THE NORSE TREATISE ALGORISMUS

Preserved in manuscript GKS 1812 4to

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ABSTRACT

The treatise *Algorismus* is a complete prose translation of the Latin hexameter *Carmen de Algorismo* into the medieval Old Norse language. *Carmen* is dated in 1202, written by the French canon Alexander de Villa-Dei. The treatise explains for the first time the Hindu-Arabic decimal place value numeral notation and calculation methods to the Norse people, Icelanders and Norwegians. *Algorismus* also relates the four Elements: Earth, Water, Air and Fire to cubic numbers and ratios. The treatise exists in four manuscripts, one of them only a fragment. The four manuscripts are compared by digital methods to show that the two oldest of them are quite similar and possibly copies of the same copy of the original translation. This paper focuses on the version in Ms GKS 1812 4to. It is a pedogogical study of the algorithms presented in the treatise, contrasting them with current day methods and the presentation in *Carmen*.

1 Introduction

The thirteenth century Old Norse arithmetic treatise *Algorismus* exists in four manuscripts: GKS 1812 4to, preserved in Reykjavík, and AM 544 4to, AM 685 d 4to, and AM 736 III 4to, preserved in Copenhagen. The bulk of the treatise is a prose translation of the Latin hexameter poem *Carmen de Algorismo*, written in France in the early thirteenth century. *Algorismus* is also related to *Algorismus Vulgaris* by Johannes de Sacrobosco, dated around 1225. In this paper we explore incidences where *Algorismus* deviates from *Carmen de Algorismo*, and we compare the four extant manuscripts of *Algorismus* with phylogenetic alignment methods.

Algorismus was published in a scientific edition 1892–1896 by Finnur Jónsson (1892–1896) based on AM 544 4to, with corrections from the other three manuscripts as applicable. The mathematician Otto B. Bekken translated *Algorismus* into modern Norwegian in 1985 and explained its text in cooperation with linguist Marit Christoffersen (Bekken and Christoffersen, 1985). Kristín Bjarnadóttir (2003, 2007) has explained the content of *Algorismus* in English and in modern Icelandic (Bjarnadóttir, 2004). Bjarnadóttir and Halldórsson (forthcoming) studied *Algorismus*, focusing the manuscript GKS 1812 4to and the phylogenetic alignment methods, as done here.

Algorismus, a treatise of nearly 3000 words, contains an explanation of the Hindu-Arabic decimal place-value notation and calculation methods in seven algorithms: addition, subtraction, doubling, halving, multiplication, division, and extraction of roots, which is further subdivided into the square root and the cubic root. These methods have been relayed to *Algorismus* via a well-known Latin hexameter, *Carmen de Algorismo*, composed by Alexander de Villa Dei (1170–1240) between 1200 and 1203 (Beaujouan, 1954). Alexander de Villa Dei was a Franciscan and a master at the University of Paris, later canon at the St. Andrew’s Cathedral in Avranches (Beaujouan, 1954). Finnur Jónsson (1892–1896, p. cxxxii) suggested that *Carmen* was translated into the Old Norse language before 1270. Jónsson referred to an analysis by Hankel (1874, p. 325) of the typeface of Arab numerals, where the typeface used in the manuscripts of *Algorismus* corresponds Hankel's examples from before 1271. According to Tropfke (1980) the typeface belongs to the West-Arab notation. Helgi Guðmundsson (1967, p. 68) deems it possible that the translation existed in the Viðey monastery in the early 14th century.

Carmen is a verbal explanation of Hindu-Arabic arithmetic, built on a translation of the work by Muhammad ibn-Mūsā al-Kwārizmī (ca. 780–850), *Kitāb al-jam΄wal tafrīq bi hisāb al-Hind* [*The Book of Bringing Together and Separating According to the Hindu Calculation*], most likely on *Liber alghoarismi de practica arismetrice*, one of its twelfth century Latin translations (Allard, 1992, p. xxxi). This conjecture is based on the order of the arithmetic operations which varies in the different translations. The cubic root is not included in the translations of Al-Khwārizmī's work and must be acquired from another source. *Algorismus* expands *Carmen* with concrete examples as well as a concluding section of unknown origin. There are also a few omissions that do not compromise the meaning.

The poem *Carmen* exists in a great number of manuscripts, preserved in libraries in France, Great Britain, the Netherlands and many other countries. It is considered to have played an even greater role in distributing Hindu-Arabic positional number notation in Northern Europe than the well-known *Liber abaci* by Leonardo da Pisa (Jacqueline Stedall, personal communication, 2009). The translation of *Carmen* into the vernacular of the Norse people in Norway and Iceland was a further effort in the distribution of knowledge.

Carmen de Algorismo, contained in the manuscript MS. Auct. F.5.29, preserved in the Bodleian Library in Oxford, dated to the thirteenth century, has been drawn on when comparing *Algorismus* in the manuscripts GKS 1812 4to and AM 544 4to. The manuscripts AM 544 4to and MS. Auct. F.5.29 have chapter headings that are neither found in the other manuscripts of *Algorismus* nor in the two known printed versions of *Carmen de algorismo* (Steele, 1988, pp. 72–80; Halliwell, 1841, pp 73–83).

In the following, chosen passages from *Algorismus* and *Carmen de algorismo* are compared and translated into English. The passages from *Algorismus* in GKS 1812 4to have been rewritten with modern Icelandic spelling.

2 Arithmetic operations in Carmen de Algorismo and Algorismus

Carmen de Algorismo is a hexameter to be recited verbally. The beginning of the poem reads as follows: [1]

> *Hec algorismus ars presens dicitur ; in qua*
> *Talibus Indorum fruimur bis quinque figuris.*
> *0. 9. 8. 7. 6. 5. 4. 3. 2. 1.*
> *Prima significat unum : duo vero secunda :*
> *Tercia significat tria : sic procede sinistre*
> *Donec ad extremam venies, que cifra vocatur ; (Steele, 1988, p. 72).*

The ten digits in the third line of the poem are the only occurrence of the then new Hindu-Arabic numerals in *Carmen*, see Fig. 2.1. Everywhere else numbers are expressed in

[1] In the following examples, the Latin texts are from "Carmen de algorismo", printed in *The Earliest Arithmetics in English* (Steele, (Ed.), 1988).

words. The poem explains algorithms that are now common without giving concrete examples. It is not known how the poem was used to aid computation but one may assume that calculations were made on tablets or a flat surface, strewn with sand, or on a wax tablet.

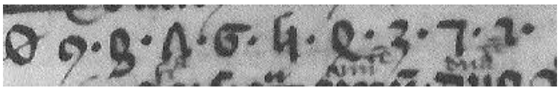

Figure 2.1: The ten digits of Hindu-Arabic numerals in *Carmen de Algorismo* in MS. Auct. F.5.29.

The initial text in *Algorismus* is a nearly literal translation of the Latin original in *Carmen*:

List þessi heitir Algorismus. Hana fundu fyrst indverskir menn með tíu stöfum er svo eru ritaðir 0 · 9 · 8 · 7 · 6 ·5 · 4 · 3 · 2 · 1. Hinn fyrsti stafur merkir einn í fyrsta stað. En annar tvo. En þriðji þrjá. Og hver eftir því sem skipaður er allt til hins síðasta er cifra heitir.[2] (*GKS 1812 4to, 13v*)[3]

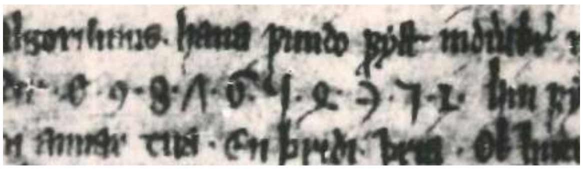

Figure 2.2: The ten digits of Hindu-Arabic numerals in *Algorismus* in GKS1812 4to.

Carmen de Algorismo is believed to be the first work where the nought (zero), *cifra*, is presented as a digit (Beaujouan, 1954). Both the Latin text of *Carmen* and that of *Algorismus* count the *cifra* as the last one in the order of numerals, and 1 as the first one in the order, which indicates that the numbers were read in the Arabic way from right to left. This is emphasised in *Carmen*: “sic procede sinistre” [proceed thus to the left].

Algorismus vulgaris by Johannis de Sacrobosco is a longer text that is related to *Carmen*. Sacrobosco’s text cites three verses from *Carmen*, e.g. on the operations:

Subtrahis aut addis a dextris vel mediabis ;
A leua dupla, diuide, multiplicaque;
Extrahe radicem semper sub parte sinistra (Steele, 1988, p. 73; Sacrobosco, 1898, p. 7).[4]

Algorismus in GKS 1812 4to:

Frá hinni hægri hendi skalt þú af taka og við leggja og skipta í helminga en frá vinstri hendi skalt þú tvöfalda og skipta og margfalda og svo draga rót undan hvorutveggju (GKS 1812, 4to, 13v).[5]

The arithmetic operations addition, subtraction and division, explained in *Carmen* and *Algorismus*, are largely similar to present day methods used in paper-and-pencil arithmetic. Multiplying two composite numbers, however, proceeds from the left, as opposed to common modern algorithms. The numbers to be multiplied are arranged so

[2] Translations from Icelandic into English were made by the author, K. B.
[3] This art we call Algorismus. It was first found by Indians and arranged by ten digits those which so are written: 0 · 9 · 8 · 7 · 6 · 5 · 4 · 3 · 2 · 1. The first digit signifies one in the first place. But the second two. But the third three. And each as it is ordered until the last one which is called cifra [the nought].
[4] Subtract or add from the right, or halving; from left double, divide, multiply; extract roots always from the left side.
[5] From the right hand you shall deduct and add and split in halves but from the left hand you shall double and divide and multiply and so extract both roots.

that the digit farthest to the right of the multiplicand is placed below the first digit (from left) of the multiplier. The multiplicand is multiplied by this digit which then disappears under the product. *Carmen* and *Algorismus* do not illustrate their algorithms on the four arithmetic operations by examples. The following example is constructed for clarification in this paper:

Multiply 523 by 217:

First 523 is multiplied by 2, and 2 disappears under the product:

217	**1046**17
523	

Next, the multiplicand is moved one place to the right so that the rightmost digit is placed below the second digit of the multiplier in the upper row and the lower number is now multiplied by that digit in the same manner. The product is written above the multiplying digit and added to the next digits to the left. In the example, the rightmost digit of the multiplicand, 3, is now placed below the second digit of the multiplier, 1, which now is the multiplying digit. The product $1 \cdot 3 = 3$ replaces the multiplier 1 and products of the other digits are added to previous products. Finally, the previously mentioned digit 3 is moved one more step to the right and the procedure is repeated. In this third step, 7 is the multiplying digit.

217	**1046**17	10**983**7	10**983**7	1**13491**
523	523	523	523	523

In general, this procedure continues until all digits of the upper number have been used as multipliers. The advantage of multiplying from the left is that the products of the digits can be added to the previous product as they are found and it is not necessary to carry.

Doubling and halving were treated as distinct algorithms. Both operations are known from antiquity, even as replacements for multiplication and division (Seppala-Holzman, 2007). Doubling was done from the left as is customary in mental arithmetic, while halving was done from the right.

Square root was drawn from the left. The method is not much different from what was customary before calculators became common in schools. In extracting the square root, one digit of the number of which the root is drawn is pulled down at the time. At the beginning, the highest possible one-digit number, squared, is subtracted. Its double, the "dufl" is preserved aside, and so is the half of the double, the "subdufl", becoming the first digit of the root. Next, as many "dufls" as possible are subtracted when the next digit has been pulled down. The multiplier is the next digit of the root. As the work progresses, the "dufls" are combined and their multiples subtracted, each multiplier constituting the next digit. The combined "subdufls", the first found digit and the multipliers, form the desired square root. An example of drawing the square root of 119,025 was composed, see Fig. 2.3.

	119025		
3^2	9	$-(100a)^2$	subdufl 3, dufl 6
	29		
6·4	24	$-2\cdot 100a\cdot 10b$	
	50		
4^2	16	$-(10b)^2$	subdufl 4, dufl 8
	342		
68·5	320	$-(2\cdot 100a + 2\cdot 10b)\cdot c$	
	25		
5^2	25	$-c^2$	subdufl 5, dufl 10

Figure 2.3: Here a = 3, b = 4, and c = 5. The square root of 119,025 is 345

The algorithm is based on the fact that

$$(100a + 10b + c)^2 = (100a)^2 + 2\cdot 100a\cdot 10b + (10b)^2 + (2\cdot 100a + 2\cdot 10b)c + c^2$$

One notices that by this method, the digits of the square root emerge gradually in a natural way without guessing.

Extracting cubic root had also separate sections in *Carmen*, *Algorismus* and *Algorismus vulgaris*. The section in *Carmen* is considered to be the first treatment of extracting cubic root in Latin (Beaujouan, 1954). It is not contained in the Latin translations of al-Kwārizmī's work but known from the work Āryabhaţīya by the Indian mathematician Āryabhaţa (born 476) (Katz, 1993, p. 202).

Extracting the cubic root is done by alternatively pulling down two digits and one digit at the time. Fig. 2.4 shows the extraction of the cubic root of 15,069,223 to reach 247. A triple digit is called "tripl", while the digit itself is called "subtripl". The algorithm is based on the identity

$$(100a + 10b + c)^3 =$$
$$(100a)^3 + (100a + 10b)\cdot 3\cdot 100a\cdot 10b + (10b)^3 + (100a + 10b + c)(3\cdot 100a + 3\cdot 10b)c + c^3$$

The procedure, shown in Fig. 2.4 is less simple than that of finding the square root. In the second step, when subtracting the term (100a + 10b)·3·100a·10b, one must search the value of b by testing its most likely value, and likewise when searching for c.

	15069223		
2^3	8	$-(100a)^3$	subtripl 2, tripl 6
	706		
24·6·4	579	$-(100a + 10b)\cdot 3\cdot 100a\cdot 10b$	
	1309		
4^3	64	$-(10b)^3$	subtripl 4, tripl 12
	124522		
247·72·7	124488	$-(100a + 10b + c)(3\cdot 100a + 3\cdot 10b)\cdot c$	
	343		
7^3	343	$-c^3$	subtripl 7, tripl 21

Figure 2.4: Here, a = 2, b = 4, and c = 7. The cubic root of 15,069,223 is 247.

3 Deviations of Algorismus from Carmen de Algorismo

The poem *Carmen* contains a description of the Hindu-Arabic number notation in general terms. The treatise *Algorismus* enhances *Carmen* by demonstrating the new system's notation. It extends the first chapters, suggesting a need to clarify the text by numerical examples, while several repetitions in *Carmen* were omitted in the translation. Following *Carmen's* explanation of decimal place value notation, examples are inserted in the Old Norse translation, shown here in bold font and square brackets.

Ergo, proposito numero tibi scribere, primo
Respicias quis sit numerus ; quia si digitus sit,
Una figura satis sibi; sed si compositus sit
Primo scribe loco digitum post articulum fac
Articulus si sit, cifram post articulum sit. (Steele, 1988, p. 72)

Ef þú vilt rita nokkra tölu þá hygg þú að ef það er fingur og rita í fyrsta stað eina hverja fígúru slíka sem þarf ***[á þessa leið, 8].*** *En ef þú vilt lið rita þá settu cifru fyrir fígúru* ***[á þessa lund, 70].*** *Vilt þú samsetta tölu rita þá settu fígúru fyrir lið* ***[sem hér, 65].*** *(GKS 1812 4to, 13v)*[6]

Notice that the order of sentences in *Algorismus* is different from the Latin version. Next even and odd numbers are presented, where the following addition is inserted in *Algorismus*:

Quolibet in numero, si par sit prima figura,
Par erit et totum, quicquid sibi continetur;
Impar si fuerit, totum sibi fiet et impar. (Steele, 1988, p. 73)

*Hverja tölu er þú ritar þá er hún jöfn ef [***tigum gegnir eða]** *jafn fingur er umfram. En öll tala er ójöfn ef ójafn fingur er umfram.* ***[Jafnir fingur eru fjórir: 2. 4. 6. 8. En ójafnir aðrir fjórir: 3. 5. 7. 9. En einn er hvorki því að hann er eigi tala heldur upphaf allrar tölu.]*** *(GKS 1812 4to, 13v)*[7]

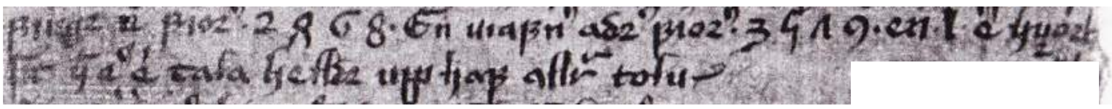

Figure 3.1: …fingur eru fjórir, 2 4 6 8. En ójafnir aðrir fjórir, 3 5 7 9, en 1 er hvorki því hann er ei tala helldur upphaf allrar tölu. (AM 685d 4to, p. 25v)

The digits inserted are written in Hindu-Arabic mode in all extant *Algorismus* manuscripts. *Algorismus* also inserts a note that one is neither even nor odd number as it is the origin of all numbers. Bekken and Christoffersen (1985, p. 27) have pointed out a likeness to the statement that one is not a number in al-Kwārizmī's *Arithmetic*, which again refers to another book on arithmetic, most likely either Euclid's *Elements*, book VII,

[6] *If you want to write some number then think if it is a digit and write in the first place each one figure such as is needed* ***[in this way, 8].*** *But if you want to write tens then put zero in front of the figure* ***[in this way, 70].*** *If you want to write a composite number, put a figure in front of the tens* ***[as here, 65]***. (This translation of the Latin term "post" by "fyrir", meaning "before", "in front of" (see e.g. Cleasby, 1957) is contradictory, but consistent with earlier counting from right to left where 1 was counted as the first in the row of the ten digits, and the cifra as the last one).

[7] *Each number that you write, then it is even if* ***[it is a multiple of ten or]*** *an even digit is extra; but the whole number is uneven if an uneven digit is extra.* ***[Even dig 6. 8. But uneven another four; 3. 5. 7. 9. But one is neither as it is the origin of all number.]***

definition 2, stating: “A number is a multitude composed of units” (Euclid, 1956, p. 277), or *Arithmetica* by the Neo-Pythagorean Nicomachus. The citation referred to is the following from the translation *Dixit Algorizmi* of al-Kwārizmī’s work:

> *Et iam patefeci in libro algebr et amucabalah, idest restaurationis et oppositionis, quod uniuersus numerus sit compositus et quod uniuersus numerus componatur super unum. Unum ergo inuenitur in uniuerso numero. Et hoc est quod in alio libro arithmetice dicitur quia unum est radix uniuersi numeri et est extra numerum :* (*al-Kwārizmī, 1992, p 1*).[8]

The next insertion to *Algorismus* is when *Carmen*’s text states that there are seven operations: addition, subtraction, doubling, halving, multiplication, division and root extraction:

> *Septem sunt partes, non plures, istius artis ;*
> *Addere, subtrahere, duplare, dimidiare ;*
> *Sextaque est diuidere, set quinta est multiplicare ;*
> *Radicem extrahere pars septima dicitur esse. (Steele, 1988, p. 73)*

Then *Algorismus* adds that root extraction has two branches, extracting square root and cubic root:

> *Í sjö staði er skipt greinum þessarar listar. Heitir hin fyrsta viðurlagning. Önnur afdráttur. Þriðja tvefaldan. Fjórða helminga skipti. Fimmta margfaldan. Sjötta skiptingin. Sjöunda að taka rót undan* ***[og er sú með tveimur greinum. Önnur er að taka rót undan ferskeyttri tölu. En önnur grein er það að draga rót undan átthyrndri tölu þeirri er verpils vöxt hefur]****. (GKS 1812 4to, p. 13v)*[9]

Sacrobosco’s elaboration of al-Kwārizmī’s work, *Algorismus Vulgaris*, states: “... radicem extractio, et haec dupliciter, quoniam in numeris quadratis et cubicis” [extraction of roots, which is twofold, since [it applies] to square numbers and cube numbers].” (Sacrobosco, 1897, p. 1). This quotation suggests that the translator may have known Sacrobosco’s text in addition to Villa Dei’s *Carmen*. Sacrobosco claims, however, that there are nine arithmetic operations, adding numeration and progression as operations number one and eight (Sacrobosco, 1897, p. 1).

Each arithmetic operation is explained in a separate chapter. In order to multiply, the reader is instructed to arrange the two numbers to be multiplied in columns such that the first digit (from the right) of the multiplier is placed below the last digit of the multiplicand as explained previously. However, one must first check the difference of the larger digit of the multiplicand from ten and then delete the smaller one from its tens as often as that difference:

> *In digitum cures digitum si ducere, major*
> *Per quantes distat a denis respice, debes*
> *Namque suo decuplo tociens delere minorem; (Steele, 1988, p. 75)*

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[8] *And I have already explained in the book on algebra and almucabalah, that is on restoring and comparing, that every number is composite and every number is composed of the unit. The unit is therefore to be found in every number. And this is what is said in another book on arithmetic that the unit is the origin of all numbers and is outside numbers* (English translation by the author, KB, after André Allard’s translation from Latin to French).

[9] *In seven parts is divided this art’s branches. The first one is called addition. Second subtraction. Third doubling. Fourth halves splitting. Fifth multiplication. Sixth the division. The seventh to take root from under [**and is that one in two branches. One is taking a root from under a square number. But another branch is drawing a root from under an eight-vertex number, the one which has cubical growth].***

Þar næst skalt þú hugsa hversu mikið hina meiri fígúru skortir á tíu þá er þú vilt margfalda. Og svo margar einingar sem áskortir á tíu svo oft skalt þú hina minni töluna þá er þú vilt margfalda taka af tigum hennar. (GKS 1812, 4to, p. 14v)[10]

Algorismus adds this explanation as the last clarification example:

Og að þú skiljir þetta margfalda sjö og níu. Níu skortir einn á tíu, því tak þú eina sjö af sjötigum. Þá verða eftir þrír og sextigir það eru sjö sinnum níu. Að slíku skapi mátt þú aðrar tölur reyna. (GKS 1812 4to, 14v)[11]

In modern notation this can be written

$$7 \cdot 9 = 10 \cdot 7 - 1 \cdot 7$$

or more generally:

$$a \cdot b = 10a - (10 - b)a \qquad (0 < a, b < 10).$$

Two conclusions may be drawn from this explanation. First, the Latin text's presentation was not considered clear enough so that an example was needed. Second, the example demonstrates that the translator/transcriber was not fully confident with Hindu-Arabic digits, so he used words. Numerals do not have a consistent representation across manuscripts. The manuscript GKS 1812 4to uses words, the AM 544 4to Roman numerals, and in the youngest manuscript containing the whole treatise *Algorismus*, the AM 685 d 4to, some numbers in this particular example are written in the Hindu-Arabic numerals while others in words or Roman numerals.

4 A chapter in Algorismus on numbers related to the Elements

In addition to the calculation examples, a separate chapter is added to the translation. It is on the cubic numbers 8 and 27 and their intermediate numbers 12 and 18, and their relation to the Elements: Earth, Water, Air and Fire. This chapter does not exist in *Carmen*, and its content is unrelated to the bulk of *Algorismus* in modern understanding. Its introduction says:

Hver ferskeytt tala hefur tvær mælingar, það er lengd og breidd. En cubicus tala hefur þrenna mæling. Það er breidd og lengd og þykkt eður hæð. Og því kalla spekingar hvern sýnilegan líkama með þessi tölu saman settan að hann hefur saman þessa mæling þrenna. Með því að eilíf speki og einn guð vildi heiminn sýnilegan og líkamlegan skapa, þá setti hann fyrst tvær hinar ystu höfuðskepnur eld og jörð. Því að ekki má náttúrlega sýnilegt vera utan þær. Þar sem eldur gerir ljós og hræring. En jörð staðfesti og hald. En með því að þau hafa þrenna ójafna huiligleika og gagnstaðlega[12] *þá var náttúruleg nauðsyn að setja nokkuð milli þeirra það er samþykkti þeirra ósætti. Og sem fyrr er sagt að eldur og jörð og það allt sem líkamlegt er er með þrefaldri tölu er vér köllum cubicum saman sett þá ritum vér þessa*

[10] *Next you are to think how much the larger figure differs from ten, the one you want to multiply. And so many units as differ from ten so often you are to take the lesser number, the one you want to multiply, from its tens.*

[11] *So that you understand this multiply seven and nine. Nine differs by one from ten, therefore, take one seven from seventies. Then remain three and sixties, that is seven times nine. In that way you may try with other numbers.*

[12] Corrected from "gang staðlega" in GKS 1812 4to. The manuscripts AM 544 4to and AM 685 d 4to have "gagn staðlega", meaning "opposite", "contrary".

tvo cubus. Ritum vér jörðina þessa leið. Tvisvar sinnum tveir tvisvar, 2, 4, 8. En eldinn svo: þrisvar þrír þrisvar, 3, 9, 27. (GKS 1812 4to, 16r–16v)[13]

Thus Earth was assigned the numerical value $2 \cdot 2 \cdot 2 = 8$ and Fire $3 \cdot 3 \cdot 3 = 27$. As no single mediator existed between these two cubes, two proportional numbers were found by taking the square (4) of the root 2 of the smaller cube, multiplied by the root 3 of the larger cube ($4 \cdot 3$): 2, 4, 12. In the same way, the root of the smaller cube (2) was multiplied by the square (9) of the larger cube ($2 \cdot 9$): 3, 9, 18. These two numbers belong equally to the two previously mentioned cubes, 8 and 27, as 27 contains 18 and half of 18, and 18 contains 12 and half of 12.

Similarly, God arranged two elements between Fire and Earth: Air and Water. Water contains two attributes and two numbers from Earth and one attribute and one number from Fire. Air contains two attributes from Fire and two numbers, but one from Earth and one number. The four Elements are thus assigned numerical values: Earth, $2^3 = 8$; Water, $2^2 \cdot 3 = 12$; Air, $2 \cdot 3^2 = 18$; Fire, $3^3 = 27$. This puts the Elements in the correct order by lightness: Fire (27), Air (18), Water (12) and Earth (8). These numbers constitute a sesquialterate progression 8:12::12:18::18:27, or in general terms: n : (n + ½n). The text of *Algorismus* concludes by saying that this can be more perfectly understood from a figure later in the manuscript, called Cubus Perfectus (GKS 1812 4to, 16v).

The idea about the Elements has a strong relation to Plato's *Timaeus*, paragraphs 31b-32c. Calcidius translated the first part (to 53C) of *Timaeus* from Greek into Latin around the year 321 CE.

[31b] … And since it [the world] was rightly to be corporeal, visible, and tangible, and there is no perception of anything visible in the absence of fire, or of anything tangible in the absence of solidity, and no solidity without earth, god laid down fire and earth as the foundations of the world body … no two things cohere firmly and indissolubly without the binding force of a third [31c] … [32a] … if the body of the world were required to have only length and width but no solidity and were of the same sort as the surface of fully formed bodies, then one mean would suffice [32b] for the cohesion of it and its extreme parts. But as it is, since the world body required solidity, and the cohesion of solids involves never one but two means, the craftsman of the world accordingly inserted air and water between fire and earth, salubriously balancing the same elements so that the relationship between air and water would be the same as that between fire and air … And so from the four material elements here named [32c] he fabricated this splendid engine as visible, tangible, and bound together by a harmonious proportion in the equilibrium of its parts … (Calcidius, 2016, p. 49-51)

Comparing texts, originally written in different languages and brought together through translations from language to language, is intriguing and needs vigilance. The texts about

[13] *Every quadratic number has two measures, that is length and breadth. But a cubic number has three measures. That is breadth, length and thickness or height. And therefore wise men call each visible body composed by this number, that it has these three measures. As eternal wisdom and one God wanted to create the world visible and corporeal, he first set the two outmost elements, fire and earth. Because nothing can be naturally visible without them. As fire makes light and motion. But earth solidity and hold. But as they have three different and contrary sets of attributes and then there was a natural necessity to add something in between them that would agree their alienation. And as said previously that fire and earth and everything that is corporeal is combined by a triple number which we call a cube then we write these two cubes. We write the earth in this way. Twice two twice, 2, 4, 8. But the fire so: thrice three thrice, 3, 9, 27.*

the four elements in *Algorismus* and *Timaeus* bear remarkable resemblance. Here, the term “staðfesti” has been translated by “solidity”, the term used in the translation of *Timaeus*, referring to that the earth is a solid and firm body.

Calcidius provided an extensive commentary to his translation of *Timaeus*. In the commentary on the passages above, Calcidius discussed the analogy of the relations between the four Elements to continuous proportions, where the air would have “two powers of fire, its fineness and mobility, and one of earth, i.e. its compactness … for air is compact, fine and mobile”. Similarly, water would have “two powers of earth, i.e. its compactness and corporeality, and one of fire, i.e. its movement, and the substance of water will emerge, that being a body compact, corporeal and mobile. And thus between fire and earth from the coalescence of extremes air and water will arise, giving binding continuity of the world” (Calcidus, 2016, p. 153).

Calcidius (2016, pp. 139–145) had already discussed continuous proportions in relation to these items. However, he did not bring up the sequence 8 – 12 – 18 – 27. That sequence, however, appeared in a manuscript of Euclid’s *Elements*, the Vat. Gr. 190 (codex P), on p. 115v, see Fig. 4.1. The Vatican Euclid (Vat. Gr. 190, called P) is a version of a Greek text of Euclid’s *Elements* dating from the ninth century.

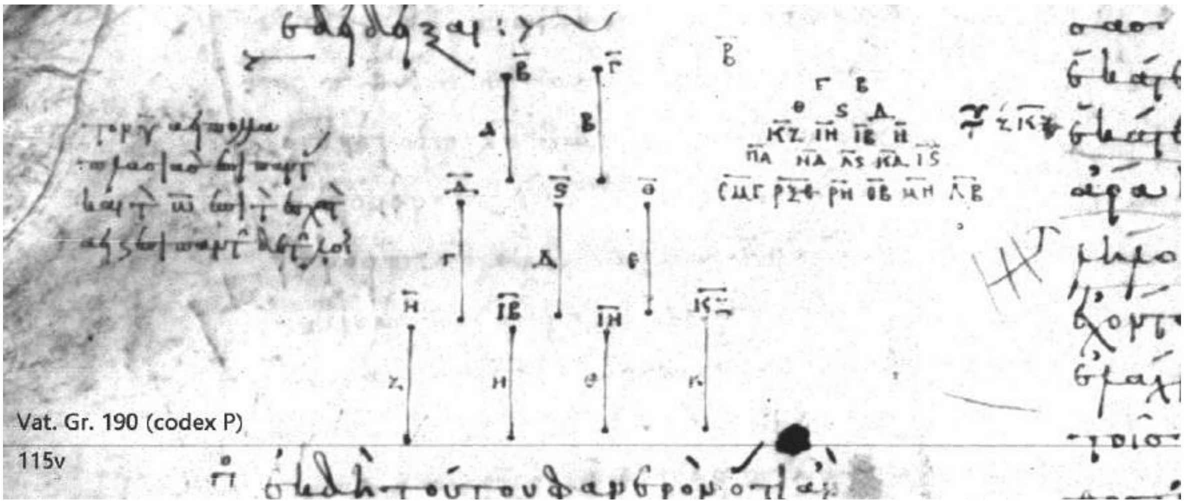

Figure 4.1: A diagram of continued proportions in the manuscript Vat. Gr. 190 (codex P), 115v.

The diagram shown in Fig. 4.1 appeared at the end of proposition VIII 2 in Euclid’s *Elements*. Proposition VIII 2 is the following:

> *To find numbers in continued proportion, as many as may be prescribed, and the least that are in a given ratio. (Euclid, 1956, p. 346)*

At the top of the vertical line segments in Fig. 4.1, marked Α, Β, Γ, Δ, etc., there are numbers, written in the ancient Greek number notation where alphabetical letters with a bar at the top denote numbers. Thus Ᾱ = 1, and the Β and Γ with bars on top of the uppermost vertical line segments denote 2 and 3. The first vertical line segment in the second row is marked Δ with a bar, denoting 4. Thus, in the top row the numbers are 2 and 3; in the middle row 4, 6, 9; and in the bottom row 8, 12, 18 and 27, the numbers appearing in *Algorismus* and in a number of medieval manuscripts, representing continuous proportions with the rate 3/2. On the right side of the diagram, another set of numbers appear, the first three rows with numbers, identical to those on the left, in a reversed order. The sequences thereafter continue up to 2 and 3 to the fifth power on each end, and the numbers in-between in continuous proportions.

5 Diagrams of the Elements

The manuscript AM 736 III 4to contains only a fragment of the treatise *Algorismus*. It does not contain the text on the Elements and their associated numbers. However, on a different leaf in the same manuscript a diagram of the four Elements is found together with their names and the texts "*bis bini tres* xii" [twice two three 12] associated to *Aqua*, Water, "*tres trium bis* xviii" [three thrice twice 18] to *Aer*, Air, and "*tres trium tres*" [three thrice three] to *Ignis*, Fire. The roundels to the right distribute three pairs of qualities: acuity (*acutus* above, *obtusus* below), density (*subtilis* above, *corpulentus* below), and capacity for motion (*mobilis* above, *immobilis* below), see Fig. 5.1. Water is associated with three qualities, being corporeal, soft and mobile. Similarly, Air is soft, mobile and light. It seems fair to conclude that we have here the Cubus Perfectus which is mentioned in the three complete copies of *Algorismus*.

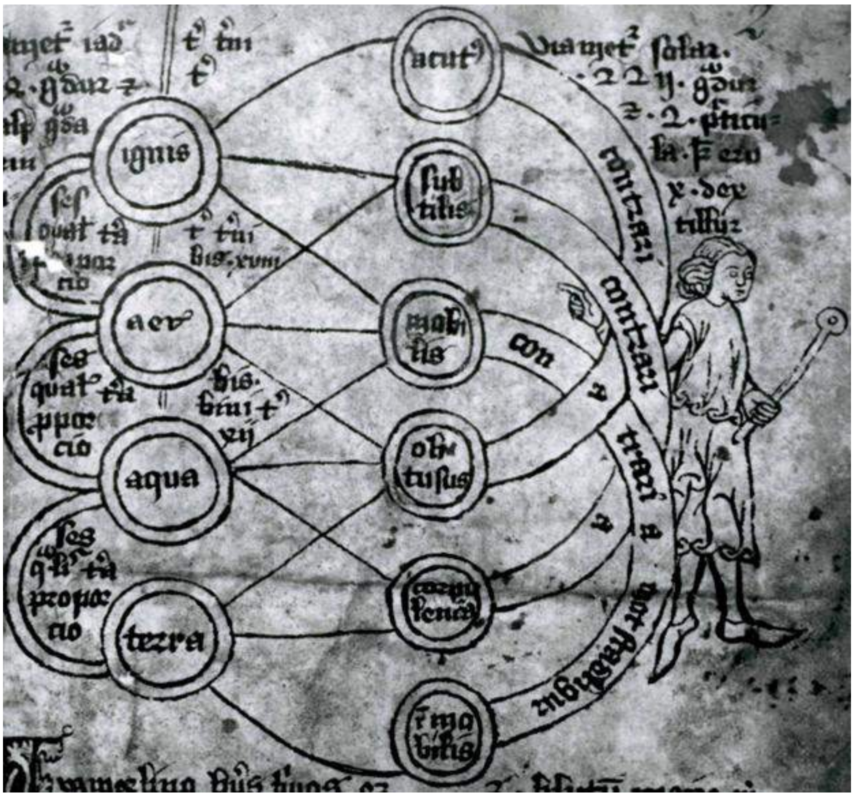

Figure 5.1: A diagram of the four Elements in the manuscript AM 736 III 4to, 2r.

Diagrams with the Elements and the four numbers exist in other foreign manuscripts on medieval cosmology, but those manuscripts are not related to *Algorismus*. For instance, the same sequence of proportions, 8, 12, 18 and 27, and Elements, *Ignis*, *Aer*, *Aqua* and *Terra*, and the same qualities appear on the right in St. John's College MS 17 (Oxford Digital Library),[14] see Fig. 5.2. A similar schema exists in an eleventh century manuscript of Boethius, Madrid Biblioteca nacional Vit. 20 fol. 54v (Bekken, 1986). It is also found in an anonymous treatise on cosmology in Bodleian Library Digby 83, fol. 3r.

[14] The Calendar and the Cloister – St. John's College MS 17, commentary.

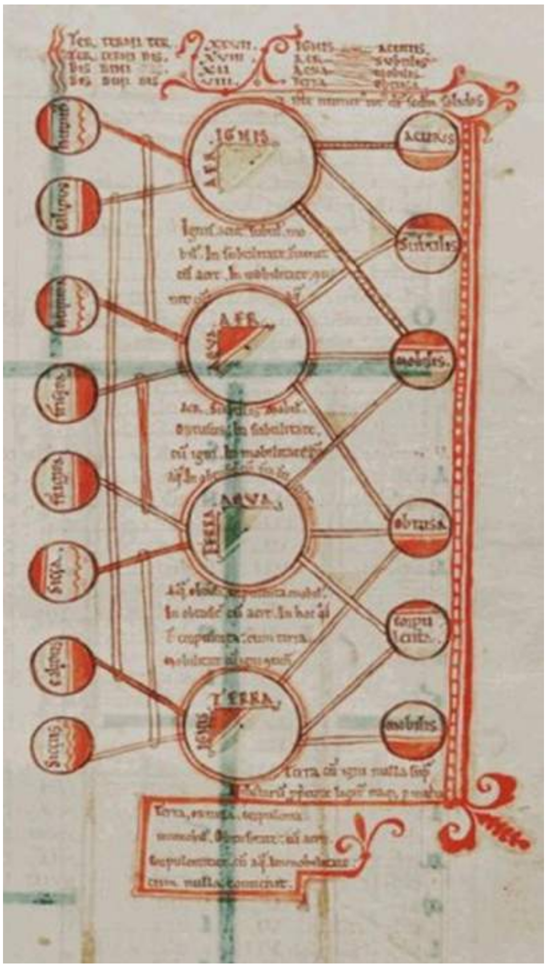

Figure 5.2: St. John's College MS 17, dated early 12th century.

6 Comparing the four manuscripts of Algorismus

The texts of *Algorismus* in the manuscripts AM 544 4to and GKS 1812 4to are identical in most respects, as is *Algorismus* in AM 685 d 4to, which however has added a 306-word long section, placed after the section on subtraction. It describes a method of halving a number, the fourth operation. This section is neither contained in *Carmen* nor in the manuscripts AM 544 4to and GKS 1812 4to, and is not discussed further in this paper. The AM 736 III 4to is only a fragment.

AM 544 4to, preserved in the manuscript collection *Hauksbók*, contains the oldest manuscript of the treatise, estimated to be written in the period 1302–1310, most likely in 1306–1308 (Karlsson, 1964). The text is divided into chapters bearing headings. Numbers are written using Hindu-Arabic numerals in the introduction and in the additions to *Carmen* with examples of place value notation and even and odd numbers, shown earlier. Numbers, however, are mainly written in Roman numerals, until in the last chapter on the Elements, which does not originate in *Carmen de Algorismo,* and where Hindu-Arabic numerals are used.

The part of **GKS 1812 4to** containing *Algorismus* is estimated to be written in 1300–1400 (*A Dictionary of Old Norse Prose – Indices*, 1989, p. 26). There are no chapter headings. Numbers are mainly written using words as in *Carmen*, exceptionally in Roman numerals. Hindu-Arabic numerals are only used in the first additions to *Carmen*, as is done in AM 544 4to, and in the chapter on the Elements.

AM 685 d 4to, is dated to 1450–1500 (*A Dictionary of Old Norse Prose – Indices*, 1989, p. 26). It has no chapter headings. Numbers are written alternately in words, Roman numerals, and Hindu-Arabic notation which is the most common. Finnur Jónsson states that the text of *Algorismus* in AM 685 d 4to is the most error free of the four texts, basing this conclusion on various spelling examples (Jónsson, 1892–1896, p. cxxxi). Furthermore, this text is the most concise of the four texts as it is often contracted, preserving a correct meaning. The text in AM 685 d 4to is also correct where other texts have an error on the origin of one half (Jónsson, 1892–1896, p. 419), called *semiss*, coming up after halving an odd number, which indicates that one of the transcribers of AM 685 d 4to understood the treatise well.

AM 736 III 4to is estimated to origin around 1550 (*A Dictionary of Old Norse Prose – Indices*, p. 26). It contains only a fragment of the text of *Algorismus*, a section on root extraction, in addition to the leaf with the diagram of the Elements in Fig. 5.1.

The adaptations made to *Carmen de Algorismo* to create *Algorismus* suggest that *Algorismus* served a role in introducing the use of Hindu-Arabic numerals in the Norse societies. In the oldest manuscript of *Algorismus*, AM 544 4to, Roman numerals are used to explain the text, or plain words are used as in *Carmen*. The use of Roman numerals indicates that the transcriber needed to shorten the text and that he was not used to Hindu-Arabic numerals.

Plain-word number notation is dominant in GKS 1812 4to. The youngest whole manuscript, AM 685 d 4to, rarely has Roman numerals, while words and Hindu-Arabic numerals are used interspersed.

6.1 Manuscript comparison - methodology

When reading the four manuscripts of *Algorismus* it is apparent that they are quite similar; sentence structure and phrasing suggests that they all derive from the same prototype. The same text insertions and deletions are made in all four manuscripts to *Carmen de Algorismo*, exemplifying that these are not different translations.

How similar are these manuscripts? Numerical methods were used to compare the manuscripts, comparable to methods used extensively in comparative linguistics (Fox, 1995) and in gene and protein comparison. In the following comparison, difference in spelling is generally not revealed as the texts of all the manuscripts have been rewritten in modern Icelandic.

The four texts were aligned using the computer programme ClustalW (Thompson, Higgins and Gibson, 1994), and a weighted number of mismatches between the manuscripts was computed. As ClustalW is designed to align protein sequences it takes as input sequences from the twenty letter alphabet of protein sequences. The Icelandic version of the Latin alphabet is larger than twenty letters, so each letter was mapped to two letters in the alphabet of protein sequences. ClustalW was then used to align the texts and the text was mapped back to the Latin alphabet. The alignment was then corrected manually, considering in particular word reorder and different forms of the imperative.

Mismatches between the manuscripts were counted and classified into three distinct classes; *single character mismatches, word reorders* and *word mismatches*.

- *Single character mismatches* were defined as:

  - Identical spelling apart from a single character difference.
  - Mismatches in writing style of the numerals; Hindu-Arabic, Roman or spelled out.
  - Mismatches in the writing of the imperative, e.g. *tak þú* - *taktu*.
- *Word reorders* were defined as parts of the manuscripts where the order of two or more words had been reordered.
- *Word mismatches* were all other types of differences such as word insertion, missing words or a different word being used.

The weighted distance between the manuscripts was used to infer the phylogeny of the manuscripts, using the assumption that it is unlikely that the same change is made more than once. One may also assume that each transcriber is equally likely to cause a distinction. Finally, a simple programme was written to count the number of differences.

6.2 Results

The manuscripts are different in length. In the following, a section in AM 685 d 4to of length 306 words, not extant in the other manuscripts, has been removed. The lengths are:

Manuscript	Words #	Characters #
GKS 1812 4to	2986	15174
AM 544 4to	2960	15110
AM 685 d 4to	2902	14772
AM 736 III 4to	630	3323

Table 6.1: No. of words and characters in the four manuscripts of *Algorismus*.

That AM 685 d 4to has fewest words of the complete manuscripts suggests that the transcriber(s) of AM 685 d 4to sometimes shortens the text.
The following weights of mismatches were used:

Word mismatches:	1.00 point
Word reorders:	0.25 point
Single character mismatches:	0.25 point

Results from counting mismatches between the three complete texts in AM 685 d 4to, AM 544 4to, and GKS 1812 4to, were:

Manuscripts	GKS 1812 4to	AM 544 4to	AM 685 d 4to
	0.00	123.25	264.50
AM 544 4to	123.25	0.00	261.00
AM 685 d 4to	264.50	261.00	0.00

Table 6.2: No. of mismatches between the three complete manuscripts of *Algorismus.*

The shortest distance between two manuscripts is between AM 544 4to, and GKS 1812 4to, 123.25 mismatches by 2986 words, or 4.1%.
The greatest distance between two manuscripts is between AM 685 4to, and GKS 1812 4to, 264.5 mismatches by 2986 words, or 8,9%.
The parts of the manuscripts that all have in common, i.e. the part also found in AM 736 III 4to, were compared separately.

The results were:

Manuscripts	GKS 1812 4to	AM 544 4to	AM 685 d 4to	AM 736 III 4to
GKS 1812 4to	0.00	26.00	55.00	57.75
AM 544 4to	26.00	0.00	52.00	57.25
AM 685 d 4to	55.00	52.00	0,00	73.25
AM 736 III 4to	57.75	57.25	73.25	0.00

Table 6.3: No. of mismatches in the part common to all manuscripts of *Algorismus*.

The distance of AM 736 III 4to is greatest from AM 685 d 4to, while it is closest to AM 544 4to, and nearly equally close to GKS 1812 4to. Clearly, AM 544 4to and GKS 1812 4to are more close to each other than the other two, which are also different from each other.

In this counting of mismatches the ratio 1 : 0,25 or 4 : 1 between *word mismatches* and other mismatches was used. Counting was also done using the ratio 3 : 1 and lead to comparable conclusions.

Fig. 6.1 exhibits the relation between the different copies of *Algorismus*. A matrix was made according to the distances between the four manuscripts, from which was constructed a phylogenetic tree with distances similar to the distances in the distance matrix. The diagram was made by the programme ATV (Zmasek and Eddy, 2001).

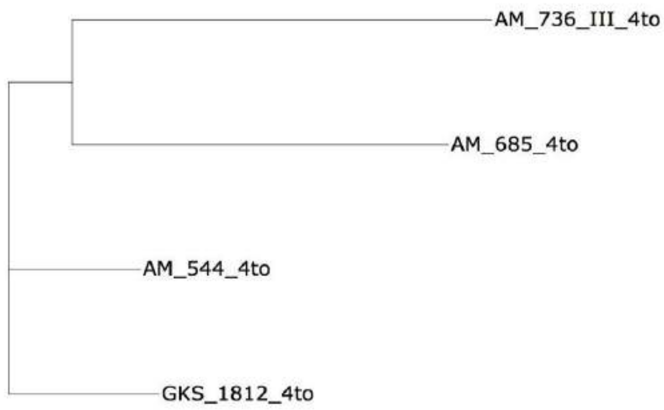

Figure 6.1. A phylogeny of the copies of the part of *Algorismus* in common to the manuscripts AM 736 III 4to, AM 685 d 4to, AM 544 4to, and GKS 1812 4to, made by the programme ATV.

The phylogeny may be interpreted such that the manuscript AM 544 4to, contains the most original copy of the treatise, and that the copy in GKS 1812 4to is closest to it. The copies in the manuscripts AM 736 III 4to and AM 685 d 4to are partly drawn from the same stem, but are further from the origin, in particular AM 736 III 4to.

Comparing copies of *Algorismus* in GKS 1812 4to and AM 544 4to we see that both copies were written about +/-50 years after Finnur Jónsson's estimated translation date, before 1270. The difference in the number of words in the two copies are 26 words where GKS 1812 4to is the longest. Of them, 18 can be ascribed to the differently expressed imperative form of the verbs, such as *skalt þú – skaltu*. The contracted form is more common in AM 544 4to while the separated form is the norm in GKS 1812 4to.

Out of the points for mismatches, 123.25, 11 may be explained by different form of the imperative and different expression of numbers. Both may be interpreted as efforts to save the precious vellum. Most other mismatches may be ascribed to personal preferences of the scribes or simple mistakes.

It is not unreasonable to conclude that the two versions of *Algorismus* are copies of the same origin, possibly the first or second copies of the original version. The two versions of *Algorismus* are similar in length. Both contain several same errors, for example in the section about doubling:

> *... en ef semiss stendur yfir uppi í ysta stað þá legg við einn því að þar var áður <u>jöfn</u> tala er í helminga var skipt.(GKS 1812 4to, 14r)*[15]

Here, *jöfn*, *even* [number] should be replaced by *ójöfn*, *uneven*/*odd* [number]. This error is not found in the copy contained in the fifteenth century manuscript AM 685 d 4to.

7 Discussion

We have explored the thirteenth century treatise *Algorismus*, written in the Old-Norse language, its content of arithmetic studies and cosmology, and compared its manuscripts. *Algorismus* was written in the transition period when Hindu-Arabic decimal place value numeral notation and associated arithmetic methods were being introduced in Europe. The four different manuscripts of *Algorismus*, written in the time span from early fourteenth century until mid-sixteenth century, reveal that the new style of number notation gradually entrenched. We may wonder how large role *Algorismus* played in that development.

What motivated the Old Norse people to translate *Carmen de Algorismo*? Certainly, they had to count their belongings and assets, e.g. for taxation, but they could have done so with the Roman numerals they knew. Writing manuscripts was an integral part of the Christian monastic culture. The reason may have been an aspiration to belong to the European cultural world. The Old-Norse-speaking population in Iceland and Norway was never large compared to populations of millions in the centres of the Christian world in the present Italy and France. Producing writings in the vernacular was an important factor in creating a common culture for this small group of people.

The additions in *Algorismus* to its original, *Carmen de Algorismo*, bear witness to a desire for learning, to understand the text as demonstrated by insertions for clarification. Comparison of the four copies of *Algorismus* of different age reveals that people continued to work on understanding the text and gradually began to use the convenient Hindu-Arabic number notation.

But it took time. According to the phylogeny and other considerations, manuscript AM 544 4to was not the original of *Algorismus*, suggesting that *Algorismus* may originally have been written in the second half of the thirteenth century, as proposed by Finnur Jónsson, or about 200 years before AM 685 d 4to, and possibly up to 300 years before AM 736 III 4to. *Algorismus* therefore played an important role in Icelandic culture until the era of printing, when printed books began to spread much more rapidly between countries than manuscripts.

Iceland was originally an independent society in close contact to Norway, but from 1397 it belonged to the Danish realm until 1944. It lagged gradually behind other European countries in educational respect. *Algorismus* appears in history whenever mathematics education was revived, serving as a monument of the proud past, when

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[15] *... but if a semiss* [symbol for one half] *is placed above in the farthest place then add one as there was earlier an <u>even</u> number divided into halves.*

Icelanders kept up with the latest global knowledge. Even the most distinguished Icelandic scholars continued to refer to *Algorismus* until the nineteenth century (see e.g. Gunnlaugsson, 1865, p. 4), paying respect to the time when Icelanders were familiar with the latest mathematical knowledge in the world and translated it to their own language.

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