488?). The manuscript of this book remained in private hands for about four centuries and was published in 1880 in Italy. "La rigle des nombres moyens" in its English version (Flegg et al., 1985, p. 91) reads as follows:

> whoever would want to find an intermediate between $3\frac{1}{2}$ and $3\frac{1}{3}$ should add 1 to 1, which are the two numerators, and they come to 2 for numerator, and then 2 to 3 which are the two denominators, making 5 for denominator. Thus I have $3\frac{2}{5}$ as an intermediate between $3\frac{1}{2}$ and $3\frac{1}{3}$. For $\frac{2}{5}$ is more than $\frac{1}{3}$ and less than $\frac{1}{2}$.

Figure 4: English translation of Chuquet's Rule of Intermediate Numbers

Had the teacher known about episodes from the history of rational numbers, and in particular Chuquet's rule (*Common Content Knowledge*), he could have saved to himself the indecision about pursuing the student suggestion, and the risk of rejecting it, which he was at the verge of doing. Fortunately, the teacher did pursue the student proposal and, thanks to his opportune decision to listen to the student, he uncovered a piece of mathematics new to him and to his students (except for MacKay…).

3.2 Learning to listen

When students genuinely engage in learning and doing mathematics, they frequently proceed in idiosyncratic yet reasonable and productive ways, which are not always aligned with what teachers expect, and such was the case of

MacKay. Thus, the important component of MKT "*Knowledge of Content and Students*" includes (a) the anticipation of (and sensitivity towards) student distinctive ways of knowing, thinking and doing and (b) the competence of listening attentively and interpreting student questions, their often unpredicted ways of reasoning, their answers and their unexpected suggestions and conjectures.

By "listening to students", we mean giving careful attention to students, trying to understand what they say and do and the possible sources and entailments thereof. Such listening should include:

- detecting, taking up and creating opportunities for students to engage in expressing freely their mathematical ideas;
- questioning students in order to uncover the essence and the sources of their ideas;
- analyzing what one hears (sometimes in consultation with peers), making the enormous intellectual effort to adopt the 'other's perspective' in order to understand it on its own merits; and
- deciding in which ways to integrate productively students' ideas into the development of the lesson.

The importance of listening as a teaching skill cannot be overstated. It may be a strong manifestation of "a caring, receptive and empathic form" (Smith, 2003, p. 498) of teacher-student interactions. If often modeled by teachers, students as well as their mathematical productions would feel respected and valued. Moreover, the habit of listening as modeled by the teacher may be internalized by the students and become a habit in their repertoire of learning techniques and interpersonal skills. Above all, listening enables teachers to understand better student thinking for the benefit of good teaching and robust learning, and may benefit the teachers themselves. "Thinking ourselves into other persons leads us to reflect about our own relationship to mathematics" (Jahnke, 1994, p. 155). In other words, effective listening may influence 'listeners,' by making them re-inspect their own knowledge, against the background of what was heard from others. Such re-inspection of the listener's own understandings may promote the re-learning of some mathematics or meta-mathematics. There are several candid self-reports of this phenomenon even by mathematicians (e.g. Aharoni, 2005; Henderson, 1996).

Listening to students poses several challenges. For example, once we understand a complex idea, we may tend to forget (or even dismiss) the process we underwent while learning that idea. Listening requires unpacking that process. Listening also requires "decentering", namely the capability to adopt another person's perspective discarding as much as possible our own.

Given the importance of 'listening' towards understanding the students' point of view and in spite of the challenges it poses, it is a learnable ability. History of mathematics can provide rich scenarios for such learning, for example by approaching certain primary sources. Primary sources often offer ways of doing mathematics different from what is common nowadays and may conceal the thinking behind them. When facing a historical source with an approach foreign to us, we cannot dismiss it as 'incorrect', in the same way that we as teachers may dismiss an unexpected student approach. When facing an initially cryptic source, an effort may be required to make sense of it, and this activity of "deciphering" requires exercising a similar kind of decentering and unpacking needed for listening to students. Thus, working with teachers on activities of reading and understanding idiosyncratic ways of doing mathematics is a way of learning to listen. Such an activity taken from an extract of the Rhind Papyrus in which, what we call today, a linear equation with one unknown is solved was tried with prospective teachers. A detailed description of the activity and the findings of the experience can be found in Arcavi & Isoda (2007). The activity of deciphering the primary source was shown as promising in supporting the effort and nurturing the capability of understanding the other's perspective.

3.3 Revisiting was taken for granted

"I have observed, not only with other people but also with myself […] that sources of insight can be clogged by automatism. One finally masters an activity so perfectly that the question of how and why is not asked any more, cannot be asked any more, and is not even understood any more as a meaningful and relevant question." (Freudenthal, 1983, p. 469)

The development of ideas in history provides a repertoire of intricacies that may illuminate aspects of mathematics around questions that "are not asked anymore". Consider, for example, the following text taken from a letter by the mathematician Antoine Arnauld (1612-1694) to a colleague Jean Prestet (1648-1691) as it appears in Schrecker (1935).

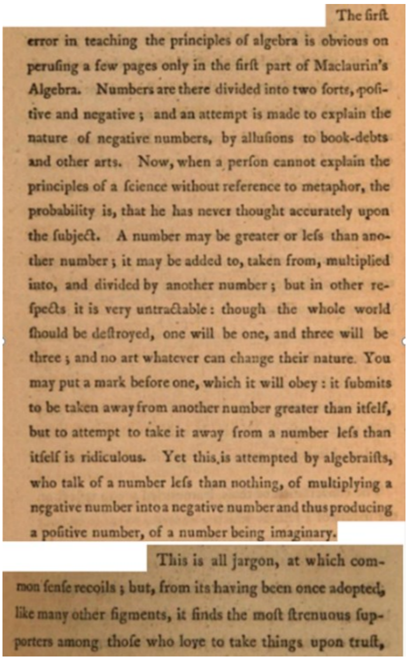
> *"…1 is to 3, as 4 is to 12. And 1 is 1/3 as 1/4 is 1/12. But I cannot adjust this to multiplications of two minus. For will we say that +1 is to -4, as -5*

is to +20? I do not see it. Because +1 is more than -4. And conversely -5 is less than +20. Whereas in all other proportions, if the first term is greater than the second, the third must be greater than the fourth.”

This text was presented to teachers in several workshops, and they were asked to formulate an answer to this contradiction between the idea of proportionality and the rule for multiplying two negative numbers, as if a student (Arcavi, 1985) posed it. This task was aimed at enhancing teachers’ Specialized Content Knowledge.

3.4 Original texts as interlocutors

The Principles of Algebra by William Frend (1757-1841) was published in London in 1796. In this book, there is a virulent attack on the use of negative numbers. The following are extracts reflecting the arguments.


The firſt error in teaching the principles of algebra is obvious on peruſing a few pages only in the firſt part of Maclaurin's Algebra. Numbers are there divided into two ſorts, poſitive and negative; and an attempt is made to explain the nature of negative numbers, by alluſions to book-debts and other arts. Now, when a perſon cannot explain the principles of a ſcience without reference to metaphor, the probability is, that he has never thought accurately upon the ſubject. A number may be greater or leſs than another number; it may be added to, taken from, multiplied into, and divided by another number; but in other reſpects it is very untractable: though the whole world ſhould be deſtroyed, one will be one, and three will be three; and no art whatever can change their nature. You may put a mark before one, which it will obey: it ſubmits to be taken away from another number greater than itſelf, but to attempt to take it away from a number leſs than itſelf is ridiculous. Yet this is attempted by algebraiſts, who talk of a number leſs than nothing, of multiplying a negative number into a negative number and thus producing a poſitive number, of a number being imaginary.

This is all jargon, at which common ſenſe recoils; but, from its having been once adopted, like many other figments, it finds the moſt ſtrenuous ſupporters among thoſe who love to take things upon truſt,

Figure 5: Extracts from Frend’s Algebra

The text’s arguments are the following:

- Rejection of “reference to metaphors” (“debts and other arts”)
- Numbers as magnitudes (“one will be one”)

- Rejection of an extended version of operations (taking away only the small number from the greater, otherwise ridiculous)
- Rejection of an extended version of number ("a number being imaginary")

These claims may open up productive discussions about the very nature of mathematics, the place of generalizing beyond the concrete, and the role of didactical resources in presenting and concretizing abstract ideas.

Frend's book has many mathematical developments in which he juggles in order to avoid the use of negative numbers. Of special interest is his treatment of the solution of quadratic equations which purposefully avoid negative numbers. Not only are his mathematical arguments worth following but also the vivid experience of how the separation of cases (to avoid negative numbers) losses the efficiency and elegance of generality.

4 Final remarks

In this presentation, I attempted to illustrate the crucial roles that history of mathematics can play in supporting and enhancing the development of Mathematical Knowledge for Teaching. Further examples and further roles can be found, explored and tried out in the many environments for teacher education and teacher professional development forums.

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