INFINITE SUMS AND THE CALCULATION OF π, AS PRESENTED BY THE SWEDISH MATHEMATICIAN ANDERS GABRIEL DUHRE IN THE EARLY 18TH CENTURY

Johanna PEJLARE
Mathematical Sciences, Chalmers University of Technology and University of Gothenburg,
SE-412 96 Gothenburg, Sweden
pejlare@chalmers.se

ABSTRACT

Anders Gabriel Duhre, an important mathematician and mathematics educator in Sweden during the 18th century, contributed with two textbooks in mathematics, one in algebra and one in geometry. Among others, he treats infinitesimals based on Nieuwentijts' theories from *Analysis infinitorum* and infinite sums based on Wallis' method of induction from *Arithmetica infinitorum*. Based on these results, Duhre develops an ingenious method to determine the area enclosed by curves by constructing a corresponding curve. He applies his method to the circle in order to find an expression of π as an infinite series. The series he finds is a modified version of the Gregory-Leibniz' series. In the present paper we consider in detail Duhre's presentation in order to further investigate the influence upon him as well as his influence on the Swedish mathematical society of his time.

1 Introduction

The Swedish mathematician and mathematics educator Anders Gabriel Duhre (c.1680–1739) was an important and influential person in the Swedish mathematical society in the early 18th century (Rodhe, 2002). He studied mathematics at Uppsala University, Sweden, and for some time he was a student of the Swedish scientist, inventor and industrialist Christopher Polhem (1661–1751) at his school *Laboratorium Mechanicum* in Stjärnsund. For some years Duhre taught mathematics to engineering students at Bergskollegium (a central agency in the mining industry) and to prospective officers at the Royal Fortification Office in Stockholm. In 1723 he opened his own school, *Laboratorium Mathematico-Oeconomicum*, outside Uppsala, where theoretical and practical subjects were taught to young boys (Hebbe, 1933). Of particular interest is that mathematics was taught in this school; Duhre had knowledge of mathematics that was not yet taught at the university, and students at the university turned to him to learn more on modern mathematics. Among his students were several of the Swedish mathematicians to be established during the 1720s and 1730s (Rodhe, 2002). Duhre taught in Swedish and early on planned to write mathematical textbooks in Swedish in order to introduce the Swedish youth to new and modern mathematics.

Duhre contributed with two textbooks in mathematics – one in algebra and one in geometry. Both were based on his lecture notes from his teaching at Bergskollegium and the Royal Fortification Office. The first book, *En Grundelig Inledning til Mathesin Universalem och Algebram* ("A thorough introduction to universal mathematics and algebra"), was edited by Georg Brandt and published in 1718. In this book, modern algebra based on Descartes' notation is presented, as well as examples from Newton's, Wallis' and Nieuwentijt's theories from the end of the 17th century. For example, he treats infinitesimals based on Nieuwentijt's theory as presented in *Analysis infinitorum* (1695) and utilizes Wallis' method of induction, as presented in *Arithmetica infinitorum* (1656), to determine the quotient of infinite series. In his second book, *Första Delen af en Grundad Geometria* ("The first part of a founded geometry"), published in 1721, Duhre takes advantage of the theories he presented earlier in his book on algebra. Of particular interest is his use of algebra in the geometrical context (Pejlare, 2017).

In this paper, we will consider Duhres' utilization of infinitesimals and infinite sums to determine the quotient between the circumference and the diameter of a circle, in order to find π expressed as an infinite series. We will first give a short introduction to Nieuwentijt's *Analysis infinitorum* and his utilization of infinitesimals, before we consider Duhre's interpretation of Nieuwentijt's work. Thereafter we will consider Wallis' *Arithmetica infinitorum* and how Duhre utilizes his method of induction to determine the quotient of infinite series. Following that, we will consider Duhre's method to find the area enclosed by curves. Finally, we will consider how Duhre utilizes this method on a circle and how he determines an expression for π.

2 Infinitesimals in Nieuwentijt's *Analysis infinitorum*

The Dutch philosopher and mathematician Bernard Nieuwentijt (1654–1718) is, in particular, known for his critique on the foundations of Leibniz' infinitesimal calculus. In 1695 he published *Analysis infinitorum*, a book "written by a beginner for beginners"[1] on elementary infinitesimal calculus. This book is primarily of a didactic character; he attempted at presenting mathematics in a systematic way as a coherent unit (Vermij, 1989). In the prologue he presents three definitions and two axioms which enable him to deduce rules for calculating with the infinite and infinitesimal quantities through more than 50 lemmas. In the chapters following the introduction, these lemmas lead to the propositions on infinitesimal calculus.

For Nieuwentijt, a quantity is infinitesimal if it is smaller than any arbitrary given quantity and it is infinite if it is greater than any arbitrary given quantity. The word infinitesimal is however not used in the definitions, axioms or lemmas. Instead, Nieuwentijt uses the expression "datâ minor" which can be translated into "the given smallest". Of central importance is his first axiom:

> Anything that when multiplied, however many times, does not equal another given quantity, however small, cannot be considered a quantity, geometrically it is absolutely *nothing*.[2]

The main peculiarity of Nieuwentijt's approach to infinitesimals is represented in Lemma 10, where it is stated that if an infinitesimalquantity is multiplied by an infinitesimal quantity, then the product is zero or nothing. The product of two infinitesimal quantities, or "the infinite small of the infinite small", can be interpreted as Leibniz' second differential. However, whereas Nieuwentijt considered squares of infinitesimals to be equal to zero, this is generally not the case with Leibniz' differentials (Mancosu, 1996).

---

[1] "Tyroni scriptum tyronibus" (Nieuwentijt, 1695, præfatio).

[2] "Quicquid toties sumi, hoc est per tantum numerum multiplicari non potest, ut datam ullam quantitatem, ut ut exiguam, magnitudine suâ æquare valeat, quantitas non est, sed in re geometricâ merum *nihil*" (Nieuwentijt, 1695, p. 2).

3 Infinitely small quantities in Duhre'stextbook on algebra

In Chapter XXVI of his book on algebra, Duhre presents an interpretation of the prologue of Nieuwentijt's *Analysis infinitorum* (1695). An infinitely small quantity is defined by Duhre as:

> If a *quantity* is divided by an infinitely big number, one should consider the received *quotient* to be infinitely small; it is something that is smaller than the smallest *quantity* that can ever be given.[3]

Thus, according to Duhre, if $\mathfrak{O}$ is an infinitely big number then the quotient $\frac{a}{\mathfrak{O}}$ is infinitely smaller than the quantity a. Duhre considers the nature of an infinitely big number to be that it is bigger than every given number and that it thus can be seen as "ceaselessly growing with no return".[4]From this it follows that $\frac{a}{\mathfrak{O}}$ is smaller than the smallest quantity that can ever be given. Duhre gives a proof by contradiction that $\frac{a}{\mathfrak{O}}$ really is "smaller than the smallest": if c is a quantity that is smaller than $\frac{a}{\mathfrak{O}}$ then the given quantity a is bigger than $\mathfrak{O}c$ and the quotient $\frac{a}{c}$ is bigger than the infinitely big quantity $\mathfrak{O}$, but this "contradicts all truth".[5] Therefore,$\frac{a}{\mathfrak{O}}$must be smaller than the smallest quantity, i.e., an infinitely small quantity.

The arguments above show that handling the infinite is problematic. Duhre treats the infinite as a fixed number, but this is in conflict with his earlier statement that an infinite number grows ceaselessly. Also, it seems easier to accept the infinitely big than the infinitely small, since the existence of the infinitely small is proven with the help of a given existence of the infinitely big.

After introducing infinitely small quantities, Duhre continues with 14 lemmas with rules for calculating with them; 10 of these are also found in Nieuwentijt's *Analysis infinitorum*. Among Duhre's lemmas we find, among others, that the sum of two infinitely small quantities is an infinitely small quantity (Lemma 1) and that the product of any number and an infinitely small quantity is an infinitely small quantity (Lemma 3). Of great importance for his later presentation on infinite sums is Lemma 4, which corresponds to Nieuwentijt's Lemma 10:

> If an infinitely small part $\frac{a}{\mathfrak{O}}$ is either *multiplied* by itself or by another infinitely small part $\frac{d}{\mathfrak{O}}$; then the received *product* $\frac{aa}{\mathfrak{O}\mathfrak{O}}$ or $\frac{ad}{\mathfrak{O}\mathfrak{O}}$ is nothing or no *quantity*.[6]

Thus, Duhre, just as Nieuwentijt, considers the square of infinitely small quantities to be equal to zero. In the proof of this lemma Duhreuses Nieuwentijt's first axiom: If the product of two infinitely small quantities is multiplied by an infinite number, this will be equal to an infinitely small quantity, i.e., $\frac{\mathfrak{O}\times aa}{\mathfrak{O}\mathfrak{O}} = \frac{aa}{\mathfrak{O}}$ and $\frac{\mathfrak{O}\times ad}{\mathfrak{O}\mathfrak{O}} = \frac{ad}{\mathfrak{O}}$, and since something multiplied by an

[3] "Om en förestäld *quantitet* hålles före wara fördehlad utaf ett oändeligen stort tahl; bör man anse then ther af komna *quotienten* för oändeligen lijten thet är för en ting som är mindre än then allerminsta *quantitet* som någonsin kan gifwas" (Brandt, 1718, p. 212).

[4] "[…] ouphörligen växande utan någon återvända" (Brandt, 1718, p. 213).

[5] "[…] stridande emot all sanning" (Brandt, 1718, p. 213).

[6] "Om en oändeligen lijten dehl $\frac{a}{\mathfrak{O}}$, antingen warder *multiplicerad* med sig sielf eller med någon annan oändeligen lijten dehl $\frac{d}{\mathfrak{O}}$; at then ther af komna *producten*$\frac{aa}{\mathfrak{O}\mathfrak{O}}$ eller $\frac{ad}{\mathfrak{O}\mathfrak{O}}$ måtte wara alsintet eller ingen *quantitet*" (Brandt, 1718, p. 214).

infinite number is equal to an infinitely small number then this something is not a quantity and geometrically is nothing.

In this proof Duhre does not seem to have a problem handling the infinite; it is no problem for him to shorten the expression with the infinitely big number $\mathfrak{D}$. He uses Lemma 4 in Lemma 14 where he deals with how infinitely small quantities can be handled in equations. He concludes that in an equation involving infinitely small quantities, the infinitely small quantities can be omitted, since, if the equation is divided by an infinitely big number $\mathfrak{D}$, then it follows from Lemma 4 that these can be considered as nothing. Algebraically this lemma can be interpreted as $x + \frac{a}{\mathfrak{D}} = x$ since $\frac{x}{\mathfrak{D}} + \frac{a}{\mathfrak{D}\mathfrak{D}} = \frac{x}{\mathfrak{D}}$.

4 Wallis' *Arithmetica infinitorum*

After considering the introduction of Nieuwentijt's *Analysis infinitorum*, Duhre, in Chapter XXVII of his book on algebra, proceeds with studying John Wallis' (1616–1703) *Arithmetica infinitorum* from 1656. *Arithmetica infinitorum* was an important text in the 17th century, in particular regarding the transition from geometry to algebra and regarding infinite series (Stedall, 2005). For example, Isaac Newton (1642–1727) was influenced by Wallis in his work towards integral calculus. Introducing new methods and concepts, Wallis' purpose was to find a general method of quadrature, i.e., finding the area enclosed by curves, or rather the ratios of those areas to inscribed or circumscribed rectangles. He achieved this by drawing together ideas from René Descartes' (1596–1650) algebraic geometry and Bonaventura Cavalieri's (1598–1647) theory of indivisibles. Wallis' results were based on the summation of indivisibles or infinitesimal quantities, where an indivisible can be considered to have at least one dimension equal to zero, as for example a line or a plane, while an infinitesimal is considered to have an arbitrarily non-zero width or thickness. Wallis was however not concerned with the distinction between indivisibles and infinitesimals and generally spoke of infinitely small quantities.

In order to find the area enclosed by curves, Wallis reduced the geometric problem to the summation of arithmetic sequences (Stedall, 2004). Two important mathematical methods he developed were *induction* and *interpolation*. Wallis' method of induction relied on intuition; he believed that if a pattern was established for a few cases then it could be assumed to continue indefinitely. Also, in his method of interpolation he relied on intuition; for example, he assumed continuity regarding sequences of numbers in order to interpolate intermediate values. One example of this is when he used his method of interpolation between the triangular numbers 1, 3, 6, 10 … Another example of interpolation is when he, in Proposition 191, found the ratio of a square to an inscribed circle: $\frac{4}{\pi} = \frac{3\times3\times5\times5\times7\times7 etc.}{2\times4\times4\times6\times6\times8 etc.}$.

5 Infinite sums in Duhre's textbook on algebra

We now turn our attention to Duhre's textbook on algebra again. We will here only consider those parts when Duhreuses Wallis' method of induction in order to deal with infinite sums. Duhre begins Chapter XXVIIby determining that the proportion of the sum of infinitely many squares with the roots 1, 2, 3, 4, 5 et cetera to the *summan totidem terminorum maximo æqualium* equals the proportion of 1 to 3. The *summan totidem terminorum maximo æqualium* is explained to be "the sum of the greatest term as many times as there are terms in the progression"[7]. Thus, in modern notation the proportion to be determined can be interpreted as:

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k^2}{(n+1)n^2} = \frac{1}{3}$$

Duhre proves this proportion using Wallis' method of induction, as presented in *Arithmetica infinitorum*. To do this, he first examines the proportion when *n* equals 1, 2, 3, 4, and 5 in the expression above:

$$\frac{0+1}{1+1} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{0+1+4}{4+4+4} = \frac{1}{3} + \frac{1}{12}$$

$$\frac{0+1+4+9}{9+9+9+9} = \frac{1}{3} + \frac{1}{18}$$

$$\frac{0+1+4+9+16}{16+16+16+16+16} = \frac{1}{3} + \frac{1}{24}$$

$$\frac{0+1+4+9+16+25}{25+25+25+25+25+25} = \frac{1}{3} + \frac{1}{30}$$

Duhre examines the pattern of the partial proportions and concludes that the denominators 6, 12, 18, 24, 30 et cetera form an arithmetical sequence. As long as the number of squares is finite the proportion is bigger than $\frac{1}{3}$. However, if we have infinitely many ($\mathfrak{D}$) squares, the proportion will be $\frac{1}{3} + \frac{1}{\mathfrak{D}}$, but since $\frac{1}{3} + \frac{1}{\mathfrak{D}} = \frac{1}{3}$ according to Lemma 14 in Chapter XXVI (see Section 3), the proportion will be $\frac{1}{3}$. Therefore, he concludes, the proportion of the sum of infinitely many squares with the roots 1, 2, 3, 4, 5 et cetera to the *summan totidem terminorum maximo æqualium* equals the proportion of 1 to 3.

In this presentation, Duhre closely follows Wallis, but unlike Wallis who in his following propositions offers geometrical interpretations of this result, Duhre does not do so. According to Wallis, the above proportion 1 to 3 geometrically corresponds to the proportion of the complement of half a parabola to the parallelogram completed by the same half parabola and its complement (Wallis, 1656, Prop. XXIII). Furthermore, Wallis' method of induction would not be an accepted method of induction today, since only a limited number of cases for $n = 1, 2, 3, \ldots$ were tested and the induction step (i.e., if the property is assumed to be true for $n = k$ it should be proven to be true for $n = k + 1$) was not included.

Duhre proceeds by proving the corresponding proportion for cubes with the help of Wallis' method of induction. In modern notation, he proves the following:

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k^3}{(n+1)n^3} = \frac{1}{4}$$

[7] "[…] en summa innehållande then största ledamoten så ofta som progressionens ledamöter äre" (Brandt, 1718, p. 77).

After these two proofs, using Wallis method of induction, Duhre states that, again interpreted in modern notation, the following proportions are true:

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k^4}{(n+1)n^4} = \frac{1}{5}$$

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k^5}{(n+1)n^5} = \frac{1}{6}$$

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k^6}{(n+1)n^6} = \frac{1}{7}$$

6 Duhre's method of finding the area enclosed by curves

Let us now turn to Duhre's textbook on geometry. We will consider Duhre's method of finding the area enclosed by curves in order to see how he uses the proportions including infinite sums that he considered in his *Algebra*. In Chapter XXX Duhre formulates a proposition where he considers the curve $\boldsymbol{ABCD}$ and from it constructs the curve $\boldsymbol{AIOM}$ such that the area of the segment $\boldsymbol{ADCBA}$ is equal to half of the area $\boldsymbol{AEMOIA}$ (see Figure 6.1). The curve $\boldsymbol{AIOM}$ is constructed in the following way: Let $\boldsymbol{AS}$ be a tangent at the point $\boldsymbol{A}$, parallel to the ordinate $\boldsymbol{DE}$ and for every point $\boldsymbol{C}$ on $\boldsymbol{ABCD}$with a tangent $\boldsymbol{CG}$ where $\boldsymbol{G}$ is a point on $\boldsymbol{AS}$, the ordinate $\boldsymbol{OK}$ is equal to the line $\boldsymbol{AG}$.

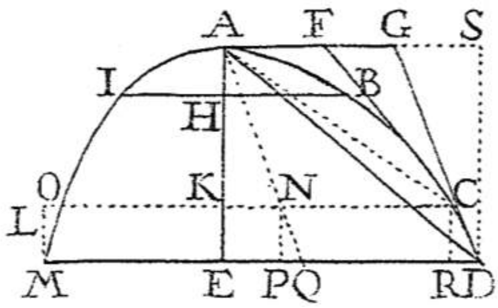


Figure 6.1: The area of the segment $ADCBA$ is equal to half of the area $AEMOIA$ (Duhre, 1721, p. 572).

Duhre proves this proposition without using algebra, only considering geometrical properties. First, he draws a few help lines. He draws the line $\boldsymbol{AQ}$ parallel to $\boldsymbol{DG}$ such that $\boldsymbol{ADGQ}$ is a parallelogram. If the point $\boldsymbol{C}$ is considered to be infinitely close to the point $\boldsymbol{D}$, he concludes that the line $\boldsymbol{CD}$ can be considered to be a straight line and thus it can be considered to be a part of the tangent $\boldsymbol{DG}$. Then he draws the line $\boldsymbol{CL}$ parallel to $\boldsymbol{DM}$ and the lines $\boldsymbol{CR}$, $\boldsymbol{LM}$ and $\boldsymbol{NP}$ parallel to $\boldsymbol{AE}$. Finally, he draws the line $\boldsymbol{AC}$. The proof of the proposition follows:

Since the two parallel lines $\boldsymbol{DM}$ and $\boldsymbol{CL}$ are infinitely close to each other, the points $\boldsymbol{L}$ and $\boldsymbol{O}$ are infinitely close to each other, and thus the mixed lines figure $\boldsymbol{EMOK}$ must be the same as the parallelogram$\boldsymbol{EMLK}$. Furthermore, the lines $\boldsymbol{EM}$, $\boldsymbol{AG}$ and $\boldsymbol{CN}$ are equal to each other and hence the parallelogram $\boldsymbol{EMLK}$ equals the parallelogram $\boldsymbol{PNCR}$, which in turn equals the parallelogram $\boldsymbol{QNCD}$. Now, if $\boldsymbol{CD}$ is considered as a base, the parallelogram $\boldsymbol{QNCD}$ is twice as big as the triangle $\boldsymbol{ACD}$, since the lines $\boldsymbol{CD}$ and $\boldsymbol{AQ}$ are parallel. This implies that also the mixed lines figure$\boldsymbol{EMOK}$ and the parallelogram$\boldsymbol{PNCR}$ are twice as big as the triangle $\boldsymbol{ACD}$.

Finally, if other lines parallel to the line $\boldsymbol{DM}$ are drawn, each of the resulting mixed lines figures are twice as big as the corresponding triangles for the same reason that the mixed lines figure $\boldsymbol{EMOK}$ is twice as big as the triangle $\boldsymbol{ACD}$. Therefore, the figure $\boldsymbol{AEMOIA}$, which is the composite of the mixed lines figures, equals twice the sum of the corresponding triangles that forms the segment $\boldsymbol{ADCB}$, which is what Duhre wanted to prove.

7 Duhre's method applied to the circle

In order to calculate the decimals of $\boldsymbol{\pi}$, or more specifically, in order to show that the proportion between the diameter and the circumference of a circle is approximately the same as 100 to 314, Duhre now wants to apply the proposition from Chapter XXX to a circle, i.e., instead of considering the circumference he considers the area of a circle. He begins Chapter XXXI with considering a half circle; the area under the corresponding curve to a half circle should be equal to the area of a full circle (see Figure 7.1). However, the corresponding curve$\boldsymbol{ASM}$ to the half circle $\boldsymbol{ACB}$in fact is an asymptote to the line $\boldsymbol{BV}$, and thus the "indescribable width"[8] of the area contained by the "indescribable" line $\boldsymbol{ASM}$ is equal to the area of the circle. However, the "undescribable width" is too difficult for Duhre to consider further. Therefore, he instead considers a quarter of a circle $\boldsymbol{ACD}$and its corresponding curve$\boldsymbol{ASR}$. Doing this, the area $\boldsymbol{ADRH}$ equals twice of the area of the segment $\boldsymbol{ACE}$ according to the proposition in Chapter XXX. By adding half of this area to the area of the triangle $\boldsymbol{ADC}$ and multiplying the expression by four, an expression of the area of the circle will be given.

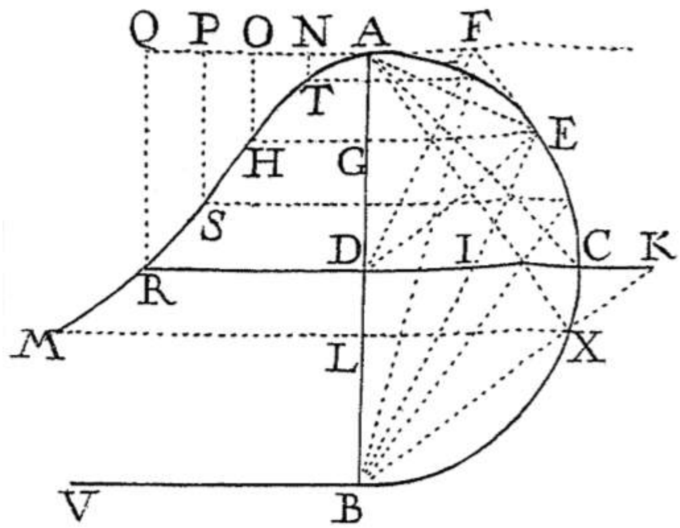


Figure 7.1: The area $ADRH$ equals twice of the area of the segment ACE (Duhre, 1721, p. 574).

Instead of calculating the area of the figure$\boldsymbol{ADRH}$, Duhre's idea is to calculate the area of the figure $\boldsymbol{ARQ}$. He states that the line $\boldsymbol{AQ}$, which is equal to the line $\boldsymbol{AD}$, can be divided into infinitely many equal parts, and the lines $\boldsymbol{NT}$, $\boldsymbol{OH}$, $\boldsymbol{PS}$ et cetera proceeding from these points of intersection will fill up the figure $\boldsymbol{ARQ}$.

Now Duhre introduces the variables $\boldsymbol{a}$, $\boldsymbol{x}$ and $\boldsymbol{y}$. He lets $\boldsymbol{AB} = \boldsymbol{2a}$, i.e., the radius of the circle equals $\boldsymbol{a}$, the ordinate $\boldsymbol{GH} = \boldsymbol{AF} = \boldsymbol{DI} = \boldsymbol{x}$ and $\boldsymbol{AG} = \boldsymbol{y}$. He wants to find an

[8] "[…] obeskrifweliga widden" (Duhre, 1721, p. 110).

expression for $\boldsymbol{y}$, which can be considered as a length that varies.He does this using proportional reasoning: He first concludes that $\boldsymbol{BG = 2a - y}$ and, because of properties of the circle the square of $\boldsymbol{GE}$ equals $\boldsymbol{AG \cdot BG}$ which is the same as $\boldsymbol{2ay - y^2}$. Considering the two uniform triangles $\boldsymbol{BDI}$ and $\boldsymbol{BGE}$, Duhre concludes that since$\boldsymbol{BD}$, $\boldsymbol{DI}$, $\boldsymbol{BG}$ and $\boldsymbol{GE}$ are geometrical proportional, i.e., $\boldsymbol{BD, DI :: BG, GE}$, the squares $\boldsymbol{BDq}$, $\boldsymbol{DIq}$, $\boldsymbol{BGq}$ and $\boldsymbol{GEq}$ will also be geometrical proportional, i.e., $\boldsymbol{BDq, DIq :: BGq, GEq}$.[9] From this it follows that $\boldsymbol{aa, xx :: 4aa - 4ay + yy, 2ay - yy}$, which can be simplified into $\boldsymbol{aa, xx :: 2a - y, y}$. He now uses the fact that the product of the two utmost in a geometrical progression equals the product of the two inners, i.e., $\boldsymbol{aay = 2axx - xxy}$. By adding $\boldsymbol{xxy}$ and dividing by $\boldsymbol{aa + xx}$ on both sides, Duhre now finally finds the expression $\boldsymbol{y = \frac{2axx}{aa+xx} = AG}$. This quotient can be expressed as an infinite series:

$$AG = y = \frac{2axx}{aa + xx} = \frac{2xx}{a} - \frac{2x^4}{a^3} + \frac{2x^6}{a^5} - \frac{2x^8}{a^7} \&c.$$

Furthermore, he concludes that if $GH = 2x$ then

$$AG = \frac{8xx}{a} - \frac{32x^4}{a^3} + \frac{128x^6}{a^5} - \frac{512x^8}{a^7} \&c.,$$

if $GH = 3x$ then

$$AG = \frac{18xx}{a} - \frac{162x^4}{a^3} + \frac{1458x^6}{a^5} - \frac{13122x^8}{a^7} \&c.,$$

and so on. Since $AQ = a$ is divided into infinitely many equal parts, where the first one is $AN = x$, $AO = 2x$, $AP = 3x$, and so on, the expressions above give the corresponding lengths of$AG = y$. These lengths could also be denoted NT, OH, PS according to Figure 7.1. The last of these lengths is $QR = a$.The infinitely many lengths together fill up the figure AQR, and therefore Duhre now has to compute the infinite sum of these infinitely many series. In order to compute the sum, i.e., the area of the figure AQR, Duhre now collects all terms of the same power of x. Thus, the area AQR will be:

$$\frac{2}{a}(xx + 4xx + 9xx \,\&c.) - \frac{2}{a^3}(x^4 + 16x^4 + 81x^4 \&c.) + \frac{2}{a^5}(x^6 + 64x^6 + 729x^6 \&c.)\&c.$$

In modern notation this expression can be interpreted as

$$\frac{2}{a}\lim_{n\to\infty}\sum_{k=1}^{n}(kx)^2 - \frac{2}{a^3}\lim_{n\to\infty}\sum_{k=1}^{n}(kx)^4 + \frac{2}{a^5}\lim_{n\to\infty}\sum_{k=1}^{n}(kx)^6 - \cdots$$

To compute these sums, Duhre uses the results on infinite sums from his text book on algebra (see Section 5). First, he has to determine the *summa totidem terminorum maximo æqualium*. The *summa totidem terminorum maximo æqualium* to the infinite sum$xx + 4xx + 9xx\&c.$ must be $a \cdot aa$, since he considers a to be the number of terms in the infinite sum and aa to

[9] In modern notation: $\frac{BD}{DI} = \frac{BG}{GE}, i.e., \frac{BD^2}{DI^2} = \frac{BG^2}{GE^2}$.

be the biggest term in the sum. It follows that, in modern notation, $\lim_{n\to\infty}\sum_{k=1}^{n}(kx)^2 = \frac{1}{3}a^3$. In the same way $\lim_{n\to\infty}\sum_{k=1}^{n}(kx)^4 = \frac{1}{5}a^5$, $\lim_{n\to\infty}\sum_{k=1}^{n}(kx)^6 = \frac{1}{7}a^7$ and so on. Therefore, the infinite sum of the infinite series above, i.e., the area of the figure AQR, will be equal to

$$\frac{2}{a}\left(\frac{1}{3}a^3\right) - \frac{2}{a^3}\left(\frac{1}{5}a^5\right) + \frac{2}{a^5}\left(\frac{1}{7}a^7\right)\&c. =$$

$$= \frac{2}{3}aa - \frac{2}{5}aa + \frac{2}{7}aa - \frac{2}{9}aa\ \&c.$$

Duhre can now easily find an expression for the area of the figure ARD; he just has to take the area of the square of AQ, i.e., a^2, and subtract the area of the figure AQR. Thus, the area of the figure ARD will be

$$aa - \frac{2}{3}aa + \frac{2}{5}aa - \frac{2}{7}aa + \frac{2}{9}\&c.$$

According to the method presented in Chapter XXX (see Section6), the area of the figure ARD is twice the area of the segment ACE, and therefore it follows that the area of the segmet ACE will be

$$\frac{1}{2}aa - \frac{1}{3}aa + \frac{1}{5}aa - \frac{1}{7}aa + \frac{1}{9}aa\ \&c.$$

Now, adding the area of the triangle ADC to this expression and multiply with four will finally give an expression for the area of the circle with radius a:

$$4aa - \frac{4}{3}aa + \frac{4}{5}aa - \frac{4}{7}aa + \frac{4}{9}aa\ \&c.$$

Duhre modifies this expression even further, in order to find an expression for the circumference of the circle. Since the area of a circle equals the area of a triangle where the base equals the circumference of the circle and the height equals the radius of the circle, he concludes that he will find an expression of the circumference of the circle if he divides the area of the circle with half of its radius, i.e., $\frac{1}{2}a$. Thus, he gets the following series expressing the circumference of the circle:

$$8a - \frac{8}{3}a + \frac{8}{5}a - \frac{8}{7}a + \frac{8}{9}a\&c.$$

Duhre now lets the diameter of the circle, i.e., $2a$, equal 1 and finds that the proportion between the diameter of a circle and its circumference is as one to the following series:

$$4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9}\&c.$$

He finally modifies this series by merging the terms pairwise:

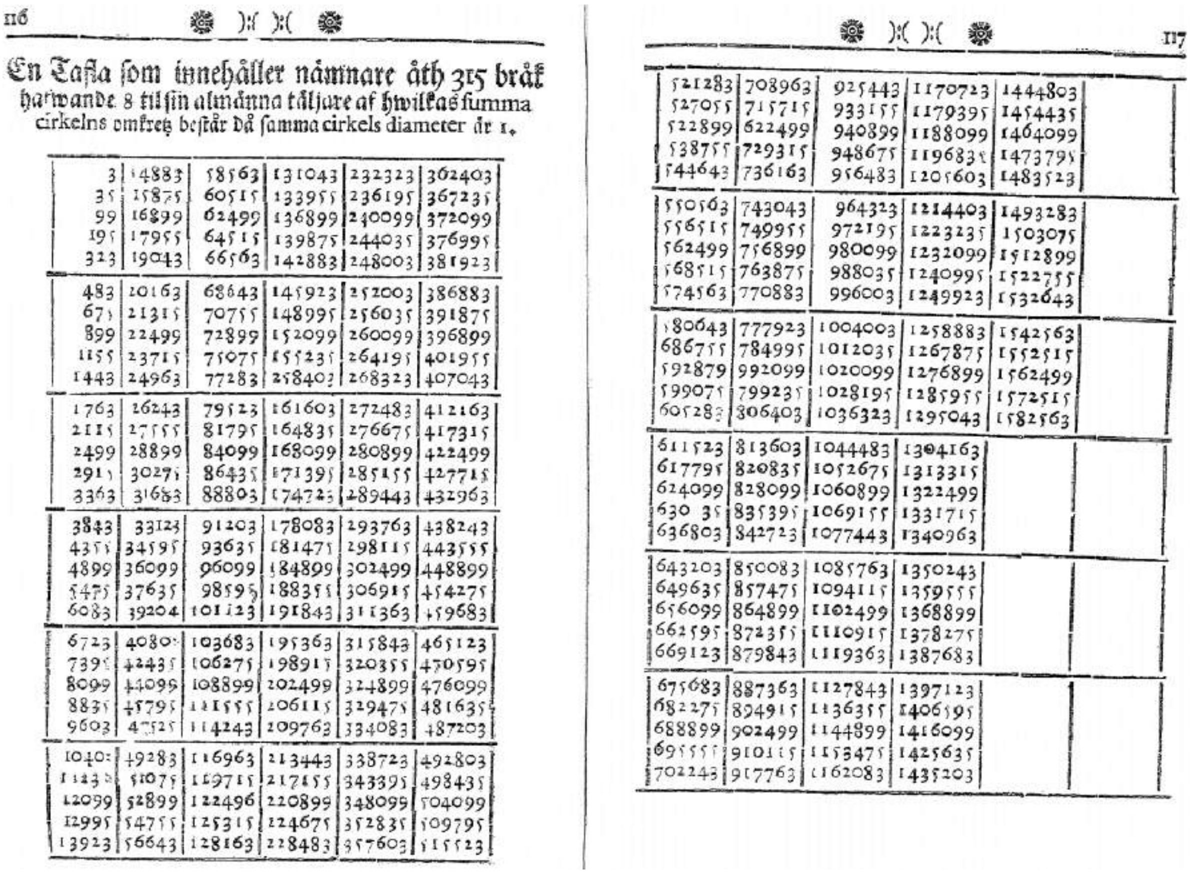
$$\frac{8}{3}+\frac{8}{35}+\frac{8}{99}+\frac{8}{195}+\frac{8}{323}+\frac{8}{483}+\frac{8}{675}\,\&c.$$

In modern notation we can interpret this result as

$$\pi=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{8}{(4k-2)^2-1}.$$

8 Duhre’s calculation of π

After finding the proportion of the diameter of the circle to its circumference, Duhre proceeds with computing this proportion. He starts with constructing a table (see Figure 8.1) with the first 315 denominators of the series $\sum_{k=1}^{n}\frac{8}{(4k-2)^2-1}$ .This table is actually not completely correct, possibly due to typesetting errors. For example, for k=100 it says 258.403 instead of 158.403 and for k= 50 it says 39.204 instead of 39.203.



116

En Tafla som innehåller nämnare åth 315 bråk hafwande 8 til sin almänna täljare af hwilkas summa cirkelns omkrets består då samma cirkels diameter är 1.

3	14883	58563	131043	232323	362403
35	15875	60515	133955	236195	367235
99	16899	62499	136899	240099	372099
195	17955	64515	139875	244035	376995
323	19043	66563	142883	248003	381923
483	20163	68643	145923	252003	386883
675	21315	70755	148995	256035	391875
899	22499	72899	152099	260099	396899
1155	23715	75075	155235	264195	401955
1443	24963	77283	258403	268323	407043
1763	26243	79523	161603	272483	412163
2115	27555	81795	164835	276675	417315
2499	28899	84099	168099	280899	422499
2915	30275	86435	171395	285155	427715
3363	31683	88803	174723	289443	432963
3843	33123	91203	178083	293763	438243
4355	34595	93635	181475	298115	443555
4899	36099	96099	184899	302499	448899
5475	37635	98595	188355	306915	454275
6083	39204	101123	191843	311363	459683
6723	40803	103683	195363	315843	465123
7395	42435	106275	198915	320355	470595
8099	44099	108899	202499	324899	476099
8835	45795	111555	206115	329475	481635
9603	47525	114243	209763	334083	487203
10403	49283	116963	213443	338723	492803
11235	51075	119715	217155	343395	498435
12099	52899	122496	220899	348099	504099
12995	54755	125315	224675	352835	509795
13923	56643	128163	228483	357603	515523

117

521283	708963	925443	1170723	1444803
527055	715715	933155	1179395	1454435
522899	622499	940899	1188099	1464099
538755	729315	948675	1196835	1473795
544643	736163	956483	1205603	1483523
550563	743043	964323	1214403	1493283
556515	749955	972195	1223235	1503075
562499	756899	980099	1232099	1512899
568515	763875	988035	1240995	1522755
574563	770883	996003	1249923	1532643
580643	777923	1004003	1258883	1542563
686755	784995	1012035	1267875	1552515
592879	992099	1020099	1276899	1562499
599075	799235	1028195	1285955	1572515
605283	306403	1036323	1295043	1582563
611523	813603	1044483	1304163	
617795	820835	1052675	1313315	
624099	828099	1060899	1322499	
630 35	835395	1069155	1331715	
636803	842723	1077443	1340963	
643203	850083	1085763	1350243	
649635	857475	1094115	1359555	
656099	864899	1102499	1368899	
662595	872355	1110915	1378275	
669123	879843	1119363	1387683	
675683	887363	1127843	1397123	
682275	894915	1136355	1406595	
688899	902499	1144899	1416099	
695555	910115	1153475	1425635	
702243	917763	1162083	1435203	

Figure 8.1: The table containing the first 315 denominators in Duhre’s infinite series of π (Duhre, 1721, pp. 116–117).

Duhre proceeds with constructing a second table, containing the first 315 terms and partial sums of the series (see Figure 8.2). However, he does not want to consider decimals and therefore he considers a circle with diameter 100.000.000 instead of 1, i.e., the general numerator in the series will be 800.000.000 instead of 8. In modern notation this new series can be written as

$$100.000.000\pi = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{800.000.000}{(4k-2)^2 - 1}.$$

In this way the partial sums, after approximations, will be natural numbers. In the table in Figure 8.2 we can see that the proportion of the diameter of a circle to its circumference will be approximately as 100.000.000 to 314.000.528, or as 100 to 314.

266666667	312236366	313178896	313501379
22857143	71206	18852	8544
8080808	66121	18141	8325
4102564	61562	17469	8114
2476780	57459	16834	7911
1656315	53753	16233	7716
305840277	312546467	313266425	313541989
1185185	50394	15663	7528
889878	47340	15123	7346
692641	44556	14611	7171
554400	42010	14124	7003
453772	39677	13661	6840
309616153	312770444	313339607	313577877
378250	37532	13220	6682
320128	35557	12800	6531
274442	33734	12400	6384
237383	32047	12019	6242
208171	30484	11655	6105
311035027	312939798	313401701	313609821
183697	29033	11307	5972
163299	27683	10974	5844
146119	26424	10656	5719
131514	25250	10352	5599
118994	24152	10060	5482
311778650	313072340	313455050	313638437
108181	23125	9780	5369
98778	22161	9513	5260
90549	21257	9255	5153
83307	20407	9009	5050
76901	19606	8772	4950
312236366	313178896	313501379	313664219

313664219	313777588	313848706	313897483
4853	2891	1917	1363
4759	2848	1893	1349
4668	2805	1870	1335
4579	2764	1848	1322
4492	2723	1825	1308
313687570	313791619	313858059	313904160
4408	2684	1804	1295
4327	2645	1782	1282
4247	2607	1761	1269
4170	2569	1740	1256
4095	2533	1720	1244
313708817	313804657	313866866	313910506
4022	2497	1700	1231
3951	2462	1680	1219
3881	2428	1661	1207
3814	2395	1642	1196
3748	2362	1623	1184
313728233	313816801	313875172	313916543
3684	2330	1605	1173
3622	2298	1587	1161
3561	2267	1569	1150
3501	2237	1552	1139
3443	2207	1535	1128
313746044	313828140	313883020	313922294
3387	2178	1518	1118
3322	1250	1501	1107
3278	2212	1485	1097
3226	2095	1469	1087
3175	2068	1453	1077
313762432	313838753	313890446	313927780
3125	2041	1438	1067
3076	2016	1422	1057
3028	1990	1407	1047
2981	1965	1392	1038
2036	1941	1378	1028
313777588	313838706	313897483	313933017

313933017	313956008	313974757	313990339
1019	823	678	569
1010	816	673	565
1001	810	668	561
992	803	664	557
983	797	659	554
313938022	313960057	313978099	313993145
975	790	654	550
966	784	649	546
958	778	645	543
949	772	640	539
941	766	635	536
313942811	313963947	313981322	313995859
933	760	631	532
925	754	627	529
917	748	622	525
909	742	618	522
902	757	613	519
313947397	313967688	313984433	313998486
894	731	609	515
886	726	605	512
879	720	601	509
872	715	597	506
864	709	592	
313951792	313971289	313987437	314000528
857	704	588	
850	699	584	
843	694	580	
836	688	577	
830	683	573	
313956008	313974757	313990339	

Figure 8.2: Duhre's table showing the first 315 approximated termsand partial sums in the series $\sum_{k=1}^{n} \frac{800.000.000}{(4k+2)^2-1}$ (Duhre, 1721, pp. 119–121).

Duhre concludes Chapter XXX by noting that in practice, when minor computations have to be made, the proportion 100 to 314 or the Archimedean proportion 7 to 22 can be used, the requested proportion being smaller than the former and bigger than the latter. If larger computations have to be performed, however, he suggests that the proportion 100.000 to 314.159 should be used. Nevertheless, he does not perform the computations needed to find this proportion.

9 Concluding remarks

The series $\mathbf{4} - \frac{\mathbf{4}}{\mathbf{3}} + \frac{\mathbf{4}}{\mathbf{5}} - \frac{\mathbf{4}}{\mathbf{7}} + \frac{\mathbf{4}}{\mathbf{9}}$&$c$. which Duhre received before he merged the terms pairwise, we recognize as a Maclaurin series for $\mathbf{4}\tan^{-1}\boldsymbol{x}$ for$\boldsymbol{x} = \mathbf{1}$. Since $\mathbf{4}\tan^{-1}\mathbf{1} = \boldsymbol{\pi}$, we can conclude that Duhre's series is correct. However, it converges very slowly. This series is known as the Gregory–Leibniz' series after James Gregory (1638–1675) and Gottfried Wilhelm Leibniz (1646–1716). Leibniz was concerned with the quadrature and when he applied his method to the circle he received the series $\frac{\boldsymbol{\pi}}{\mathbf{4}} = \mathbf{1} - \frac{\mathbf{1}}{\mathbf{3}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{7}} + \cdots$. Leibniz found this result in 1673, but already in 1671 Gregory, who was concerned with infinite series representations of transcendental functions, had found the corresponding Taylor series. Also, an Indian mathematician, whose identity is not definitely known, found the series for $\mathbf{tan^{-1}}\,\boldsymbol{x}$ during the 15th century (Roy, 1990). This series, written in Sanskrit verse, is usually ascribed to Kerala Gargya Nilakantha (c.1450–c.1550) and can be found in the book *Tantrasangraha* composed around 1500.

Since Duhre follows Wallis' method of induction when he considers the infinite series, it may be surprising that he in his book on geometry does not proceed with studying Wallis' interpolation method to find the area of a circle in order to find an expression for $\boldsymbol{\pi}$. However, Duhre's method, where he from the circle constructs a corresponding curve where he can use the previously found infinite sums to find the enclosed area, is indeed ingenious. In his search for $\boldsymbol{\pi}$ Duhre also uses modern algebra that cannot be found in Wallis' *Arithmetica infinitorum*. Duhre considers algebra to be helpful, since it enables complicated expressions to be transformed into simpler ones, and thus convenience in calculations is obtained.

While Duhre primarily was an educator, his main pioneering achievement was that he brought knowledge of modern mathematics into the Swedish mathematical community. Of particular value is his choice to write in Swedish in order to find a greater audience. Twice he applied for a position as professor at Uppsala University, without success, but he still succeeded in inspiring several among the next generation of Swedish mathematicians. Certainly, also his students at Bergskollegium and the Royal Fortification Office had the opportunity to be introduced into modern mathematics thanks to Duhre.

Acknowledgement: This work was supported by the Swedish Research Council [Grant no. 2015-02043].

REFERENCES

Brandt, G. (1718). *En Grundelig Inledning til Mathesin Universalem och Algebram – Efter Herr And. Gabr. Duhres håldne prealectioner*. Stockholm: Kongl. Antiquit Archiv.

Duhre, A. G. (1721). *Första Delen af en Grundad Geometria – Bewijst uti de Föreläsningar som äro håldne på Swänska Språket Uppå Kongl. Fortifications Contoiret*. Stockholm: Kongl. Antiquit. Archiv.

Hebbe, Per (1933). *Anders Gabriel Duhres "Laboratorium mathematico-oeconomicum". Ett bidrag till Ultunas äldre historia*. Stockholm: Kungliga Lantbruksakademien.

Mancosu, P. (1996). *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*. New York, NY: Oxford University Press.

Nieuwentijt, B. (1695). *Analysis infinitorum, seu curvilineorum proprietates ex polygonorum natura deductæ.* Amstelædami: Joannem Wolters.

Pejlare, J. (2017). On the relationships between the geometric and the algebraic ideas in Duhre's textbooks of mathematics, as reflected via Book II of Euclid's Elements. In: K. Bjarnadóttir, F. Furinghetti, M. Menghini, J. Prytz, & G. Schubring (Eds.). *"D where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education* (pp. 263–273). Rome: Edizioni Nuova Cultura.

Rodhe, S. (2002). *Matematikens utveckling i Sverige fram till 1731*. Uppsala: Diss. Uppsala Universitet.

Roy, R. (1990). The discovery of the series formula for π by Leibniz, Gregory and Nilakantha. *Mathematics Magazine 63*(5), 291–306.

Stedall, J. (2004). Introduction: The arithmetic of infinitesimals by Jacqueline A. Stedall, In: J. A. Stedall (Ed.), *The arithmetic of infinitesimals: John Wallis 1695. Translated from Latin to English with an introduction by Jacqueline A. Stedall*. New York: Springer.

Stedall, J. (2005). John Wallis, Arithmetica infinitorum (1656), In: I. Grattan-Guinness, & R. Cooke

(Eds.), *Landmark Writings in Western Mathematics 1640–1940* (pp. 23–32). Elsevier Science Limited.
Vermij, R. H. (1989). Bernard Nieuwentijt and the Leibnizian Calculus. *Studia Leibnitiana 21*(1), 69–86.
Wallis, J. (1656). *Arithmetica infinitorvm, sive nova methodus inquirendi in curvilineorum quadraturam, aliaq*; *difficiliora matheseos problemata*. Oxoniense: Leon: Lichfield academiz typographi impensis Tho. Robinson.