THE FIRST NORWEGIAN TEXTBOOKS IN MATHEMATICS

A story of independence and controversy

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ABSTRACT

Norway got its Constitution in 1814 after being subject to Denmark for 400 years. As a result of the Napoleonic wars, we had to go into a union with Sweden that lasted until 1905. However, the years after 1814 were a time for national awakening, where Norway had to rely on its own resources.

The first Norwegian university was functional from 1813, and there were learned schools that prepared students for the university. These schools had mostly been using Danish textbooks, but in 1825, Bernt Michael Holmboe wrote the first Norwegian textbook in mathematics for the learned schools. Holmboe wrote textbooks in arithmetic, geometry, stereometry and trigonometry, and most of them came in several editions.

Holmboe became a very influential person in the development of school policy in Norway, and he was a close friend of the mathematician Niels Henrik Abel. The learned community was small in Norway in the beginning of the 19th century, but that did not prevent Holmboe's textbooks from meeting opposition. His textbook in geometry caused a long and bitter controversy with the only other Norwegian professor in mathematics at that time.

1 Introduction

Towards the end of the 18th century, a great effort was done to establish mathematics as a school subject in the higher education in Norway, and a school reform that was introduced around year 1800 lead to proper teaching in mathematics. The mathematical community in Norway at that time was small, and all the participants did necessarily become significant members of the society. This was a time of considerable development in the subject of mathematics, and this also influenced the debate about mathematics education.

My focus in this paper is the textbooks in mathematics written by Bernt Michael Holmboe (1795–1850), and my motivation is to look at the development of mathematics education and the didactic debate in the early 19th century, and what significance Holmboe played. The topics in school textbooks for the learned schools during this period were arithmetic and algebra, geometry, trigonometry and stereometry, and there was a growing demand for rigour in mathematics. An intuitive understanding of numbers was, as an example, not sufficient anymore, and this reflected on the textbooks. The teaching of mathematics was also a motivation for this demand for rigour.

I will in this paper try to understand these textbooks based on the historical period, and the political and social conditions in which they were written. I will describe the content of the textbooks, I will illuminate issues like how updated they where with regards to the development of mathematics, where the inspiration came from, and a relevant question will be what we may learn today from these textbooks and the didactical debate in early 19th century.

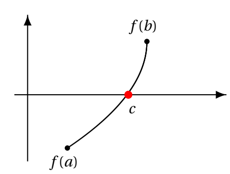
Issues addressed in this paper, and earlier research about the textbooks of Bernt Michael Holmboe, may be found in Christiansen (2009, 2010, 2012a,b, 2015b,a) and in Bjarnadóttir et al. (2013). All translations from Norwegian and Danish-Norwegian to English are made by the author.

1.1 The demand for rigour – an example

An intuitive and geometrical interpretation of real numbers did not satisfy the 19th century mathematicians' demand for purity in methodology, and the need for a new understanding of real numbers did arise in connection with the proof of the *intermediate value theorem*. This theorem was proven by Bolzano (1817), and both continuity of functions and convergence of infinite series are defined and used correctly in this paper, as it is understood in a modern sense. Bolzano's comprehensive paper was later translated into English (Russ 1980, 2004), and the latest of these two translations forms the basis of the following short description.

According to Russ (1980: 157), Bolzano's paper includes "*the criterion for the (pointwise) convergence of an infinite series, although the proof of its sufficiency, prior to any definition or construction of the real numbers, is inevitably inadequate*". This criterion is, however, *not* concerning the definition of convergence, but the Bolzano-Weierstrass theorem, which states that every bounded sequence has a convergent subsequence.

Bolzano's only strict requirement is "*that examples never be put forward instead of* proofs *and that the essence* [Wesenheit] *of a deduction never be based on the merely figurative use of phrases or on associated ideas, so that the deduction itself becomes void as soon as these are changed*"[1] (Russ 2004: 256).



In modern words, Bolzano's explains his point by considering two formulations. First we have a geometric theorem in intuitive form, which states that if a continuous line, where the ordinates are first negative and then positive – or conversely – then this line must necessarily intersect the axis of abscissas somewhere between those ordinates. Next we have a purely analytical formula – the intermediate value theorem – which states that if we have a function f, continuous on $[a, b]$ such that $f(a) < 0,\ f(b) > 0$ then it follows that $\exists c \in (a, b)$ such that $f(c) = 0$. Bolzano then states that we *cannot* conclude the intermediate value theorem from the geometric theorem in intuitive form. We may, on the contrary, conclude the mentioned geometric theorem from the intermediate value theorem. (Russ 2004: 254–255)

2 A short historical background

Some historical events had a significant impact on how Norway developed as a nation.

The black plague played a very significant role in the development of the Norwegian society. It reached Norway in the summer of 1349, and it is believed that it killed 2/3 of the country's population. This was one of several reasons why Norway entered a union with Denmark in 1380, a union that throughout the ages had many different forms. The Danish era lasted 434 years, until 1814.

[1] Bolzano also criticised Gauss's original proof of the fundamental theorem of algebra of 1799, because Gauss here used geometrical considerations to prove an algebraic theorem (Otte 2009: 53). Bolzano did not doubt the validity of the theorem, but he criticised the "impurity" of the method.

The Reformation started in Germany in 1517, and was literally forced on Norway by the Danish King in 1536. Norway had been in a *union* with Denmark, sharing the same king, but in 1536 the Norwegian *National Council* [Riksråd] was dissolved. This meant that Norway was no longer a nation, but became a part of Denmark, and this had of course a profound importance on the development of the country. The reformation had, however, a positive effect on the development of the educational system in Norway.

Much happened in Norway in and around 1814. Denmark-Norway was on the losing side of the Napoleonic Wars, and at the peace treaty in Kiel in January 1814, it was agreed between the Danish King and the Swedish Crown Prince that Norway should enter into a union with Sweden. In Norway there was great dissatisfaction with this agreement, and representatives of the Norwegian elite gathered at Eidsvoll in February in an attempt to take control of country's own destiny. It was agreed that the provision in Kiel should be rejected and that Norway should be declared an independent state. The Norwegian Constitution was drafted and was signed on May 17, 1814. On the same day, a Danish Prince, Christian Fredrik, was chosen as king. As a reaction to this, the Swedish Crown Prince went to war against Norway in the summer of 1814, a war that lasted no more than three weeks. The new king had to abdicate and left the country in October. Norway had to adjust the Constitution, and went into a union with Sweden that lasted till 1905.
(Kunnskapsforlaget 2006)

3 Educational system

Since the Middle Ages, there have been schools for the children of the elite. There was a growing religious pietism in Norway after the Reformation, and in the 18th century there was a demand to educate children of common people, mainly to make them able to read the Bible.

3.1 Schools for regular people

In the 1730s there were several laws that regulated children's confirmation in the church, and the teaching that led to it. Schools for common people were introduced where children could learn Christianity, reading, writing and calculations. Schools were established in cities and towns, but in the rural areas the situation was a bit different.

The system that developed in rural areas is something we call "*omgangsskole*", where the teacher moved from area to area, stayed at one farm, and taught all the children in the neighbourhood. After a while, the teacher moved on. In a year, the children could get around two months of schooling.
(Kunnskapsforlaget 2006)

3.2 Learned schools

Cathedral schools [Kathedralskoler] were schools from the medieval time that were connected to cathedrals, and were meant to give a theologic education to the future priests.

Table 1: Some Cathedral schools from the Middle Ages

SCHOOL	ESTABLISHED	REMARKS
Oslo	1152	
Bergen	1152	
Trondhjem	1152	
Hamar	1152	Merged with *Oslo Kathedralskole* in 1602. Re-esablished in Hamar in 1876
Stavanger	13th century	Moved to Christiansand in 1686. Stavanger had a smaller latin school that was closed in 1739, and re-established in 1824
Christiansand	1686	

All cathedral schools were turned into *Latin schools*, or *grammar schools* [latinskoler], when the reformation was introduced in Norway in 1536, and it was mandatory for every town to have a one. The Latin schools, together with the old cathedral schools, constituted the so-called *learned schools*. Some of the schools did, however, keep their old names. Many of these Latin schools were of a poor quality, so in reality, the higher education preceding the university in 1814 was only *four* cathedral schools.

Table 2: Some learned schools after 1814

SCHOOL	ESTABLISHED
Christiania Cathedralskole	1152
Bergens Cathedralskole	1152
Trondhjems Cathedralskole	1152
Christiansands Cathedralskole	1686
Drammens Latinskole	1817
Fredrikshalds Latinskole	1822
Skiens Latinskole	1822
Stavangers Latinskole	1824

By a governmental decree in 1809, the pupils started at the learned schools at the age of 9 or 10 years, and the duration was normally eight years consisting of four two-year grades, and each day at school was seven hours – four before noon and three after. The *university qualifying examination* [examen artium] were arranged by the university.

The learned schools gave a classic education, and a higher education in scientific subjects at the same standard as the learned schools could be achieved at the *Military Academy*.[2] This school admitted pupils from the age of 12–14.

Several *intermediate schools* [middelskoler] were established in smaller towns after 1814, and they were learned schools without the upper two-year grade.
(Andersen 1914; Kunnskapsforlaget 2006)

[2] Den frie mathematiske Skole (1750–1798), Det norske militære Institut(1798–1804), Det kongelige Norske Landcadetcorps' skole (1804–1820) and Den Kongelige Norske Krigsskole after 1820.

The kingdom Denmark-Norway introduced a new school reform around the year 1800 which in many ways strengthened the position of the discipline of mathematics, and from now on may we talk about proper teaching in mathematics in the higher education (Piene 1937). There was much work done in the last decades of the 18th century to reintroduce mathematics as a subject in school.

3.3 University

The first Norwegian University was established in Christiania in 1811, and came into function in 1813. The name was *The Royal Fredriks University* [Det Kongelige Frederiks Universitet (Universitas Regia Fredericiana)], named after the Danish King Fredrik the 6th, and it was changed to *The University of Oslo* [Universitetet i Oslo] in 1939. The only use of mathematics the first years was for the *examen philologico-philosophicum* – a preparatory exam for other subjects. The lectures in mathematics were on trigonometry, stereometry, basic algebra, and later applied mathematics after Christopher Hansteen's appointment. (Holst 1911; Kunnskapsforlaget 2006)

3.4 Niels Treschow (1751–1833)

The philosopher and politician Niels Treschow was an important person in the learned society of Norway, and he was professor in philosophy at the new university at the time when Holmboe was a student. Treschow published several books, among them a textbook in *Common Logic* (Treschow 1813), clearly influenced by Immanuel Kant. Treschow has a rigorous classification of statements – in *direct* and *indirect* statements, and in *analytic* and *synthetic* statements – based on the relation between subject and predicate in the statements, and on the nature of the statements (Treschow 1813: 161–162). Exactly the same classification is described in the introduction chapter of Holmboe's textbook in arithmetic (Holmboe 1825: 1–3).

A *statement* [sætning] is connection of two concepts. The first thought of, in the connected concepts in a statement is called *subject* [Subject], and the second is called *predicate* [Prædicat]. Statements are called *direct* [umiddelbar] if one perceives the subject's connection to the predicate *without* regarding other statements, and *indirect* [middelbar] if one uses other statements when considering the connection between subject and predicate. In the latter case, the indirect statement will then be a consequence of the statements used to elucidate it. The presentation of the conclusions used to elucidate the subject's connection to the predicate in indirect statements is called a *proof* [Beviis] (Holmboe 1825: 1–2).

The statements are divided in *synthetic* or *practical* statements, expressing that a connection between concepts shall be made, and *analytic* or *theoretic* statements, expressing a connection between concepts that already exists. Both synthetic and analytic statements may be direct as well as indirect, and Holmboe is therefore introducing four types of statements (Holmboe 1825: 2–3):[3]

Direct synthetic statements which he calls *Postulates* [Fordringssætninger]. A postulate expresses that two concepts shall be connected.

Direct analytic statements which he calls *Fundamental Statements* [Grundsætninger] or *Axioms*. An axiom expresses the relations between two connected concepts.

[3] All these definitions are derived from Treschow (1813: 161pp), but it seems like Holmboe has introduced the synonyms *practical* for *synthetic*, and *theoretic* for *analytic*.

Indirect synthetic statements which he calls *Problems* [Opgaver]. A problem expresses that two concepts shall be connected using already existing connections.

Indirect analytical statements which he calls *Theorems* [Læresætninger]. A theorem expresses a connection between two concepts which is proven to be a consequence of preceding connections.

There are in Holmboe's textbook in arithmetic 10 direct statements called *Fundamental Statements*, there are 27 indirect synthetic statements, or problems, and there are 30 theorems and 163 corollaries. The preface of Holmboe (1825) may seem unnecessarily complicated and difficult for a textbook meant for young students, but Holmboe clearly states that his textbook is not meant for self-study, but requires a skilled teacher.

4 Bernt Michael Holmboe (1795–1850)

Bernt Michael Holmboe was born on the 23rd of March 1795 in Vang in Valdres, centrally situated in Southern Norway, and he died on the 28th of March 1850, at the age of 55 years and 5 days. Holmboe was a teacher at Christiania Kathedralskole from 1818 till 1826, and after that he was lecturer at the university until 1834, when he was appointed professor in mathematics, a position he had until his untimely death. Holmboe's home burnt down shortly after his death, and some of his works and letters were lost, in addition to some of the works and letters from Niels Henrik Abel (1802–1829). (Bjerknes 1925: 56,79)

Among Holmboe's students we find mathematicians like Niels Henrik Abel, Ole Jacob Broch and Carl Anton Bjerknes. Holmboe proclaimed that in no other subject did novices complain more than in mathematics (Piene 1937), and his aim was to make the students familiar with mathematical signs *before* a more methodical study. He further stated that unless pupils engaged in "uninterrupted practice" ["idelig øvelse"] even persons with several years of education would find that mathematics is "something of a mind consuming and boring matter" ["noget åndsfortærende og kjedsommelig tøi"]. The lecture notes of Carl Anton Bjerknes shows, however, that Holmboe's teaching was characterized by *pre-abelian* times, in spite of his knowledge of Abel and his works (Bjerknes 1925).

Holmboe wrote in a letter to the then 24 years old Carl Anton Bjerknes (Holmboe 1849) that he is inspired by the great French mathematician Joseph-Louis Lagrange (1736–1813). Bjerknes had asked Holmboe to advise him about studies in mathematics, and Holmboe wrote "*The best I have to state in this respect is to inform you about some notes from Lagrange and some rules and remarks by him regarding the study of mathematics, which I found in Lindemanns and Bohnebergers* Zeitschrift für Astronomie *about 30 years ago … Those who really want, should read Euler, because in his works all is clear, well said, well calculated, because there is an abundance of good examples, and because one should always study the sources*".

4.1 Holmboe's textbooks

The following table shows an overview over Holmboe's textbooks, their Norwegian titles and their various editions.

Table 3: An Overview over Holmboe's textbooks

TITLE	EDITION	YEAR	EDITED BY	PUBLISHER
Lærebog i Mathematiken	1st	1825		Jacob Lehmann
Første Deel, Inneholdende Indledning	2nd	1844		J. Lehmanns Enke
til Mathematiken samt Begyndelses-	3rd	1850		J. Chr. Abelsted
grundene til Arithmetiken	4th	1855		R. Hviids Enke
	5th	1860		R. Hviids Enke
Lærebog i Mathematiken	1st	1827		Jacob Lehmann
Anden Deel, Inneholdende	2nd	1833		Jacob C. Abelsted
Begyndelsesgrundene til Geometrien	3rd	1851	Jens Odén	R. Hviids Enke
	4th	1857	Jens Odén	J. W. Cappelen
Stereometrie	1st	1833		C. L. Rosbaum
	2nd	1859	C. A. Bjerknes	J. Chr. Abelsted
Plan og sphærisk Trigonometrie	1st	1834		C. L. Rosbaum
Lærebog i den høiere Mathematik Første Deel	1st	1849		Chr. Grøndahl

As a general rule throughout the first four textbooks, theorems and proofs have this structure, as described in Holmboe (1825: 3):

Theorem [Læresætning *or* Theorem] — A verbal description without any mathematical notation.

Condition [Betingelse *or* Hypothesis] — A small number of theorems have conditions set in algebraic notation. The condition is a theorem showing a preceding connection.

Algebraic statement [Sats *or* Thesis] — The theorem represented in pure algebraic notation.

Proof [Beviis] — A regular proof of the theorem using algebraic notation and/or written text.

A theorem that is a direct consequence of a preceding theorem, is called a **Corollary** [Tillæg], or a supplement. Some of the *corollaries* are followed by a proof, but I have not found any instances where a corollary is written as an algebraic statement.

4.2 Arithmetic

Volume one of Holmboe's textbook in mathematics contains the introduction to arithmetic and algebra. The book came in a total of five editions from 1825 through 1860, but only the first three (1825; 1844; 1850) were published in Holmboe's lifetime, and these three editions are the attention of this description.

In the first three editions, there is an interesting development in the definition and use of irrational numbers, which is in accordance with the development of mathematical analysis.

Table 4: Definitions of irrational numbers

EDITION	DEFINITION
1825	Any number, that cannot be expressed either as a whole number or as a fraction, whose numerator and denominator are whole and finite numbers, is called an *irrational* number.
1844 1850	Any magnitude, that cannot be expressed either as a whole number or as a fraction, whose numerator and denominator are whole and finite numbers, but whose value always falls between two fractions $\frac{t}{n}$ and $\frac{t+1}{n}$, where t and n are whole numbers, and where one can make n larger than any given number, is called an *irrational* number.

The definition from 1825 tells us what an irrational number *isn't*, it doesn't tell us what it *is*. Opposed to irrational numbers, all whole numbers and fractions, whose numerator and denominator are whole and finite numbers, are called *rational* numbers. The definition of 1844 and 1850 may indicate an influence by Bernard Bolzano and his definition of *measurable numbers* (Russ 2004: 347–349,360–361), and Holmboe now specifies *magnitudes*, and not numbers.

The following table illustrates some of the use of irrational numbers in corollaries and in the definition of the sum where at least one of the addends is irrational.

Table 5: Use of irrational numbers

EDITION	COROLLARY / DEFINITION
1825	One can always find a rational number, whose value approaches the value of a given irrational root, so that the difference between them is less than any given unit fraction.
1844 1850	If two irrational, [a] positive magnitudes P and Q, both independent of n and between boundaries of the form r and $r+\frac{a}{n}$, are in such a way that $r<P<r+\frac{a}{n}$ and $r<Q<r+\frac{a}{n}$, where one can make n larger than any given number and a is finite. Then $P=Q$.
1850	If one or both of two magnitudes, x and y, are irrational, and where $\frac{t}{n} \leq x < \frac{t+1}{n}$ and $\frac{p}{n} \leq y < \frac{p+1}{n}$, and one can make n larger than any given number. The sum $x+y$ is then to be understood as the common boundary for the sums $\frac{t}{n}+\frac{p}{n}$ and $\frac{t+1}{n}+\frac{p+1}{n}$, whose difference is $\frac{2}{n}$, which disappears when n grows infinitely.

[a] In the 1850 edition, the specification of *irrational* is taken out. The statement is now valid for *real* numbers or magnitudes – rational and irrational.

Bernard Bolzano introduces *measurable numbers* in his *Pure Theory of Numbers* ["Reine Zahlenlehre"][4] (Russ 2004: 347–49, 360–61). According to the definition, S is *measurable*

[4] Was not published until late 20th century (Russ 2004: 681). Bolzano wrote in a letter, dated 5th of April, 1835, to one of his former students, that he had "*one book near completion with the title* Pure Theory of Numbers *consisting of two volumes: an* Introduction to Mathematics, *the first concepts of the general theory of quantity, and then the* Theory of Numbers *itself*" (Russ 2004: 347).

when

$$\forall q \in \mathbb{N} \quad \exists p \in \mathbb{Z} \quad \text{such that} \quad S = \frac{p}{q} + p_1 = \frac{p+1}{q} - p_2 \quad \text{where} \quad p_1 \geq 0, \quad p_2 > 0$$

In other words

$$\frac{p}{q} \leq S < \frac{p+1}{q}$$

Bolzano explains that p_1 and p_2 denotes a pair of strictly positive number expressions, the former possibly denoting zero (Russ 2004: 361).[5]

The measurable number may be used to measure, or determine by approximation, the magnitude or quantity. Bolzano called the fraction $\frac{p}{q}$ the *measuring fraction*, and the fraction $\frac{p+1}{q}$ the *next greater fraction*. p_1 is called the *completion* of the measuring fraction since $S = \frac{p}{q} + p_1$. Every *rational number* is a measurable number where $p_1 = 0$, and indeed a complete measure.

Abel is the only known reference that Bolzano was known already in the 1820s, as he mentions Bolzano in his Paris notes (Schubring 1993: 45). Schubring (1993: 50) writes that during the four months Niels Henrik Abel stayed in Berlin in 1825, he was in close contact with August Leopold Crelle and his mathematical circle, where he was engaged in intensive conversations on all mathematical issues. Crelle had Bolzano's three booklets in his personal library, and Abel's reading of Bolzano was part of this process of communication.

4.3 Geometry

The textbook in basic geometry (Holmboe 1827) starts with several definitions of basic concepts. The very first definition describes *geometry* as a science about the *coherent magnitudes*. Coherent magnitudes are the space with all available dimensions and time. According to Solvang (2001), Holmboe's way of organizing the subject matter was influenced by Adrien-Marie Legendre's (1752–1833) introduction to geometry (Legendre 1819). The geometry of Legendre is constructed mainly the same way as Euclid, and starts with a long list of what he calls *explanations*, similar to what Euclid calls *definitions*.

The first definition in Legendre (1819) defines *geometry* as a *science which has for its objects the measure of extension. Extension has three dimensions, length, breadth, and thickness.* With reference to classification of coherent magnitudes in space and time, Holmboe classifies geometry in two parts:

1. *The real geometry* defined by the relations between the various magnitudes in space, without considering their changes in time.
2. *Mechanics*, defined by the changes the magnitudes goes through in time. All changes on a magnitude through time are called motion, and it is conditioned by force.

It is postulated that the space stretches indefinitely.[6]

[5] Bolzano denoted them p^1 and p^2, but the superscripts only distinguished, they where not powers (Russ 2004: 349). I have chosen to use subscripts to avoid confusion.

[6] Geometrie er en Videnskab om de sammenhængende Størrelser. Sammenhengende Størrelser er ere Rummet med enhver deri forekommende Udstrækning og Tiden. Med Hensyn til de sammenhængende Størrelsers Inddeling i Rum og Tid, inddeles Geometrien i 2 Dele. (1) Den egentlige Geometri, der bestemmer de i Rummet forekommende Størrelsers Forhold til hinanden uden Hensyn til deres Forandring i Tiden. (2) Mekanik, der bestemmer de Forandringer, som Størrelserne undergaae i Tiden. Anm. Enhver Forandring, som en Størrelse i Tiden undergaaer, kalles Bevægelse, hvis betingelse kaldes Kraft. Fordringssætning. Rummet maa tænkes udstrakt i det Uendelige. (Holmboe 1827: 1)

Holmboe advises the teacher to show moderation in the review of proofs, and to show examples using numbers before the examination of the proof. This practical advice contradicts the structure of his textbooks, which is strictly Euclidean. There are few exercises and numerical examples, and the notion of construction means to elucidate the concept, and not to use compass and ruler. Holmboe does not give any detailed instructions on how to use ruler and compass in this book, nor does he mention geometric locus, but writes about elucidative [anskueliggjørende] objects, magnitudes and concepts. His idea may be that the mathematics teaching shall educate the students with respect to formal logic, by encouraging them to think and conclude.

Holmboe's first definition is a description of a classification – any bounded part of the space is called a *body* [Legeme], the boundary of the body is called a *face* [Flade], the boundary of a face is called a *line* [Linie], and the boundary of a line is called a *point* [Punkt]. A body has three extensions, called *length* [Længde], *breadth* [Brede] and *height* [Høide], a face has two extensions, called *length* and *breadth*, a line has one extension, called *length*, and a point has no extensions. Holmboe then states without any explanation, but with an illustration,

> "*Fundamental concept. A straight line.*" [Grundbegreb. En ret Linie.]

A *curved line* [krum Linie] is a line of which no part is a straight line, and a *straight plane* [ret Flade] is a plane where one, between two arbitrary points may draw a straight line. The fundamental statements of the straight line is that a straight line may be prolonged infinitely, one may always draw one straight line between two points, and one may never draw more than one straight line between two points. The part of the straight line that lays between the two points is the shortest of all lines drawn between the points, and it is called the *distance* [Affstanden] between the points. (Holmboe 1827: 2–4)

Two of the chapters are called "*About two straight lines intersected by a transversal*",[7] and "*About parallel lines*".[8] The first of these chapters gives a thorough description of all pairs of angles this situation produces. This is followed by the consequences of two corresponding angles being equal, and vice versa, the situations which have the consequence that the corresponding angles are equal (Holmboe 1827: 11–16).

The chapter "*About parallel lines*" has a theorem with proof which states that when two straight lines are intersected by a transversal, such that an outside angle is equal to its corresponding interior angle, that is $\angle r = \angle p$ on the first figure on page 11, then the two straight lines cannot intersect no matter how far they are prolonged in both directions (Holmboe 1827: 45). The structure of the proof is that if the two lines cross on one side of the transversal, then the two lines and the transversal form a triangle, where $\angle r$ is an outside angle. Holmboe has already demonstrated that an outside angle of a triangle is always grater than any of its interior angles (Holmboe 1827: 34), so therefore $\angle r > \angle p$, which contradicts the condition.

This is followed by Holmboe's definition of *parallel lines*:

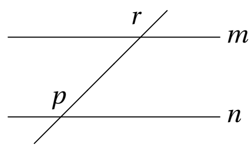
> "*Two straight lines in the same plane that do not intersect when prolonged indefinitely to both sides, are* parallel *to each other, or the one is parallel to the other*"[9] (Holmboe 1827: 46)

[7] Om to rette Linier, som overskjæres af en tredie

[8] Om parallele Linier

[9] To rette Linier i samme Plan, som til begge Sider forlængede i det Uendelige ikke skjære hinanden, siges at være *parallele* med hinanden, eller den ene at være parallel med den anden.

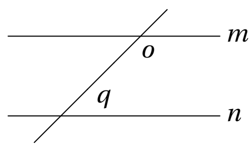
In two following theorems, using the same situation of two straight lines intersected by a transversal, he demonstrates first that if the outside angle is greater than the interior, $\angle r > \angle p$, then the two straight lines are *not* parallel. He next proves that *if* the two lines are parallel, then $\angle r = \angle p$. This last proof is done by assuming that $\angle r \neq \angle p$, and showing that the lines then are *not* parallel. (Holmboe 1827: 50)



$$\begin{aligned} \angle r = \angle p &\implies m \parallel n \\ \angle r > \angle p &\implies m \nparallel n \\ m \parallel n &\implies \angle r = \angle p \end{aligned}$$

Figure 1: Theorems about parallel lines

In a following corollary he then states that if two lines are parallel, and intersected by a transversal, then the sum of the two interior angles equals 2R. This is a consequence of the previous theorem that proves that $\angle r = \angle p$. This is followed by another corollary stating that if the sum of the two interior angles is *not* equal to 2R, then the two lines are *not* parallel. (Holmboe 1827: 51–52)



$$\begin{aligned} m \parallel n &\implies \angle o + \angle q = 2R \\ \angle o + \angle q \neq 2R &\implies m \nparallel n \end{aligned}$$

Figure 2: Corollaries about parallel lines

These two corollaries carry many characteristics of corresponding angles in the original text. It is the last one mentioned here that has the same wording as Euclid's parallel postulate, but it is not emphasized in any way. Holmboe is in his textbook very true to the ideas of the *Elements* in the way of introducing and presenting the subject matter, but without ever referring to or even mentioning Euclid.

Holmboe's textbook in geometry came in a total of four editions, but only the two first were published in Holmboe's lifetime. There are very few differences from the first edition to the second, and none concerning the concepts discussed in this paper.

4.4 Trigonometry

Most of Holmboe's textbooks came in several editions, but the textbook in plane and spherical trigonometry (Holmboe 1834: 3) was his only textbook in basic mathematics that came in one edition only, in 1834. Holmboe starts by defining *trigonometric lines* to an arc, and with this definition, lines in spherical geometry also have trigonometric lines. Trigonometric values calculated by ratios of sides in right angled triangles are results, deducted from the definitions, and presented in his textbook as a theorem with proof.

Holmboe defines *plane trigonometry* (Holmboe 1834: 3) to be that part of geometry that may be used for "solving the triangle". By this Holmboe means that there are six magnitudes in a triangle, three sides and three angles, and when a necessary and sufficient number of these six magnitudes are known to unambiguously define the triangle. The task is to find the remaining magnitudes. There is a similar definition where Holmboe defines *spheric trigonometry* (Holmboe 1834: 45) to be the task to unambiguously define a spherical triangle.

Trigonometric lines

"The dependencies between sides and angles in a triangle, or in general between straight lines and angles, or the circular arcs that measure the angles, may be expressed by certain lines, called trigonometric lines" (Holmboe 1834: 3). These lines are geometric objects shown on the constructions in figures 3 and 4, and the numerical values of their lengths are what a modern reader understand by sines, cosines, etc. With a current understanding of straight lines, we would say that the trigonometric lines were *line segments*. It was, however, not un-common in 19th century and earlier to use the word *line* (Linie or linje) for a line segment.

There is also another important definition that implies that Holmboe is assuming Euclidean geometry. §2 defines that if the sum of two angles equals 90°, they are called *complementary*, and if the sum of two angles equals 180°, they are called *supplementary*. A corollary to §2 states that each of the acute angles in a right angled triangle is the complement of the other acute angle, and that each of the angles in a right angled triangle is the supplement of the sum of the other two angles (Holmboe 1834: 3–4). This corollary refers to §44, with corollaries 1 and 2, in his textbook in geometry (Holmboe 1827: 55–56).

Figure 3 is presented in the textbook (Holmboe 1834: Final page, Figure 1). It is interesting to note that in Holmboe's figure, quadrant 1 is ACH, quadrant 2 is HCK, and so on – mirrored from what we use today.

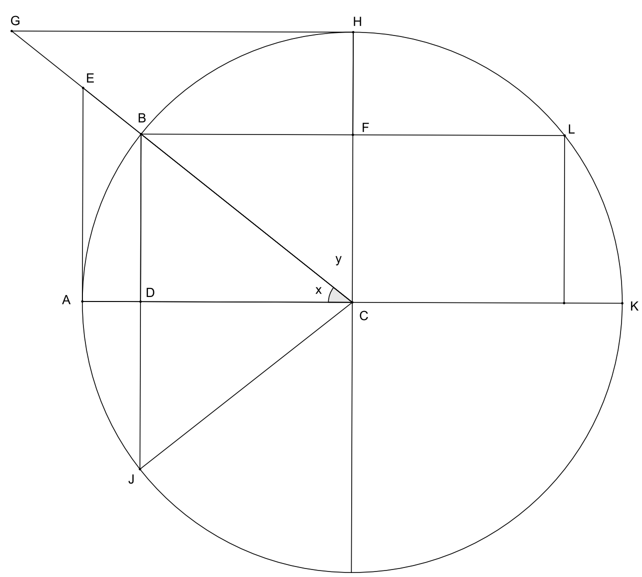


Figure 3: Construction of trigonometric lines

The trigonometric lines to the arc AB are:

Sine – BD The sine of an arc, or of the central angle when the radius is 1, is the perpendicular from the end point of the arc (B) to the diameter through the other end point of the arc (D).

Cosine – DC The cosine of an arc, or of the central angle when the radius is 1, is the sine of the complementary angle.

$BF = DC = \sin y = \cos x = DC$
A consequence of this is that $\cos x = \sin(90° - x)$

Tangent – AE The tangent of an arc, or of the central angle when the radius is 1, is the perpendicular on the diameter through one of the end points of the arc (A), to the point of intersection with the prolonged diameter through the other end point of the arc (E).

Cotangent – GH The cotangent of an arc, or of the central angle when the radius is 1, is the tangent of the complementary angle.
$GH = \tan y = \cot x$

Secant – EC The secant of an arc, or of the central angle when the radius is 1, is the distance between the centre (C), which is the angle vertex, and the end point of the tangent line of the arc, outside the periphery of the circle (E).

Cosecant – GC The cosecant of an arc, or of the central angle when the radius is 1, is the secant of the complementary angle. The distance between the centre (C) and the cotangent line outside the periphery of the circle (G).
$GH = \tan y = \cot x \implies GC = \sec y = \csc x$

Versed sine – AD The versed sine of an arc, or of the central angle when the radius is 1, equals $1 - \cos x$

Versed cosine – HF The versed cosine of an arc, or of the central angle when the radius is 1, equals $1 - \sin x$

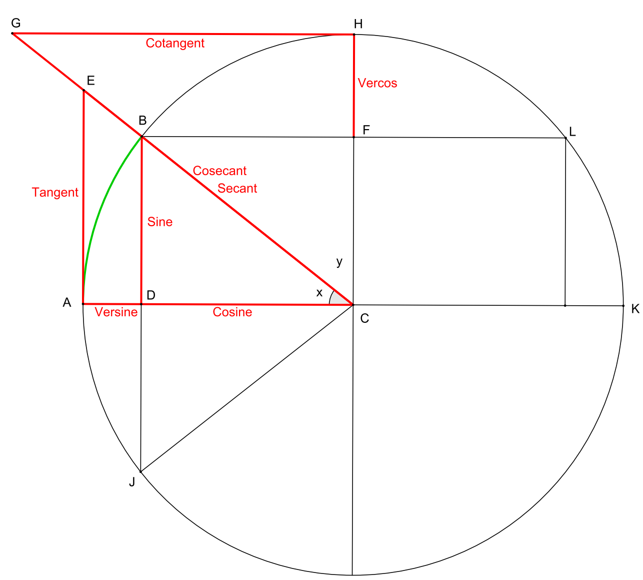


Figure 4: Trigonometric lines

Holmboe (1834: 24–25) demonstrates that $\sin x < AB < \tan x$, when $0 < x < \frac{\pi}{2}$, a point that is used later in calculating the trigonometric values. The sine line to a specific arc is half the chord of the double arc, and an arc is always longer that its chord, therefore $AB > \sin x$. The triangle CAE is greater than the segment CAB, therefore $AB < \tan x$.

Right-angled triangles

Holmboe states a theorem (Holmboe 1834: 15–16) where the trigonometric lines may be found as ratios between the sides in a right-angled triangle, that is the magnitude of one side divided by the magnitude of another side. In today's textbooks in trigonometry these ratios are the definitions of trigonometric values.

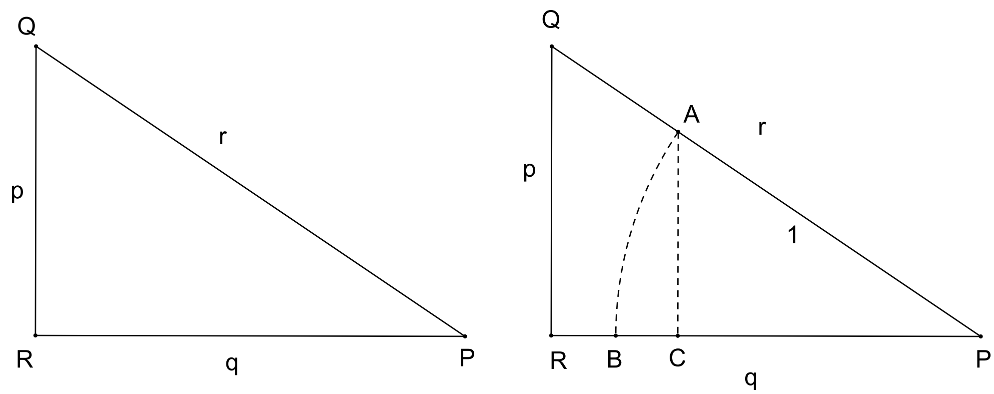


Figure 5: Trigonometry in right-angled triangles

Theorem 1
Consider the triangle △ PQR, ∠R is a right angle, and the sides are lower-case p, q, and r, as shown in Figure 5.

$$p/r = \sin P, \qquad q/r = \cos P$$
$$p/q = \tan P, \qquad q/p = \cot P$$
$$r/q = \sec P, \qquad r/p = \csc P$$

Proof With radius $PA = 1$, draw arc AB. Construct AC perpendicular on PB, as shown in Figure 5.

$$\implies \quad AC = \sin P \quad \text{and} \quad CP = \cos P$$

The ratios between the corresponding sides of the triangles $\triangle PQR$ and $\triangle PAC$ are the same, so we have

$$r : 1 = p : \sin P$$
$$\implies \quad \sin P = p/r$$

and correspondingly for all the other trigonometric values. □

Calculation of trigonometric values

Holmboe (1834: 25–27) describes how to calculate trigonometric values with the precision required. In a unit circle, the length of the full periphery equals 2π, and the problem is to find a value for $\sin x$ with an arbitrary x. He has already established in a theorem that every arc x between 0 and $\pi/2$ is larger than its sine and smaller than its tangent (Holmboe

1834: 24–25), as shown in figure 4 on page 13. Holmboe states in a theorem (Holmboe 1834: 25–26) that for every arc x between 0 and $\pi/2$, the sine must be greater than $x/\sqrt{1+x^2}$. The proof for this is purely algebraic, but my geometric understanding of $x/\sqrt{1+x^2}$ is shown in Figure 6.

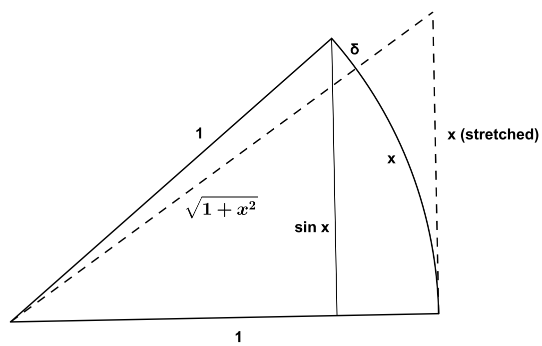


Figure 6: A geometric understanding of $x/\sqrt{1+x^2}$

The magnitude δ is varying continuously and increasing between $\delta_{\min} = 0$ when $x = 0$, and $\delta_{\max}$ when $x = \frac{\pi}{2}$. From the figure we then see that $\sin(x-\delta) = x/\sqrt{1+x^2}$, and since $x > x - \delta$ we have that $\sin x > \sin(x-\delta)$. We may therefore conclude that

$$0 < x < \frac{\pi}{2} \quad \Longrightarrow \quad \sin x > \frac{x}{\sqrt{1+x^2}}$$

The following assignment is then presented.

Assignment 2

> *To calculate the value of trigonometric lines to any given arc with a certain number of decimals.*

Solution 3

$$x - \frac{x}{\sqrt{1+x^2}} < 1 \text{ decimal unit}$$

$$x > \sin x > \frac{x}{\sqrt{1+x^2}}$$

By *decimal unit* is here meant "decimal unit of a certain order after the decimal point", and Holmboe gives the following example.

Example

$$x = 1 \text{ arcminute}$$
$$= \frac{\pi}{180 \cdot 60}$$
$$= 0.00029088$$

$$\sqrt{1+x^2} = 1.0000000423$$
$$\frac{x}{\sqrt{1+x^2}} = 0.00029087$$
$$\sin 1' = 0.00029088$$

The value of π was known with a sufficient number of correct decimals to make these calculations accurate, and $\sin 1'$ has the first eight decimal places in common with $1'$.

The solution is that the difference $x - x/\sqrt{1+x^2}$ will be smaller than 1 decimal unit of the least value that one requires. Then $\sin x$ will be between x and $x/\sqrt{1+x^2}$ where x is the length of the arc.

Holmboe's motivation for calculating sine for arcs as small as 1' is that *Sine of every arc less than 1' has the first eight decimals common with the arc; the smaller the arc is, the smaller is the difference between x and* $x/\sqrt{1+x^2}$, *and* $\sin x$ *is always between these two magnitudes* (Holmboe 1834: 27). If one needs an accuracy of eight correct decimals, and the arc is less than 1', one may calculate the length of the arc instead of finding the sine value. To calculate larger angles with the same number of accurate decimals, one may use angles where the sine and cosine values are known, and one may use known formulas.

5 Christopher Hansteen (1784–1873)

Christopher Hansteen (1784–1873) was born in Christiania in Norway. He was first a law student in Copenhagen, but became interested in the natural sciences after he met the physicist H. C. Ørsted. He became a teacher in applied mathematics at the university in Christiania in 1814, and he was professor from 1816 till 1861. Hansteen was very productive, and wrote about terrestrial magnetism, northern lights, meteorology, astronomy, mechanics, etc. He was a well-known scientist, and received further international recognition after an expedition to Siberia in 1828–30 to study the geomagnetism. In 1835, Hansteen wrote a textbook in geometry where he challenged the traditional Euclidean geometry.

5.1 Plane geometry

In 1835, Hansteen published a textbook in basic geometry (Hansteen 1835), which in many ways challenged Holmboe's textbooks. Hansteen's book was 278 pages, which is a lot more than what is expected of a textbook in elementary geometry. The author is intentionally trying to tear down the walls that existed between the classical geometry on one side, and the newer analytical geometry and the infinitesimal geometry on the other. The basis of the textbook is real life, with references to artifacts like corkscrews, stove pipes and hourglasses. The presentation of the subject matter is very unlike Euclid's Elements. The style is narrative and written in the first person, sometimes very lengthy, and there are many numerical examples. Hansteen tried to expand Euclid's definition of straight lines and of parallel lines, and Euclid's parallel postulate (Euclid 1956).

Hansteen's textbook contains a comprehensive preface which also contains definitions of fundamental concepts. The first concept to be defined is the straight line (Hansteen 1835: III–IV), which is also, according to Hansteen, "*the foundation of geometry*" [Geometriens Grundvold]. It is of great importance that this concept is clearly defined, especially in a science that demands a consistent and logic practice. Hansteen presents five different ways a straight line may be defined

- "*A straight line is a line which lies evenly with the points on itself*" from Euclid (1956). Close to this is also Baron Wolff's definition stating that "*a line is straight when a part is similar to the whole*" [*Linae recta est, cujus pars quæcunque est toti similis*].
- Archimedes, and most French geometers after him, defined the straight line as "*the shortest trajectory between two points*".
- Some geometers regard the straight line as a hereditary concept that only needs to be mentioned to be understood, and defines a straight line as "*those things which is known to be a straight line*" [*Qvæ linea recta dicatur notum est*].
- Abraham Kästner says that "*a straight line is that whose points all bear against one trace*" [en ret Linie er den, hvis Punkter alle ligge hen mod een Egn], and he adds that "*no one will learn the straight line to know from an explanation, and no one needs to; but one may say something about it, that guides the attention to a closer attention to what makes it a straight line*".
- Finally, others say that "*when a point moves continually in the same direction, then its trajectory is a straight line*".

According to Hansteen, after such definitions, all geometers introduces a postulate which states that "*one may create a straight line between two given points, and prolong such a given straight line in any direction in both directions as one pleases*". Hansteen makes noteworthy objections to such a postulate by asking with what tool such a prolonging shall be made, and how to make sure that the line made by such a tool is homogeneous, or that is satisfies the demands made in the various definitions of a straight line (Hansteen 1835: VI–V).

Hansteen elaborates towards a definition where he let lines be produced by the movement of a point, and there are two kinds. One kind has the quality that when two points of a part of the line are placed on two arbitrary points on the whole line, then all points of the part of the line will coincide with points in the whole line – analogous, if we let a part of a line move along the whole line, and the part always fits with the whole line. Such lines are called *homogeneous lines* [eensartede Linier], and there are two types – *straight* and *curved* lines. A homogeneous line has all over the same curvature, and all perpendiculars of any plane homogeneous line will, when duly prolonged, either intersect in one point, or not intersect at all. There are in other words only two types of homogeneous lines in a plane – the straight line and the circle. When a point moves from one place to another in a space, then it describes a line. If this line is straight, it is called the direction of the motion (Hansteen 1835: IV–V,7,9,35–38). From the concept of the straight line we may derive the concept of the plane, and from these definitions we may prove that a line is straight when all the points in the line remains unchanged in the same position is the line is rotated around two arbitrary points on the line (Hansteen 1835: 9), and that a straight line between two points is shorter than any curved or broken lines between these two points (Hansteen 1835: 40).These two statements are not axioms, but theorems.

It is more proper that a "mechanical artist" derives rules for his practice from the definitions and theorems of the geometry, than that the theoretical geometer shall direct his concepts and definitions towards this practice. The carpenter's planer and the metalworker's file are tools that are suitable for producing homogeneous planes and lines, and the geometer should not neglect to acquire the theoretical principles on which these methods are based. A *ruler* is described as a tool – made by wood or metal – by which one may produce straight lines in a plane. (Hansteen 1835: VIII,13)

The cause for the much discussed controversy Hansteen's textbook made was the handling of parallel lines. Hansteen states very clearly that the Euclidean definition of parallel straight lines, embraced by nearly all geometers, has all the logic errors a definition may have. He states correctly that parallel lines are defined, according to Euclid, by a *negative* quality, and not a *positive*. He continues by stating that the quality by which the parallel lines are defined, is *outside all experience and test*, as it points towards the infinite. Euclid's definition may also not be used on curved lines, which may also be parallel – according to Hansteen. "*No one will hesitate in declaring two concentric circles reciprocal parallel*". There is a definition stating that if two lines in a plane never intersect, no matter how far they are prolonged in any direction, does not make an angle (Hansteen 1835: 28). There is, however, no mentioning that these lines are parallel.

Hansteen argued for an understanding of parallel lines where one let a perpendicular to any kind of line move along this line, in such a way that it always is a perpendicular. Any point on this perpendicular then describes a line, where any point's smallest distance to the original line all over is the same (Hansteen 1835: IX). Consequently, Hansteen has a definition of parallel lines

> "*Any line that is being described by a point on the perpendicular to a given line, when it moves along the same with an unaltered angle, is said to be parallel to the directrix*[10]" (Hansteen 1835: 59)

where the characteristics of a line, parallel to another, are

- it always cuts off equal parts of all its perpendiculars
- any perpendicular to one of these lines is also a perpendicular to the other

and a parallel to a straight line has in addition the following characteristics

- the parallel is also a straight line
- as these straight lines never intersect, they form no angle with each other
- if the parallel lines are intersected by a transversal, then the alternate interior angles are equal, the corresponding angles are equal, and the consecutive interior angles equals 2R

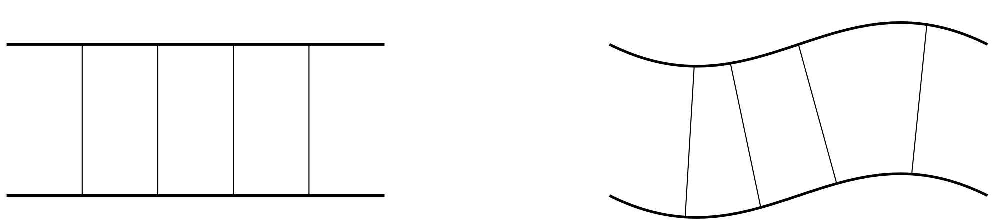


Figure 7: Hansteen's definition of parallel lines

[10] a fixed line used in describing a curve

Hansteen's textbook was published in one edition only, but one reason may be that it contained much subject matter outside the school curriculum. He explains that because of a limited production of textbooks in Norway, he has added subject matter that is outside the curriculum of the learned schools, but should be of interest for students that want to prepare themselves for a study of the higher mathematics (Hansteen 1835: XVIII). It is also worthwhile to mention, as a curiosity, that Hansteen in his textbook introduces and describes *Metre* as a new unit of length (Hansteen 1835: 81).

6 Disagreement and controversy

Holmboe's textbooks were more or less controlling the market regarding textbooks in mathematics in the first half of the 19th century. Hansteen's textbook challenged Holmboe's textbooks, and was the cause of a bitter controversy between the two professors in mathematics.

A newspaper polemic between Holmboe and Hansteen about Hansteen's textbook in geometry took place in *Morgenbladet* from December 1835 till January 1836, and in *Den Constitutionelle* from June till September 1836.[11]

The core of the debate that followed was whether one in mathematics education should let utilitarian considerations overrule logical deduction and theoretical thinking. Hansteen declared that proofs should not be used in the elementary teaching before it was necessary for the students. This, he said, invited the students to memorizing without understanding. To this, Holmboe replied that you either have to prove all or nothing, as half a proof is worse than no proof.

The polemics between Holmboe and Hansteen has later been called the "dispute about parallelism" and they both published booklets where they justified their views (Holmboe 1836; Hansteen 1836).

The main article on the 5th of December, 1835, was written by Holmboe and called "On Professor Hansteen's new understanding of parallel lines" [Om Professor Hansteens nye Parallellære]. It was a review of Hansteen's textbook and it was very critical to Hansteen's definitions of straight and parallel lines. Ten days later there was an unsigned article titled "Concerning Professor B. Holmboe's article in Morgenbladet: 'On Professor Hansteen's new understanding of parallel lines'".[12] The author praised Holmboe for his "touch of thoroughness", but he continued that it was too much to expect from a man who had too long occupied himself with obsolete knowledge, to be an impartial judge of new knowledge. Hansteen's signed reply to Holmboe's article was published on the 18th of December. He stated that Holmboe had reviewed his textbook in a very unseemly manner, and that Holmboe considered Hansteen's textbook dangerous, and teachers should be warned against it, so that young people not are led astray from the rigour of pure and orthodox geometry, into heresy and delusion. A short declaration from Hansteen appeared a week

[11] See Morgenbladet (1835) and Den Constitutionelle (1836).
The newspaper *Morgenbladet* was established in 1819, and was until 1857 a substantial voice for the opposition against the establishment, both literary and political. It was also the first daily newspaper in Norway, and it exists now as a weekly newspaper with a liberal, radical and intellectual profile. *Den Constitutionelle* existed as a daily newspaper in Norway between 1836 and 1847. The idea was to establish a newspaper on a considerably higher intellectual level than Morgenbladet. Den Constitutionelle made high demands on the journalistic content, and it introduced daily editorials. (Kunnskapsforlaget 2006)

[12] Angående Professor B. Holmboes i Morgenbladet No. 339, 1835, indrykkede Stykke: "Om Professor Hansteens nye Parallellære".

later, where he admitted that he probably never would agree with Holmboe about how a good mathematics textbook should be, and that he will publish a booklet the following week. Then there was a short notice signed by Hansteen, dated 18th of January 1836, titled "To the owners of my textbook in geometry" [Til Eierne af min Lærebog i Geometrie], where he admitted that some explanations in his textbook may be simplified. He had therefore made some new pages that by the end of the week would be available at the publisher, free of charge, to the owners of the book.

There was an unsigned paragraph in Den Constitutionelle on the 15th of June, 1836, informing that a professor Jürgensen of Copenhagen had written a review of Hansteen's textbook in the *Monthly Journal for Literature* [Maanedsskrift for Literatur]. This review took no part in the controversy, but asserted the intention of making Hansteen's textbook known in Denmark. Three weeks later there was an article signed Hansteen, titled "On the teaching of mathematics in the schools" [Om den mathematiske Underviisning i Skolerne], where he indicated that the reviewer had been unfortunate with his review. Holmboe now rejoined the fray. In an article he opposed Hansteen by asserting that Hansteen claimed that the only controversies that had been proposed against his textbook in ge-ometry was mainly the question if it is allowed to define a concept before one can prove its possibilities. Holmboe wrote that this is not the case. Hansteen now wrote a long and final article, titled "Farewell to Professor Holmboe" [Afsked til Professor Holmboe]. Hansteen ended his article with an anecdotal remark about Frederick II of Prussia, complaining over the difficulties of being at war with the Russians – "not only did you have to shoot them, you also had to knock them over with your rifle butt", meaning that you not only had to kill them once, you had to kill them twice. Hansteen concluded that he would leave Holmboe standing upright until he got tired – he would not have the trouble of knocking him down. This was the last newspaper article from Hansteen in this matter. Holmboe replied that he was surprised that Hansteen continued this polemics, even though he long time ago said that he would not. Holmboe also asserted that Hansteen had not read his booklet published after the controversy in Morgenbladet. (Morgenbladet 1835; Den Constitutionelle 1836)

6.1 Summary of the controversy

Both Holmboe and Hansteen published booklets where they justified their views. Hansteen wrote a booklet (Hansteen 1836) titled "Investigation of Mr. Professor B. Holmboe's review of my Plane Geometry, Morgenbladet no. 339, 5th of Dec. 1835",[13] dated 26th of December 1835, which means that it was written towards the end of the period the polemics was active in Morgenbladet. Holmboe's name appears only in the title, later he is only referred to "the reviewer". In addition to defending his own textbook, Hansteen also criticizes Holmboe's arguments in the review, and he attacks Holmboe's textbook in geometry (Holmboe 1827).

Hansteen's booklet is organized in five sections, labeled A till E where he focuses on five complaints from Holmboe's review.

A. *Absence of contingency proof* [Forsømmelse af Muelighetsbeviset]. Holmboe's complaint is that Hansteen uses the attributes of lines and surfaces *before* he defines them. Hansteen starts his textbook by classifying lines as *homogeneous* [eensartede] and *heterogeneous* [ueensartede]. Hansteen blames Holmboe for

[13] Belysning af Hr. Professor B. Holmboes Anmeldelse af min Plangeometrie, Morgenbladet No. 339, 5 Dec. 1835.

not respecting authorities like Newton and Laplace, and he is attacking the definitions of basic concepts in Holmbo's textbook in geometry. Hansteen justifies his presentation of the subject matter by the fact that his book has been used for half a year at Christiania Kathedralskole.

B. *Definition of a straight line.* Hansteen is accused of not using accurate descriptions and terms, and he argues by the fact that the textbook is written for children, and their only previous knowledge is their language, and names of concepts from their everyday life. Therefore one has to use a language that stimulates the imagination.

C. *"A circle is a circle".* A vital error, according to Holmboe, is that he states that there exist only two homogeneous lines in a plane, *the straight line* and *the circle*, at a stage where it is unfounded.

D. *Theory of parallelism.* The definition of parallel lines in Hansteen's textbook states that *a line parallel to another has the characteristics that it cuts equal parts of its perpendiculars.* This relates to straight as well as curved lines, and it follows that they will never cross no matter how long you extend them.[14] This definition is, according to Holmboe, not generally correct, as parallel curved lines may cross one another.

E. *Euclidean definition of parallel lines.* Hansteen states that it is better for a concept to be defined by a positive property than by a negative one, and parallel lines are by Euclid defined by a property that lies beyond our experience, and it refers our minds towards the infinite. He also attacks Holmboe's statement that "to construct is to elucidate the specified concepts of the definition of a magnitude",[15] and he finds it paradoxical that thorough knowledge of geometry does not assume the use of compass and ruler. How may such a mental construction elucidate the shape of a curved line, if it is defined by an equation between its coordinates, he asks. He also claims to have met students that didn't know one end of a compass from the other. Holmboe calls the use of compass and ruler an *insignificant requirement* [uvæsentlig fordring] which should not be included in a textbook, and he claims that he has not found these instruments mentioned in textbooks by Lacroix, Legendre, Kästner, Wolff, or Vega. Only the textbooks by Hansteen and Thomas Bugge mention the use of compass and ruler.

Towards the end of his booklet, Hansteen recommends that a new edition of Lindrup's textbook[16] should be made, if one wants easily understood textbooks in arithmetic and geometry that does not frighten students away from studies in mathematics.

Holmboe responded by writing a booklet (Holmboe 1836) titled "*Retort provoked by Mr. Professor Hansteen's enlightenment of my review of his textbook in geometry, containing: 1) Defense of the review containing proofs collected by a continued review of his textbook. 2) Refutation of his attack on my textbook in mathematics*",[17] and this was dated the 8th

[14]Den almindelige Charakter for en Linie, som er parallel med en anden, er altsaa: At den overalt affskjærer ligestore Stykker af dennes Normaler; hvoraf altsaa følger for alleslags parallele Linier, saavel rette som krumme, at de, i hvor langt de end forlænges, aldrig kunne skjære hinanden.

[15]at construere er at anskue det ved en Størrelses Definition fastsatte Begreb

[16]The Danish teacher of mathematics, Hans Christian Linderup (1763–1809) published a textbook in basic mathematics in 1807.

[17]Gjenmæle fremkaldt ved Hr. Professor Hansteens Belysning af min Anmeldelse af hans Lærebog i Geometrien, indeholdende: 1) Forsvar for anmeldelsen med Beviser hentede ved en fortsat Recention over hans Lærebog. 2) Gjendrivelse af hans Angreb paa min Lærebog i Mathematiken.

of March 1836. It was written in the period between the two polemics in Morgenbladet and Den Constitutionelle. Throughout the booklet, Hansteen is referred to as "the author". Holmboe's booklet is structured in the same five sections as Hansteen's, and section D is – not surprisingly – the most comprehensive. Holmboe shows a wide knowledge of the subject matter by quoting Klügel's definition of curved parallel lines from 1763, in addition to the textbook "*Theorie des lignes courbes*" by Lacroix. The latter does not call curved lines parallel. Holmboe admits that Hansteen is correct in his objection against Euclid's definition of parallel lines, that it declares a property that is beyond all experience, in the sense that the definition appears before it is proven that two straight lines in a plane could have such a location that they will never cross if they are prolonged indefinitely. Holmboe is very clear in adding that Hansteen's theory of parallel lines is in obvious conflict with the existing theory, which states that a curved line at a certain point is parallel to another curved line at a certain point, only if the tangents through each of the two points are parallel. The better part of Holmboe's booklet is a defense against the attacks made by Hansteen on his textbooks, and Holmboe is constantly referring to Legendre and his definitions.

7 After the conflict

The first textbook by Holmboe that was published after the conflict with Hansteen was the second edition of the textbook in arithmetic in 1844 and on the reverse page of the title page, the following signed declaration is printed:[18]

> **No. 200**
>
> The second edition of the present textbook's 1st part, or the arithmetic, is printed in 1050 copies. Each copy has a specific number, in such a way that the copies are labelled with the numbers in their order from 1 to 1050. If a copy is not numbered in this manner, and if the reverse side of the title page does not contain this declaration signed by the author, then that copy is illegal, and will be dealt with in accordance with the existing legal provisions.
>
> *B.M. Holmboe*

and this is the only publication by Holmboe that contains such a declaration.

8 Conclutions

The first half of the 19th century was in many ways a turning point for higher education in mathematics in Norway. The position of mathematics as a school subject was strengthened through school reforms at the turn of the century, and the first university was established in Norway in 1811. Bernt Michael Holmboe's textbooks in mathematics were the ones that were predominantly used in the learned schools at that time. His textbooks were, as we have seen, not without opposition – an opposition addressing the use of proofs in elementary mathematics, and whether the introduction of geometry should be in a traditional Euclidean way, using logical deductions and theoretical thinking – as in the case

[18]**No. 200.** Af nærværende Lærebogs 1ste Deels eller Arithmetikens andet Oplag er trykt 1050 Exemplarer. Hvert Exemplar har sit særskilte Nummer, saaledes at Exemplarerne ere nummererede med Tallene efter deres Orden fra 1 til 1050. Saafremt noget ikke saaledes nummereret Exemplar, og hvis Titelblad paa Bagsiden ikke er forsynet med nærværende af Forfatteren underskrevne Erklæring, maatte forefindes, er samme ulovligt, og vil blive behandlet overeensstemmende med de for Eftertryk gjældende Lovbestemmelser. *B.M. Holmboe* (sign.)

of Holmboe – versus a more "informal" way using everyday language and terms. Holmboe is in his textbooks very true to the Euclidean tradition in presenting the subject matter, and there was an ongoing debate about the use of Euclidean ideas in textbooks in geometry. When Hansteen published his textbook in geometry, it was evidently a controversial issue and his textbook was seen as an attack on the Euclidean textbooks.

Hansteen encouraged using a language that stimulated the pupils imagination, and he also used a definition of parallel lines, which – according to Holmboe – was not generally correct. The issues addressed in the newspaper polemics were both the mathematical topics of geometry, and didactical issues – how geometry should be presented to the pupils. Today we find it impressing that a debate like this reached as far as the public press, and that it was given so much attention and space in the papers. This shows, more than anything else does, the position professors at the university had in the society. Hansteen states in the preface of his textbook that there is no lack of good geometry books in the Danish-Norwegian language, but they are all very true to Euclid. It is Hansteen's stated intension to differ from not only Euclid, but also other textbooks. The world of mathematics had been through a development towards strong demands on rigour in definitions and methods (Christiansen 2010), and Hansteen turned against this in his way of presenting the subject matter. Hansteen concretized the mathematical objects, and talked about straight lines as something one could make with a ruler, while the mathematical objects for Holmboe were something one had to elucidate in one's mind, and not to construct. For Holmboe, the mathematical correctness was the most important, while Hansteen had, what we would call today, a much more didactical approach. Hansteen's textbook was only published in one edition, but in addition to being untraditional, Hansteen's textbook also contained much subject matter outside the school curriculum. Even if Hansteen's way of presenting the subject matter would be closer to how we today view didactics, tradition was stronger and Holmboe's books were used in the learned schools until they were replaced by textbooks by Ole Jacob Broch about a decade after Holmboe's death. (Christiansen 2009)

A question was formulated in the introduction regarding what we may learn from these textbooks and the didactical debate from the early 19th century. I have in this paper described important and relevant parts of Holmboe's textbooks in arithmetic, geometry and trigonometry. I have also described parts of Hansteen's textbook in plane geometry and made comparisons between Holmboe's and Hansteen's way of presenting the subject matter. The feud between the two is also a good and well documented example of the didactical debate. The pupils in the learned schools were normally somewhere between 12 and 20 years of age, and a newspaper debate about school mathematics today would probably look completely different. The knowledge level in the textbooks used by the pupils of early 19th century is considerably higher than in modern textbooks, but Holmboe's textbooks were very encyclopaedic, and only contained the subject matter. Modern textbooks may also be characterized by a didactic angle that is missing in the old textbooks, with a possible exception of Hansteen's textbook in plane geometry.

Norwegian pupils are scoring average to low in student assessments by OECD (PISA), and one explanation may be that the content knowledge in mathematics is too low. Mathematics is a comprehensive subject, and pupils will inevitable be encouraged to memorize without understanding if their body of knowledge is too low.

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