# WHAT HISTORY TRAINING FOR FUTURE MATHEMATICS TEACHERS?

## PERSONAL EXPERIENCES AND REFLECTIONS

**Dominique TOURNÈS**
University of Reunion Island, Laboratory of Computer Science and Mathematics
dominique.tournes@univ-reunion.fr

## ABSTRACT

In this paper, I present my way of training future teachers in the use of history to improve students' mathematical learning. The strategy that I recommend includes three components in constant interaction. First of all, there is the manipulation of artefacts that the teacher can use in a process of semiotic mediation to favour the appropriation of the mathematical knowledge embedded in these objects. This importance given to gestures, procedures and instruments is reinforced by an opening towards ethnomathematics. Secondly, there is the study of short original texts taken mainly from six major works covering almost completely the contents of secondary education. This study is done in close connection with specific curriculum items and the conception of scenarios for the classroom. Finally, the future teachers are responsible for designing pedagogical sequences inspired by history and experimenting with them in their classes during the internships. This devolution phase, which is the subject of didactic analyses, seems to me essential for a sustainable integration of the training's achievements. The main objective of this triptych of activities is not to train future teachers in history, but to train them in the teaching of mathematics by deepening their knowledge of the discipline on the cultural, epistemological and didactic levels.

## *1 Context and guidelines of my training practice*

As a historian of mathematics, my research focuses on the history of numerical analysis in the broadest sense of the term. I am interested in the methods and instruments of calculation in the period before the computer, between the middle of the 18th century and the middle of the 20th century. In particular, I have worked on differential equations, graphical calculus, nomography, numerical tables and engineering mathematics (Tournès, 2022).

For many years I have been involved in the training of secondary school teachers in the history of mathematics. In this training, I do not see history as an end in itself, but as an entry point to change teachers' and students' representations of mathematics, and to foster students' learning. The aim of this paper is to give a testimony of the personal way in which I put this teacher training into practice.

### 1.1 My environment: Two French islands in the Indian Ocean

Before getting into the heart of the matter, I would like to complete these elements of context with a few words about the places where I teach, which are located in the Indian Ocean, the third largest ocean on the planet, bordered by many states with very different cultures and languages. My immediate environment is in the southwest, where there are about one hundred and fifty islands grouped into five states: Madagascar, the Seychelles, Mauritius, the Comoros and France.

All these islands are multi-ethnic, multi-cultural and multi-lingual societies, with populations from Asia, Africa and Europe. Two of these islands are French departments: Reunion, which is part of the Mascarene archipelago, and Mayotte, which is part of the Comoros archipelago. In these two islands, the educational system, the curricula, the university structures and the teacher training system are the same as in France, so what I am going to talk about is representative of what is currently being done in France to integrate the history of mathematics into teaching.

### 1.2 My institutions and sources of inspiration

In Reunion Island, teacher training is carried out within INSPE (National Higher Institute of Teaching and Education), an institute belonging to the University of Reunion Island. As in other parts of France, this university also has an IREMI (Research Institute for Mathematics and Computer Science Education), where action research is carried out to support training. In Mayotte, there is a University Centre and an IREMIS (IREMI + S as Science), but teacher training depends on the University of Reunion Island. That's why I also teach there. In both islands, I am in charge of a history of mathematics unit in the master's degree course for secondary school teachers. In each of the islands, there are about twenty students in each year of the master.

The training that I have designed and that I am implementing is the result of a long experience. It has been constantly nourished by my participation in the work of the inter-IREM commission in France, the HPM meetings and the

European Summer Universities at international level. Numerous studies presented at HPM and ESU meetings have examined the epistemological, methodological, practical and didactic objections that could slow down the integration of history into mathematics teaching. In return, these studies have highlighted the cognitive, epistemological, didactic, affective and cultural contributions of the integration of history, and have addressed the crucial problem of rigorously evaluating the effects of this integration on teachers' practices and students' mathematical learning (for a synthetic overview of these studies, see Clark et al., 2019; Chorlay et al., 2022).

### 1.3 An initial assessment

My aim here is not to go back over these high-quality theoretical works, to which I would have little new and relevant to add, but to explain how I have appropriated them in order to put them into practice effectively in my activity as a secondary school teacher trainer. As a starting point, I would simply like to take up the data and analysis presented by Marc Moyon at the 2021 HPM meeting (Moyon, 2021, 2022).

Based on a survey of 646 in-service secondary school teachers, he showed, first of all, that the vast majority of teachers do not introduce history of mathematics in their lessons or only introduce it very occasionally, fewer than three times a year. Only 9% of teachers use it regularly, in almost every lesson. If we take a closer look at what they do, it is principally to introduce a teaching sequence, in particular by means of anecdotes, short biographies or elements of context, in other words, historical snippets with little didactic impact on mathematical learning. On the other hand, when teachers were asked if they were interested in the history of mathematics and if they would like to use it, a large majority (71%) answered yes. Teachers are therefore willing, even enthusiastic, to use elements of history in their lessons.

We are thus faced with a complex challenge: how to get teachers to move from intention to action. To do this, as Marc Moyon has shown, we must meet their expectations by providing them with historical knowledge, of course, but also with didactic reflections, exchanges of practices, and an evaluation of the effects of history on the mathematical representations and learning of pupils. In this task, I think that we must remain modest. The right solution could be to start from the ordinary practices of teachers, their teaching programmes, their school textbooks and the usual documents at their disposal. Without upsetting their familiar pedagogical context, we can help them to develop a critical view of the historical data present in their primary resources and provide them with ways to transform and enrich these resources.

## 2 *My overall strategy*

The strategy I advocate has three constantly interacting components that complement my course. First of all, there is the manipulation of artefacts that the teacher can exploit in a process of semiotic mediation to favour the appropriation of the mathematical knowledge embedded in these objects. This importance given to gestures, procedures and instruments is reinforced by an opening towards ethnomathematics.

Secondly, there is the study of short original texts. This work is done in close connection with specific curriculum items and the design of scenarios for the classroom. Here, it seems important to me not to limit ourselves to textual sources. In the history of mathematics, the sources are, on the one hand, texts and textual inscriptions, but also artefacts and instruments whose manufacture and use testify to mathematical activity.

Finally, the future teachers are asked to design pedagogical sequences inspired by history and to experiment with them in their classes during their practical training. This devolution phase, which is the subject of a priori and a posteriori didactic analysis, seems to me to be essential for a lasting integration of the training's achievements.

The evaluation of my teaching unit includes two tests:

- A three-hour written examination consisting of three to five exercises on original artefacts or texts, each with historical, mathematical and pedagogical questions.
- The writing and defence of a memoir on the design, classroom experiment, and didactic analysis of an educational sequence inspired by history.

### 2.1 The outline of my course

We will go through these three components of my training in turn. Before, here is the general outline of the course (see figure 1). As I only have a limited amount of time, I have decided to organise my course around six mathematicians and six books which mark crucial stages in the evolution of mathematics and which allow me to cover almost all the contents which appear in the secondary school curricula. These six mathematicians are:

- Euclid, in link with the axiomatic and deductive method, and the essential results of elementary geometry and number theory;

- al-Kwhārizmī for the constitution of algebra as an autonomous discipline;

- Descartes who, in a way, synthesised Greek geometry and Arabic algebra to give birth to coordinate or analytical geometry;

- Newton for the development of infinitesimal calculus;

- Cauchy and the constitution of the classical analysis based on the notion of limit;

- and finally, Jakob Bernoulli for his founding work of probability theory.

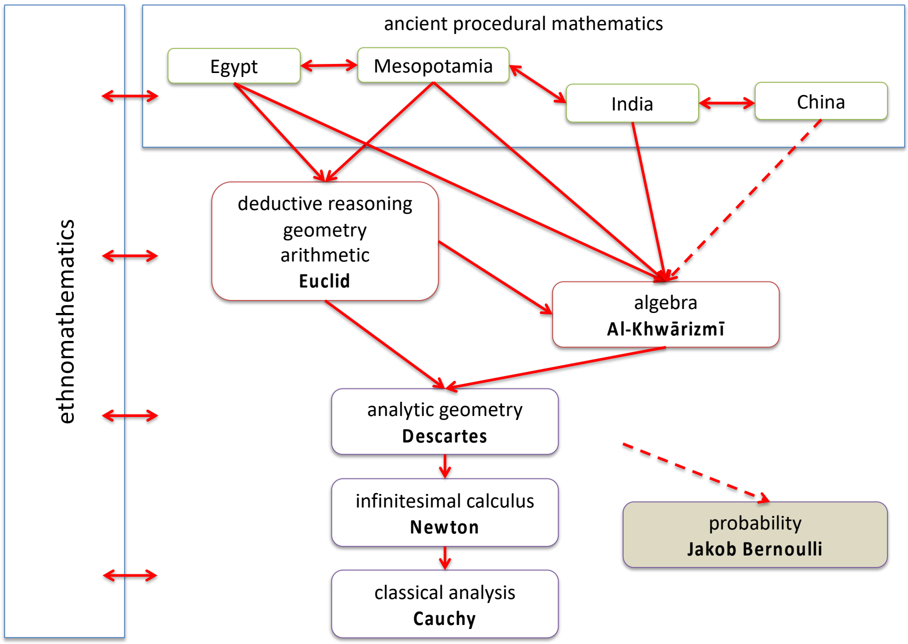


**Figure 1.** The outline of my course

If I focus my course on these six mathematicians so as not to disperse myself, this does not mean that I am talking only about them. I am, of course, placing them in a wider context. For example, in my lecture on Descartes, I also talk about Fermat, who at the same time formalised coordinate geometry. Or, for another example, I don't give my lecture on Newton without talking about Leibniz, the other inventor of the infinitesimal calculus.

Alongside this content, I have included in the top margin the ancient mathematical traditions of Egypt, Babylon, India and China. For lack of time, I have chosen not to devote specific lectures to them. However, of course, when I talk about Greek mathematics, I mention its Egyptian and Babylonian origins, and when I talk about Arabic mathematics, I mention in addition the influence of Indian mathematics. Students also have the opportunity to explore these ancient traditions on their own and to talk about them to the other students when defending their memoirs.

The margin on the left concerns ethnomathematics. Originally, ethnomathematics was seen as the study of mathematical ideas and practices in societies without writing. Now the definition has been broadened. It is, in general, the study of culturally specific uses of mathematical concepts and knowledge, especially those developed outside the scholarly and institutional field. In this sense, we can say that, insofar as they are inscribed in a specific culture, society and language, Egyptian, Babylonian, Indian or Chinese mathematics are ethnomathematics. In another direction, the mathematical practices specific to certain professional circles that have not studied academic mathematical content, such as sculptors, stone cutters or carpenters, are part of ethnomathematics. Finally, ethnomathematics is concerned with the primary mathematical knowledge acquired by children in their family or their cultural environment, a knowledge that may conflict with school learning.

Ubiratan d'Ambrosio (2006), one of the founders of ethnomathematics, has highlighted that in the history of mathematics, three types of knowledge should be taken into account: firstly, the scholar mathematics developed by mathematicians, secondly, the mathematics taught at school, and thirdly, the mathematics used in the street and in the workshop. This triptych seems to me very important: when we conceive a school activity, we must take into account, on the one hand, the scholarly knowledge and its didactic transposition, and on the other hand, the cultural roots, representations and primary knowledge of the children.

## 2.2 Correspondence between texts, curriculum and artefacts

As I said, the historical sources that I use, whether texts or instruments, are closely linked to items in the secondary school curriculum. I hope that the following table makes it possible to see more clearly the interest of the six books that I exploit in my training.

- The *Elements* of Euclid allow us to work on elementary geometry, including the Pythagorean theorem and the intercept theorem, which are at the heart of Middle School geometry. They also provide an opportunity to review the main concepts of arithmetic such as divisors, greatest common divisor, Euclid's algorithm and prime numbers. The associated artefacts are, of course, the ruler and the compass, which correspond directly to Euclid's geometry, puzzles to demonstrate the Pythagorean theorem, and cultural patterns, such as rosettes, friezes and pavings, which constitute a good entry into the elementary objects of geometry and usual geometric transformations.

| **Texts** | **Curriculum of Middle School and High School** | **Artefacts** |
|---|---|---|
| Euclid<br>*The Elements* | Elementary geometry<br>Pythagorean theorem<br>Intercept theorem<br>Divisors, gcd, prime numbers | Ruler and compass<br>Puzzles<br>Cultural patterns |
| Al-Khwārizmī<br>*al-Kitāb al-Mukhtaṣar fī Ḥisāb al-Jabr wal-Muqābalah* | Indian calculation<br>Elementary algebra<br>Linear and quadratic equations | Token abacuses |
| Descartes<br>*La Géométrie* | Coordinate geometry<br>Equations of straight lines, circles, and other curves | Ruler, compass and conics<br>Linkages<br>Nomograms |
| Newton<br>*The Method of Fluxions and Infinite Series* | Numerical solution of equations<br>Tangent and quadrature problems<br>Extremum problems | Planimeters<br>Integraphs |
| Cauchy<br>*Résumé des leçons données à l'École royale polytechnique sur le calcul infinitésimal* | Limits, continuity<br>Derivatives, integrals<br>Common functions | Graphic construction of differential equations |
| Jakob Bernoulli<br>*Ars Conjectandi* | Combinatorics<br>Probability, expected value<br>Law of large numbers | Dices, cards<br>Spreadsheets |

**Table 1.** Correspondence between texts, curriculum and artefacts

- Concerning al-Khwārizmī, I talk about his book on algebra, which allows us to work on the introduction to algebra and on linear and quadratic equations, but I also talk about his lost book on Indian calculation, which allows us to go back to numeracy and operating techniques. Manipulations with token abacuses, which make it possible to understand that Indian calculation is an abstract transcription on paper of the manipulations that one used to do on the abacus, prove to be fruitful, as much with future teachers as with the pupils (Daval & Tournès, 2018).

- Descartes' *Geometry* offers the opportunity to study coordinate geometry, equations of straight lines, circles and other curves. Conics are no longer directly on the French High School curriculum, but there is still the parabola as the representative curve of the square function and the hyperbola as the representative curve of the inverse function. These curves are sufficient for geometric construction activities with ruler, compass and conics, instruments that allow solving equations up to the fourth degree. Parabolic and hyperbolic nomograms are another type of instruments that can be used to solve equations up to the fourth degree (Tournès, 2018b). Finally, there is also the possibility of manipulating linkages that give access to other algebraic curves.

- In Newton's *Method of Fluxions and Infinite Series*, there are interesting extracts for introducing the numerical solution of equations, the questions raised by the handling of infinity in mathematics, tangent problems, quadrature problems and extremum problems. In parallel, one can propose manipulations with planimeters and integraphs, those mechanical instruments that perform the operations of the integral calculus in an exact manner.

- Cauchy's *Course* is particularly relevant because it is very close, in its presentation, to what is currently done in High School, where the notions of limit, derivative and integral are defined in natural language, without using the formalism that was introduced later by Weierstrass. In connection with the Euler-Cauchy method used by Cauchy for the definition of integrals and for the proof of the fundamental existence theorem for differential equations, one can propose graphical constructions of integral curves of differential equations, which allows one to acquire a kinaesthetic knowledge of what is an integral curve of a differential equation (Tournès, 2018a).

- Finally, Jakob Bernoulli's *Ars Conjectandi* provides a good reflection on the beginnings of probability theory. Classical historical problems, such as the

Duke of Tuscany problem or the Chevalier de Méré problem can be experimented with dices and then simulated with a spreadsheet.

## *3 Manipulation of artefacts to construct mathematical knowledge*

After this general presentation of my course, I will go into a little more detail about each of the types of activities proposed to future teachers. The first is the manipulation of artefacts. In this regard, I would like to mention the work of Maria Bartolini Bussi and Michela Maschietto (2007), who have written extensively on the use of artefacts to construct mathematical knowledge using semiotic mediation theory. I am particularly grateful to Michela Maschietto who trained us in this theory when she came as a visiting professor to University of Reunion Island a few years ago. It was under her influence that I introduced more and more artefact manipulations into my training. I would also like to mention the work of Pierre Rabardel (1995), which has had a great impact on mathematical instrumentation. In particular, he introduced the distinction between artefact and instrument, as well as the notions of instrumentation and instrumentalisation, and the process of instrumental genesis. In France, in a book on mathematical constructions directed by Évelyne Barbin (2014), the inter-IREM commission also worked on the consideration of gestures and instruments. On the same theme, I would like to mention a very rich work by John Monaghan, Luc Trouche and Jonathan Borwein (2018) which contains a lot of information on the relationship between material tools and mathematical concepts. Finally, recent research in ethnomathematics can also shed light on the role of gestures, procedures and instruments in the emergence and transmission of mathematical ideas.

### 3.1 Ruler and compass

With these references in mind, let's take a closer look at some of the artefacts I introduce in my training. The photos I will use to illustrate this paper sometimes show my own students, who are future teachers, and sometimes show my students' students, who are middle school or high school students. It doesn't matter, because I do with the future teachers the same manipulations that they later do with their students.

In connection with Euclid's *Elements*, a reflection on the ruler and compass is necessary. Understanding the material sources of the definition of the circle and the third postulate in the *Elements*, and conversely the practical implementation of these abstract notions, deserves reflection. On figure 2, you can see students constructing the centre of a circle with a compass alone. This is the famous Napoleon's problem. Geometric activities in macro-space and meso-space seem to be necessary, because the micro-space of the sheet of paper is already an abstraction. The examination of a compass used by a Malagasy sculptor (see figure 3) is also fruitful. We can see that the usual compass used on paper can be problematic: it is not a plane linkage, and the radius of the circle is not materialised, it remains virtual. On the other hand, with a taut rope or a rigid rod, we have a plane linkage, as simple as possible, which materialises the radius of the circle.

**Figure 2.** Napoleon's problem

**Figure 3.** The compass of a Malagasy sculptor

### 3.2 Puzzles for Pythagorean theorem

In parallel with the Euclid's proof of the Pythagorean theorem, it can be fruitful to examine other demonstrations. I studied all the puzzles I could find for the demonstration of this theorem, and I came to the conclusion that there were 11 fundamentally different ones, based on distinct ideas, the others being just variations of those (Tournès, 2017). I had these 11 wooden puzzles made with a laser machine (see figure 4).

This is how I use these puzzles with my future teachers. Each group is tasked with studying a particular puzzle by answering the following questions:

- Complete the puzzle.

- Prove that the pieces of the two small squares fill the large square exactly, without loss or overlap.

- Write a construction algorithm for the cutting of the two small squares and execute it, either with geometry instruments or with dynamic geometry software.

- Search on the internet for the historical origin of the puzzle: Author? Period? Context?

- Compare with Euclid's proof of the Pythagorean theorem.

- What pedagogical use could you make of this puzzle in the French 4th grade (13-14 years old)?

At the end of the workshop, each group presents its findings to the others.

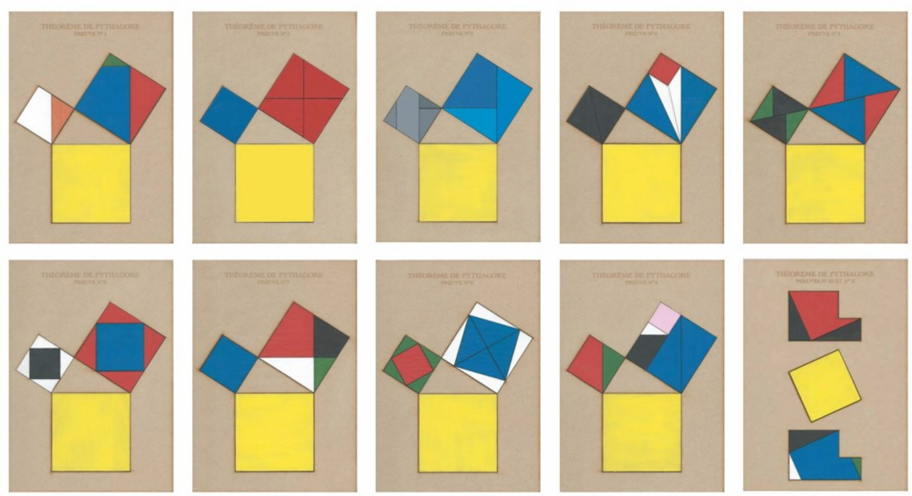

**Figure 4.** Eleven puzzles for the Pythagorean theorem

### 3.3 Cultural patterns from Reunion, Mayotte and Madagascar

In the local decorative arts, make-up, embroidery, basketry, woodcarving, etc., in Reunion, Mayotte and Madagascar, one finds patterns that allow one to work on the elementary figures of geometry and on the isometries of the plane (see figure 5). In particular, rosettes, friezes and pavings can be found. It is known that there are two infinite groups of rosettes, seven groups of friezes and seventeen groups of pavings. Working on these local artefacts as part of their culture is a great motivation for teachers and students.

In passing, I would like to read a quotation from Descartes, in the *Rules for the Direction of the Mind*, which recommends that we start from the observation of craft objects with regularities to extract abstract ideas:

> "Still, since not all minds have such a natural disposition to puzzle things out by their own exertions, the message of this Rule is that we must not take up the more difficult and arduous issues immediately, but must first tackle the simplest and least exalted arts, and especially those in which order prevails – such as weaving and carpet-making, or the more feminine arts of embroidery, in which threads are interwoven in an infinitely varied pattern." (Descartes, 1985, p. 35)

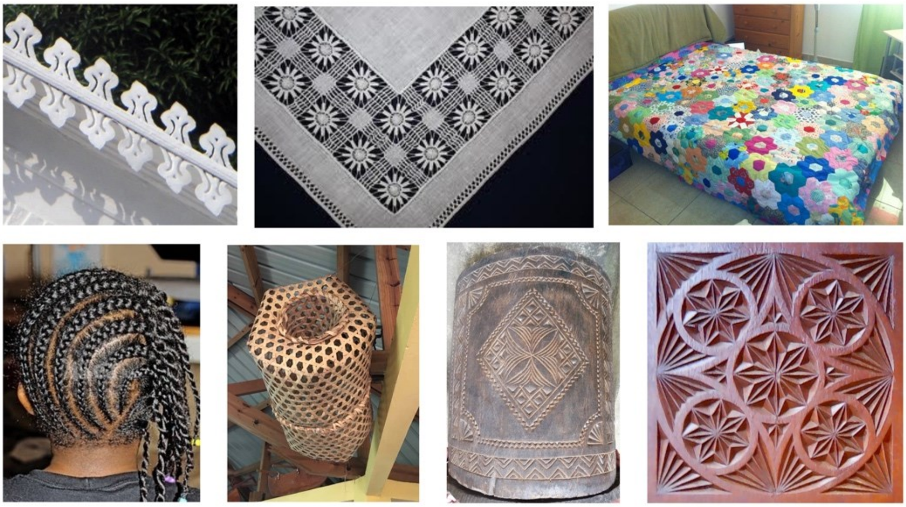

**Figure 5. Cultural patterns from Reunion, Mayotte and Madagascar**

### 3.4 Token abacuses

As I said, I use the token abacuses a lot, to make the future teachers work on numeration, operating techniques and the history of calculation, which goes from token or ball abacuses to electronic calculators through Indian written calculation. We look successively at addition, subtraction, multiplication and division. From a didactic point of view, the token abacus is more interesting than the Chinese abacus, because one can put as many tokens as one wants in a column, which makes it possible to completely separate the inscription of the numbers on the abacus from the later manipulations on these numbers. As tokens, I use Cape peas. They are quite large seeds that children can handle easily (see figure 6).

### 3.5 Nomograms

Another medium that I like very much is the graphic tables, also called nomograms. These tables played a big role in the history of numerical calculation in the 19th century and the first half of the 20th century (Tournès, 2022, Chap. 4). They were the preferred calculation tool of engineers and other professions. They are still used today in medicine. In my training, I use graphical tables formed from graduated hyperbolas or parabolas (see figure 7). A relationship between three numbers is expressed by the intersection of three lines or the alignment of three points. With this type of table, multiplication, division, extraction of square roots, and solving equations up to the fourth degree can be carried out. The manipulation of these tables involves most of the concepts of coordinate geometry present in high school, as well as the representative curves of the square function and the inverse function.

**Figure 6.** Token abacuses

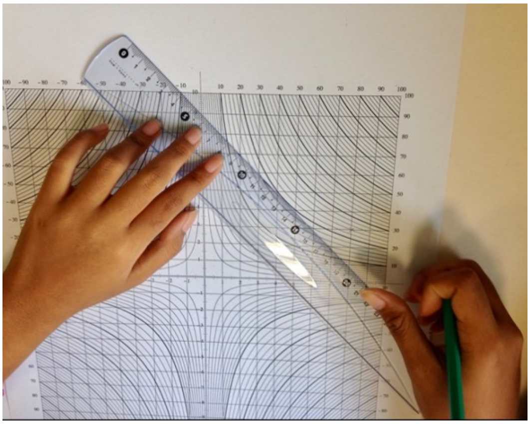

**Figure 7.** Use of a nomogram

### 3.6 Linkages

In addition to the ruler and compass, linkages allow all algebraic curves to be drawn, as suggested in Descartes' *Geometry* and demonstrated in the 19th century by Alfred Bray Kempe. Linkages can also perform geometric transformations: translation, rotation, axial symmetry, homothety, affinity, inversion, etc. We have in Reunion a collection of mathematical machines from the Laboratory of Mathematical Machines in Modena (see figure 8). Michela Maschietto taught us how to use them pedagogically.

### 3.7 Planimeters and integraphs

In the field of infinitesimal calculus, graphomechanical instruments are also available. Planimeters allow the exact calculation of the area of a surface. Integraphs make it possible to draw exactly the primitive curves of a given curve and, more generally, the integral curves of differential equations (Tournès, 2022, Chap. 6-7). These instruments played an important role in the first half of the 18th century in legitimising transcendental curves by tracing them with a single continuous motion, just as linkages had previously legitimised algebraic curves. I think it's important to talk to future teachers about this and have them manipulate it as much as possible. The difficulty is that it is not easy to have a large number of these instruments in cheap versions for concrete use in the classroom with pupils. In Reunion, we have started to make Prytz planimeters with 3D printers and to experiment with them in the classroom (see figure 9). Similar research is underway in France at the IREM des Pays de Loire (Guillet, Moureau & Voillequin, 2019; Tournès & Voillequin, 2022) and in Italy, where Pietro Milici is designing and building integraphs, which he is studying from a semiotic and didactic point of view with Michela Maschietto (Maschietto, Milici & Tournès, 2019).

**Figure 8.** Linkage

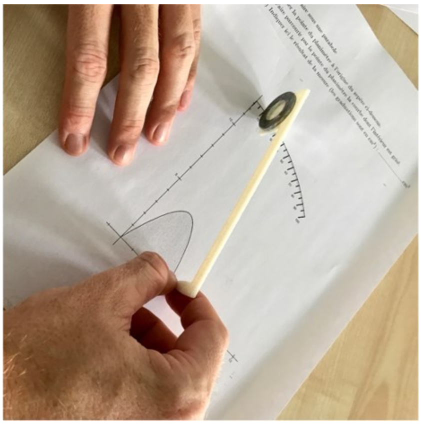

**Figure 9.** Prytz planimeter

### *4 Study of short original texts*

Let's move on to the second type of activity I offer my students. These are workshops in reading original texts. These workshops are organised in groups of three to five students. Each group is given a short excerpt from an original text directly related to curriculum items, with some questions to guide their research. Students are asked to study the text from a historical and mathematical perspective and then use it to develop a teaching scenario for a given level.

For example, I give them an extract from Descartes' *Geometry* on the geometric construction of the roots of a second-degree equation. Students should understand Descartes' text, which is rather elliptical, and justify in detail all the results it contains. Then they have to put Descartes' construction into practice on a particular case that I provide them with, the equation $z^2 = 9.1\,z - 7.2$. For that, I give them a sheet of paper on which are drawn three-line segments, one which is the unit of length and two others which measure 9.1 and 7.2 in relation to the chosen unit. They have to construct the roots of the equation with a ruler and a compass. At the end, they can measure the segments obtained, deduce approximate values of the roots and compare them with the values provided by their calculator. On figure 10 you can see some of the constructions made. In general, there are almost as many different constructions as there are students, which offers the opportunity for an interesting debate.

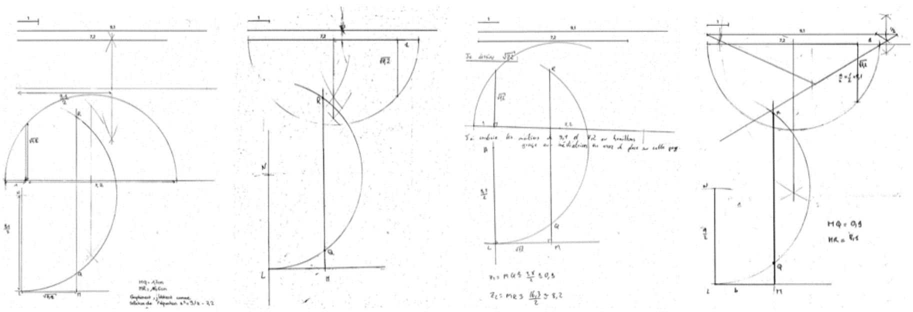

**Figure 10.** Four constructions of a second-degree equation

In the third and final stage, the students are asked to link this activity to the high school curriculum and to write the pedagogical scenario of a class session. In this case, they see that the curriculum talks about al-Khwārizmī and his solution of second-degree equations, and Descartes in relation to coordinate geometry and the equation of a circle.

The workshop ends with each group presenting its pedagogical scenario and a collective discussion on how to use Descartes' text in class. Should the text be read to the students? At the beginning or at the end of the session? In original or modernised spelling? Should they just use it as inspiration to design a meaningful problem? As an introduction to quadratic equations or as an application after studying them? Etc.

Such a workshop can last between two and three hours. This is the time needed for the students to really take ownership of the text and produce substantial output. Students are usually enthusiastic about this way of working, as they feel that they come away with simple ideas that are really applicable in the classroom.

I won't say more about my reading text workshops, because I think this example is enough for everyone to understand how they work. So now I'm going to talk about the third type of activity I offer to students to complement my course.

## 5 *Examples of memoir topics chosen by students*

In this third type of activity, each student has to write a memoir on the integration of a historical topic in the classroom. For this memoir, the student has to contextualise the topic, design a teaching sequence inspired by history, experiment with this sequence in the classroom during his or her practical training, and write a didactic report on the experimentation, including an analysis of student work.

The topics can be very diverse. The memoirs are posted on the digital platform of my teaching unit following the documents of my course, and serve as complements, especially on questions that we did not have time to deal with during the course. They are the subject of a defence during which each student presents his or her memoir to the others for a collective discussion. As there are about twenty students, each student will leave at the end of the year with about twenty ideas for historical activities experienced in the classroom.

Concerning the didactic analysis of the sessions, I ask the students to use the concepts and tools they have assimilated in their didactics course. Indeed, they follow a didactics of mathematics course with another teacher in parallel with my history of mathematics course. It is therefore natural that they analyse the lessons integrating history in the same way as any other mathematics learning situation. Thus, they can bring in the notions of didactic transposition, a priori and a posteriori analysis, framework changes, tool-object dialectic, the anthropological theory of didactics, etc.

## 5.1 From false position to algebra

I will now present two examples of the content of these memoirs. The first example is a memoir about two one-hour sessions in a French fifth grade class (students between 12 and 13 years old) whose objective is the transition from the false position method to algebra.

In the first session, the students work on two Egyptian problems taken from the Rhind papyrus. The first problem, problem 26, is formulated as follows: “A quantity, its quarter is added, that makes 15”. The second problem, Problem 24, is of the same type: “A quantity, its seventh is added, that makes 19”. Students work in groups for one hour to understand and reformulate the scribe’s solutions in their own way. To extend this activity, they are given two problems in everyday life to solve at home using the false position method to prepare for the next session. On figure 11, you can see some of the students’ work. The essays vary from student to student, which allows for fruitful discussions.

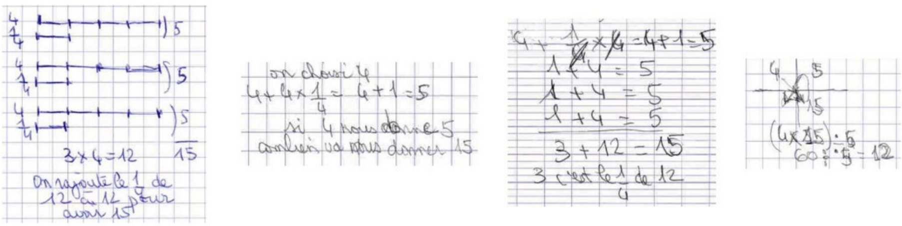


**Figure 11.** Reformulations by students of problem 26 from the Rhind papyrus

In the second session, the teacher introduced them to algebra by suggesting that they replace the numerical value used as a test with a letter and conduct the calculation similarly. In this way, they solved again the two problems of the Rhind papyrus and the two problems in everyday life. In all of this, more than the false position method or the algebra, the main obstacle came from the calculation of fractions, which was very difficult for some students.

At the end of each session, students are asked to write a few lines in their notebooks to summarise what they have done and what they have learned. Here are some examples of responses.

After the first session, a student has written:

*“During this session I learned how to do a problem by the false position method like the scribe Ahmes long ago in Egypt. I take a false starting value. I take care*

*that this value simplifies the calculations. And then by applying proportionality, I find what I am looking for."*

You can also read two personal summaries of the second session:

*"Today, I took the problem 26 that I had done with the false position, and I did it by taking a letter that represents the sought quantity. This is called the algebraic method."*

*"In this session I learned how to solve a problem using the algebraic method. We replace the quantity we are looking for with a letter, whereas in the other session, we took a false starting value. We do the calculations as if the letter was a number and by doing the calculations well, we find the quantity we are looking for."*

And here are some assessments of the whole sequence:

*"I understood better with the history, and I liked better the class and the way of working."*

*"I like history classes and I enjoyed doing history in maths. I followed it well and it helped me to understand better what to do."*

*"It was good to go back in time and see how problems were made. I liked this class."*

*"I liked to work in groups, to exchange with my classmates. We can correct errors and explain to each other or ask the teacher. I liked the history with the maths, the problem written in ancient Egyptian."*

I was very pleased with this memoir because, a month before her experimentation in the classroom, this teacher had never heard of false position methods and also had no idea about the origins of algebra. After discovering all this in my class, she herself made considerable progress in her understanding of several elementary concepts of the Middle School program and proved to be able to teach them in a relevant and effective way.

## 5.2 A cultural artefact for teaching the regular hexagon

The second example of memoir is ethnomathematical in nature. In a village in Reunion called Cilaos, there is a tradition of making carpets, bedspreads and other fabric handicrafts known as "beggar's carpets" ("tapis mendiants" in French). These objects are made from the recovery of various pieces of fabric and are presented in the form of pavings, usually square or hexagonal. The basic element is a regular hexagon, of which the seamstress has a cardboard model. She sews fabric hexagons and then assembles them seven by seven to form a kind of flower, and then these flowers are assembled into paving patterns to make, for example, a bedspread (see figure 5, photo on the top right corner).

A future teacher chose to focus her memoir on these beggar's carpets. She first showed them to the students by asking the questions: What are they? Do you know these objects? Have you seen them before? What are they used for? Where do they come from? Who makes them? What geometric shapes do you recognise? And so forth.

Once the students have identified a regular hexagon, the teacher asked them to draw it first with their hands and then with their geometry instruments. Some students used the property of equal sides, others thank of putting the hexagon in a circle, but this was not enough. Some used the ruler instead, others the compass, by trial and error. It was only after a long time, after several collective discussions, that the expert construction emerged (see figure 12). The teacher then asked students to write the program for the construction that had just been discovered.

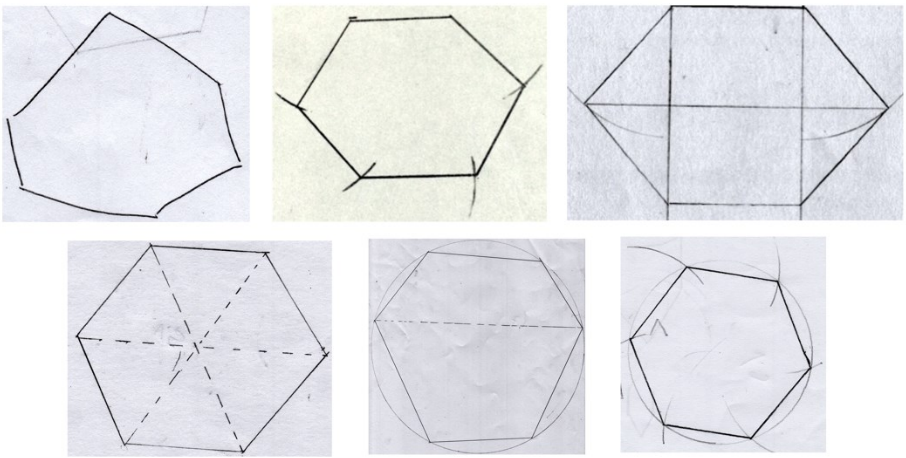

**Figure 12.** In search for a construction of the regular hexagon

The activity was also an opportunity to talk about the hexagon with equal sides and equal angles as presented in Book IV of Euclid's *Elements*, and to talk about the history of regular pavings in connection with crystallography and art history.

The sequence ended with a sewing workshop (see figure 13). To evaluate this geometry lesson based on local cultural content, the teacher asked one of her colleagues to propose a sequence on the regular hexagon in another class, but in a purely geometrical way, without any cultural context. In the class using ethnomathematics, all the pupils mastered the expert construction of the regular hexagon, whereas in the other class, only a third of the pupils demonstrated this mastery. There is no doubt here that the ethnomathematical entry created a powerful motivation and facilitated learning, both for the teacher and for her students.

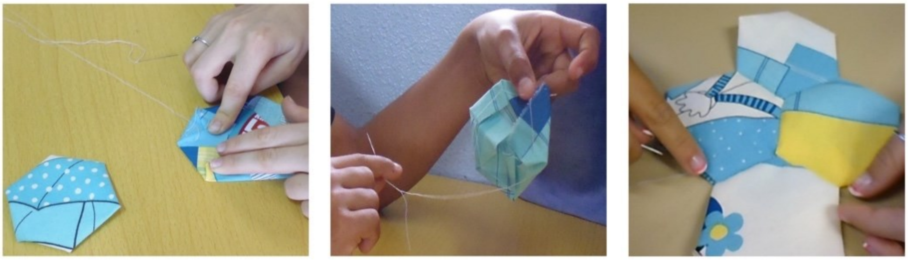

**Figure 13.** A sewing workshop

### *6 Concluding remarks*

To conclude this testimony, I could enumerate, from personal experience, some ingredients that seem to be relevant and effective in encouraging teachers to use the history of mathematics:

- consider ethnomathematics as part of the history of mathematics;
- manipulate artefacts to physically experience mathematical ideas;
- study short original texts directly related to curriculum items;
- practice with future teachers the same activities that they might offer in their classrooms;
- use the tools of the didactics of mathematics to analyse teaching and learning situations that incorporate history;
- foster a positive attitude and critical thinking towards textbooks and other historical resources;
- and… only offer very simple things in training... otherwise nothing will get into the classroom!

Finally, to find many ideas for classroom activities in the same spirit of what I have presented in this paper, I would like to point out the latest books from the French inter-IREM commission (Barbin 2010, 2012, 2018; Moyon & Tournès, 2018; Chevalarias et al., 2019).

**REFERENCES**


Barbin, É. (Ed.) (2010). *De grands défis mathématiques d'Euclide à Condorcet*. Paris: Vuibert.

Barbin, É. (Ed.) (2012). *Les mathématiques éclairées par l'histoire. Des arpenteurs aux ingénieurs*. Paris: Vuibert.

Barbin, É. (Ed.) (2014). *Les constructions mathématiques avec des instruments et des gestes*. Paris: Ellipses.

Barbin, É. et al. (2018). *Let History into the Mathematics Classroom*. New York: Springer.

Bartolini Bussi, M. G., & Maschietto, M. (2007). *Macchine matematiche: dalla storia alla scuola*. New York: Springer.

Chevalarias, N., Gandit, M., Morales, M., & Tournès, D. (Eds) (2019). *Mathématiques récréatives. Éclairages historiques et épistémologiques*. Les Ulis: EDP Sciences & Grenoble: UGA Éditions.

Chorlay, R., Clark, K. M., & Tzanakis, C. (2022). History of mathematics in mathematics education: Recent developments in the field. *ZDM Mathematics Education*, *54*(7). https://doi.org/10.1007/s11858-022-01442-7.

Clark, K. M., Kjeldsen, T. H., Schorcht, S., & Tzanakis, C. (2019). History of mathematics in mathematics education – An overview. *Mathematica Didactica*, *42*(1), 1–26.

D'Ambrosio, U. (2006). *Ethnomathematics. Link between Traditions and Modernity*. Rotterdam: Sense Publishers.

Daval, N. & Tournès, D. (2018). De l'abaque à jetons au calcul posé. In M. Moyon & D. Tournès (Eds), *Passerelles. Enseigner les mathématiques par leur histoire au cycle 3* (pp. 39-64). Bouc-Bel-Air: ARPEME.

Descartes, R. (1985). *The Philosophical Writings of Descartes, Vol. 1*. Translated by J. Cottingham, R. Stoothoff & D. Murdoch. Cambridge: Cambridge University Press.

Guillet, C., Moureau, M.-L. & Voillequin, I. (2019). Mathematics and experiment: How to calculate areas without formulas. In É. Barbin et al. (Eds.), *Proceedings of the Eighth European Summer University on History and Epistemology in Mathematics Education* (pp. 227-243). Oslo: Oslo Metropolitan University.

Maschietto, M., Milici, P. & Tournès, D. (2019). Semiotic potential of a tractional machine: A first analysis. In U. T. Jankvist, M. van den Heuvel-Panhuizen & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (Utrecht, 5-10 February 2019)* (pp. 2133-2140). Utrecht: Utrecht University and ERME

Monaghan, J., Trouche, L., & Borwein, J. M. (2018). *Tools and Mathematics*. New York: Springer.

Moyon, M. & Tournès, D. (Eds). (2018). *Passerelles. Enseigner les mathématiques par leur histoire au cycle 3*. Bouc-Bel-Air: ARPEME.

Moyon, M. (2021). I would like to introduce history in my mathematics lesson but I do not know how to do it! A challenge for in-service teachers training. *History and Pedagogy of Mathematics ICME14 Satellite Meeting (19 July 2021, session online)*. https://www.youtube.com/watch?v=NTIi44eCMLM&list=PLtabqCKbZVB1sFeEpzcUpWPAQpRKz5SIQ&index=5. Accessed 21 November 2022.

Moyon, M. (2022). Desire of teachers and realities in textbooks: dealing with history of mathematics in the new French curriculum and its impact on teacher training. *ZDM Mathematics Education, 54*(7). https://doi.org/10.1007/s11858-022-01427-6.

Rabardel, P. (1995). *Les hommes et les technologies. Approche cognitive des instruments contemporains*. Paris: Armand Colin.

Tournès, D. (2017). *Onze puzzles pour le théorème de Pythagore*. IREM de la Réunion. https://irem.univ-reunion.fr/spip.php?article954#forum1317. Accessed 28 November 2022.

Tournès, D. (2018a). A graphic approach to Euler's method. In É. Barbin et al., *Let History into the Mathematics Classroom* (pp. 87-100). New York: Springer.

Tournès, D. (2018b). Calculating with hyperbolas and parabolas. In É. Barbin et al., *Let History into the Mathematics Classroom* (pp. 101-114). New York: Springer.

Tournès, D. (Ed.) (2022). *Histoire du calcul graphique*. Paris: Cassini.

Tournès, D., Voillequin, C. & Voillequin, I. (2022). Planimètres, intégraphes, tractoriographes: les instruments de la géométrie transcendante. In N. Chevalarias (Ed.), *Géométries d'hier à demain. Pratiques, méthodes, enseignement* (pp. 185-202). Poitiers: IREM&S.